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HANKEL DETERMINANT FOR A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS

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Abstract

In this paper, we investigate the bound of the second Hankel determinant $H_2(2) = a_2a_4 - a_3^2$ for the coefficients of a function f belonging to the class $\mathcal{W}_{\alpha,\beta}(\phi)$ of all normalized analytic functions in the open unit disk $\mathbb{U}$, satisfying the following differential subordination:

$$(1-\alpha+2\beta)\frac{f(z)}{z} + (\alpha-2\beta)f'(z) + \beta z f''(z) \prec \phi(z)$$

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1 Introduction

Let $\mathcal{A}$ denote the class of analytic function $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{U},$$

where $U = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk.

We denote by $\mathcal{P}$ the class of analytic functions which satisfies the conditions: $p(0) = 1$ and $\Re(p(z)) > 0, \quad z \in \mathbb{U}$.

We also denote by $\mathcal{B}$ the class of analytic functions which satisfies the conditions: $\omega(0) = 0$ and $|\omega(z)| < 1, \quad z \in \mathbb{U}$

Suppose that functions f and g are analytic in $\mathbb{U}$. We say that function f is subordinated to function g, denoted by $f \prec g$, if there exists a function $\omega \in \mathcal{B}$, such that

$$f(z) = g(\omega(z)), \quad z \in \mathbb{U}.$$

Let $\phi : \mathbb{U} \to \mathbb{C}$ be a function with positive real part,

$$\phi(z) = 1 + A_1 z + A_2 z^2 + A_3 z^3 + \cdots, \quad A_1 > 0,\ A_1,\ A_2,\ A_3,\ \cdots \in \mathbb{R}. \tag{1}$$

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Definition 1. *A function $f \in \mathcal{A}$ is in the class $\mathcal{W}_{\alpha,\beta}(\phi),\ \alpha,\beta \in [0,1]$, if it satisfies the following differential subordination:*

$$(1-\alpha+2\beta)\frac{f(z)}{z} + (\alpha-2\beta)f'(z) + \beta z f''(z) \prec \phi(z) \tag{2}$$

We denote by $\mathcal{W}_{\alpha,\beta}(\lambda),\ 0 \le \lambda < 1$, the class of analytic functions satisfying:

$$\Re\left[(1-\alpha+2\beta)\frac{f(z)}{z} + (\alpha-2\beta)f'(z) + \beta z f''(z)\right] > \lambda$$

Clearly, for $\phi(z) = \dfrac{1+(1-2\lambda)z}{1-z}$, the class $\mathcal{W}_{\alpha,\beta}(\phi)$ becomes $\mathcal{W}_{\alpha,\beta}(\lambda)$.

The qth Hankel determinant, for $q, n \in \mathbb{N} = \{1,2,3,\cdots\}$, is defined as follows:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}$$

Pommerenke [8] investigated the Hankel determinant of areally mean p-valent functions, univalent and starlike functions. Hankel determinants have been studied by several authors for different classes of analytic, univalent or bi-univalent functions, see [1, 2, 5, 6, 7, 9] In these papers, the most discussed determinants were $H_2(2),\ H_3(1)$ and $|H_2(1)|$ which is the well-known Fekete-Szegö problem.

In this paper, we investigate the second Hankel determinant for a function f belonging to the class $\mathcal{W}_{\alpha,\beta}(\phi)$ and, for particular cases, we obtain well-known results.

In order to prove our main result, we will need the following lemmas:

Lemma 1. *[3] If function*

$$p(z) = 1 + \sum_{n=2}^{\infty} p_n z^n, \quad z \in \mathbb{U} \tag{3}$$

is in $\mathcal{P}$, then $|p_n| \le 2, \quad n = 1,2,\cdots$.
The result is sharp.

Lemma 2. *[4] If function $p(z)$ given by (3) is in $\mathcal{P}$, then*

$$\begin{aligned} 2p_2 =& p_1^2 + x(4-p_1^2), \\ 4p_3 =& p_1^3 + 2(4-p_1^2)p_1 x - p_1(4-p_1^2)x^2 + 2(4-p_1^2)(1-|x|^2)y \end{aligned} \tag{4}$$

for some x, y with $|x| \le 1$ and $|y| \le 1$.

2 Main results

Theorem 1. *Let* $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ *be in the class* $\mathcal{W}_{\alpha,\beta}(\phi)$*, where* $\alpha, \beta \in [0,1]$.

1. *If* $(2-2m)|A_2| \leq (2m-1)A_1$ *and* $\left|A_3 - m\frac{A_2^2}{A_1}\right| \leq mA_1$ *then the second Hankel determinant satisfies:*

$$|a_2a_4 - a_3^2| \leq \frac{A_1^2}{(1+2\alpha+2\beta)^2} \tag{5}$$

2. *If* $(2-2m)|A_2| \geq (2m-1)A_1$ *and* $\left|A_3 - m\frac{A_2^2}{A_1}\right| \geq (1-m)A_2 + \frac{1}{2}A_1$ *or* $(2-2m)|A_2| \leq (2m-1)A_1$ *and* $\left|A_3 - m\frac{A_2^2}{A_1}\right| \geq mA_1$ *then the second Hankel determinant satisfies:*

$$|a_2a_4 - a_3^2| \leq \frac{A_1}{(1+\alpha)(1+3\alpha+6\beta)}\left|A_3 - m\frac{A_2^2}{A_1}\right| \tag{6}$$

3. *If* $(2-2m)|A_2| > (2m-1)A_1$ *and* $\left|A_3 - m\frac{A_2^2}{A_1}\right| \leq (1-m)A_2 + \frac{1}{2}A_1$*then the second Hankel determinant satisfies:*

$|a_2a_4 - a_3^2| \leq$

$$\leq \frac{A_1}{(1+\alpha)(1+3\alpha+6\beta)}\left\{mA_1 - \frac{[(2-2m)|A_2| + (1-2m)A_1]^2}{4\left|A_3 - m\frac{A_2^2}{A_1}\right| - (1-m)(8|A_2| + 4A_1)}\right\} \tag{7}$$

where

$$m = \frac{(1+\alpha)(1+3\alpha+6\beta)}{(1+2\alpha+2\beta)^2}. \tag{8}$$

Proof. Let $f \in \mathcal{W}(\alpha,\beta)$. Then, there exists $\omega \in \mathcal{B}$ such that

$$(1-\alpha+2\beta)\frac{f(z)}{z} + (\alpha-2\beta)f'(z) + \beta zf''(z)) = \phi(\omega(z)) \tag{9}$$

We define

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots . \tag{10}$$

Since $\omega \in \mathcal{B}$, it follows that $p \in \mathcal{P}$,thus

$$\omega(z) = \frac{1}{2}p_1z + \frac{1}{2}\left(p_2 - \frac{1}{2}p_1^2\right)z^2 + \frac{1}{2}\left(p_3 - p_1p_2 + \frac{1}{4}p_1^3\right)z^3 + \cdots \tag{11}$$

From (1) and (11), we obtain

$$\phi(\omega(z)) = 1 + \frac{1}{2}A_1p_1z + \left[\frac{1}{4}A_2p_1^2 + \frac{1}{2}A_1\left(p_2 - \frac{1}{2}p_1^2\right)\right]z^2 + \\ + \left[\frac{1}{2}A_1\left(p_3 - p_1p_2 + \frac{1}{4}p_1^3\right) + \frac{1}{2}A_2p_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{8}A_3p_1^3\right]z^3 + \cdots \tag{12}$$

Also, the Taylor expansion of f gives

$$(1-\alpha+2\beta)\frac{f(z)}{z} + (\alpha-2\beta)f'(z) + \beta z f''(z) = \\ 1 + (1+\alpha)a_2z + (1+2\alpha+2\beta)a_3z^2 + (1+3\alpha+6\beta)a_4z^3 + \cdots \tag{13}$$

Then, from (9), (12) and (13), we have

$$a_2 = \frac{A_1p_1}{2(1+\alpha)} \tag{14}$$

$$a_3 = \frac{1}{1+2\alpha+2\beta}\left(\frac{1}{2}A_1p_2 + \frac{A_2-A_1}{4}p_1^2\right) \tag{15}$$

$$a_4 = \frac{1}{1+3\alpha+6\beta}\left(\frac{A_1p_3}{2} + \frac{A_2-A_1}{2}p_1p_2 + \frac{A_1-2A_2+A_3}{8}p_1^3\right) \tag{16}$$

Therefore,

$$a_2a_4 - a_3^2 = \left[\frac{A_1(A_1-2A_2+A_3)}{16(1+\alpha)(1+3\alpha+6\beta)} - \frac{(A_2-A_1)^2}{16(1+2\alpha+2\beta)^2}\right]p_1^4 \\ + \left[\frac{A_1(A_2-A_1)}{4(1+\alpha)(1+3\alpha+6\beta)} - \frac{A_1(A_2-A_1)}{4(1+2\alpha+2\beta)^2}\right]p_1^2p_2 \\ + \frac{A_1^2}{4(1+\alpha)(1+3\alpha+6\beta)}p_1p_3 - \frac{A_1^2}{4(1+2\alpha+2\beta)^2}p_2^2 \tag{17}$$

We can assume that $p_1 = p > 0$. Substituting the values of p_2 and p_3 from Lemma 2 in the above expresion, we obtain

$$|a_2a_4 - a_3^2| = \Bigg|\left(\frac{A_1A_3}{16(1+\alpha)(1+3\alpha+6\beta)} - \frac{A_2^2}{16(1+2\alpha+2\beta)^2}\right)p^4 \\ + \left(\frac{A_1A_2}{8(1+\alpha)(1+3\alpha+6\beta)} - \frac{A_1A_2}{8(1+2\alpha+2\beta)^2}\right)p^2(4-p^2)x \\ + \left[\left(\frac{-A_1^2p^2}{16(1+\alpha)(1+3\alpha+6\beta)} - \frac{A_1^2(4-p^2)}{16(1+2\alpha+2\beta)^2}\right)\right](4-p^2)x^2 \\ + \frac{A_1^2}{8(1+\alpha)(1+3\alpha+6\beta)}p(4-p^2)(1-|x|^2)y\Bigg| \tag{18}$$

Replacing $|x|$ by μ and $|y| \leq 1$ in the above equality, we obtain

$$\begin{aligned} |a_2a_4 - a_3^2| \leq & \frac{A_1}{16(1+\alpha)(1+3\alpha+6\beta)} \left\{ \left| A_3 - \frac{(1+\alpha)(1+3\alpha+6\beta)}{(1+2\alpha+2\beta)^2} \frac{A_2^2}{A_1} \right| p^4 \right. \\ & + 2\frac{(\alpha+\beta)^2+\beta(3\beta-2)}{(1+2\alpha+2\beta)^2} |A_2| p^2(4-p^2)\mu \\ & + \left[p^2 + (4-p^2)\frac{(1+\alpha)(1+3\alpha+6\beta)}{(1+2\alpha+2\beta)^2} - 2p \right] A_1(4-p^2)\mu^2 \\ & \left. + 2A_1p(4-p^2) \right\} = F(p,\mu). \end{aligned} \tag{19}$$

In order to maximize function $F(p,\mu)$, for $p \in [0,2]$ and $\mu \in [0,1]$, we shall differentiate it partially with respect to μ as follows:

$$\begin{aligned} \frac{\partial F}{\partial \mu} = & \frac{A_1}{16(1+\alpha)(1+3\alpha+6\beta)} \left\{ 2\frac{(\alpha+\beta)^2+\beta(3\beta-2)}{(1+2\alpha+2\beta)^2} |A_2| p^2(4-p^2) \right. \\ & \left. + \left[p^2 + (4-p^2)\frac{(1+\alpha)(1+3\alpha+6\beta)}{(1+2\alpha+2\beta)^2} - 2p \right] 2A_1(4-p^2)\mu \right\} \end{aligned} \tag{20}$$

We observe that $\frac{\partial F}{\partial \mu} > 0$ for $\mu \in [0,1]$ and $p \in [0,2]$, thus $F(p,\mu)$ is an increasing function of μ, having its maximum value:

$$\max F(p,\mu) = F(p,1) = G(p). \tag{21}$$

Afer some computation, we have:

$$\begin{aligned} G(p) = & \frac{A_1}{16(1+\alpha)(1+3\alpha+6\beta)} \left\{ \left[\left| A_3 - \frac{(1+\alpha)(1+3\alpha+6\beta)}{(1+2\alpha+2\beta)^2} \frac{A_2^2}{A_1} \right| \right. \right. \\ & \left. - \frac{(\alpha+\beta)^2+\beta(3\beta-2)}{(1+2\alpha+2\beta)^2} (2|A_2| + A_1) \right] p^4 \\ & + \left[8\frac{(\alpha+\beta)^2+\beta(3\beta-2)}{(1+2\alpha+2\beta)^2} |A_2| + 4\left(1 - 2\frac{(1+\alpha)(1+3\alpha+6\beta)}{(1+2\alpha+2\beta)^2} \right) A_1 \right] p^2 \\ & \left. + 16\frac{(1+\alpha)(1+3\alpha+6\beta)}{(1+2\alpha+2\beta)^2} A_1 \right\} \\ = & \frac{A_1}{16(1+\alpha)(1+3\alpha+6\beta)} \left(Pt^2 + Qt + R \right), \quad p^2 = t \in [0,4]. \end{aligned} \tag{22}$$

where

$$\begin{aligned} & P = \left| A_3 - m\frac{A_2^2}{A_1} \right| - (1-m)\left(2|A_2| + A_1 \right), \\ & Q = 8(1-m)|A_2| + 4A_1(1-2m), \\ & R = 16mA_1 \end{aligned} \tag{23}$$

and m is given by (8).

We know that

$$\max_{0\le t\le 4}\left(Pt^2+Qt+R\right)=\begin{cases} R, & Q\le 0,\ P\le -\dfrac{Q}{4};\\ 16P+4Q+R, & Q\ge 0,\ P\ge -\dfrac{Q}{8}\ \textit{or}\\ & Q\le 0,\ P\ge -\dfrac{Q}{4};\\ \dfrac{4PR-Q^2}{4P}, & Q>0,\ P\le -\dfrac{Q}{8}. \end{cases} \tag{24}$$

Thus, after a simple computation, we obtain the result stated in the theorem. □

For $\phi(z)=\dfrac{1+(1-2\lambda)z}{1-z}$, we obtain the following corollary:

Corollary 1. *Let $f(z)=z+a_2z^2+a_3z^3+\cdots$ be in the class $\mathcal{W}_{\alpha,\beta}(\lambda)$, where $\alpha,\beta,\lambda\in[0,1]$.*

1. *If $m\ge\frac{3}{4}$, then the second Hankel determinant satisfies:*

$$|a_2a_4-a_3^2|\le\frac{(2-2\lambda)^2}{(1+2\alpha+2\beta)^2} \tag{25}$$

2. *If $m<\frac{3}{4}$, then the second Hankel determinant satisfies:*

$$|a_2a_4-a_3^2|\le\frac{(2-2\lambda)^2}{(1+\alpha)(1+3\alpha+6\beta)}\left[\frac{1}{8(1-m)}+1-m\right] \tag{26}$$

where m is given by (8)*.*

Remark 1. For $\alpha=1,\beta=0$ and $\phi(z)=\dfrac{1+z}{1-z}$, Theorem 1 reduces to the result stated in [5], *that is:* $|a_2a_4-a_3^2|\le\frac{4}{9}$.

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