Bulletin of the *Transilvania* University of Braşov • Vol 12(61), No. 1 - 2019
Series III: Mathematics, Informatics, Physics, 95-106
https://doi.org/10.31926/but.mif.2019.12.61.1.8

AN ANALYSIS OF $(0,1;0)$ INTERPOLATION BASED ON THE ZEROS OF ULTRASPHERICAL POLYNOMIALS

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Abstract

The aim of this paper is to construct an interpolatory polynomial (0,1;0) with special types of boundary conditions. Here the nodes $\{x_i\}_{i=1}^n$ and $\{x_i^*\}_{i=1}^{n-1}$ are the roots of $P_n^{(k)}(x)$ and $P_{n-1}^{(k+1)}(x)$ respectively, where $P_n^{(k)}(x)$ is the Ultraspherical polynomial of degree n. In this paper, we prove, existence, explicit representation and order of convergence of the interpolatory polynomial.

2000 *Mathematics Subject Classification:* 41A10, 97N50.
Key words: Lagrange interpolation, Ultraspherical polynomial, Explicit form, Order of convergence.

1 Introduction

The Ultraspherical polynomial $P_n^{(k)}(x)$ of degree n and order k is defined by

$$P_n^{(k)}(x) = \frac{\Gamma(n+2k)\,\Gamma\left(k+\frac{1}{2}\right)(-1)^n}{\Gamma(2k)\,\Gamma\left(n+k+\frac{1}{2}\right)2^n n!} \times (1-x^2)^{-k+\frac{1}{2}} \frac{d^n}{dx^n}[(1-x^2)^{n+k-\frac{1}{2}}]$$

for n=0, 1, 2, ... In 1979, J.S.Hwang [6] studied the Turan's problem of (0,2) interpolation on the zeros of Jacobi polynomials. Later, A.M.Chak and J.Szabados [1] introduced the similar problem of (0,2) interpolation on the zeros of Laguerre polynomials $L_n^{(\alpha)}(x)$ $(\alpha > 1)$. He considered, the (0,2) interpolating polynomial $R_m(f,x)$ of degree at most $2n-1$ associated with $f(x)$,which are defined by the relations.

$$R_m(f, x_k) = f(x_k),$$

$$R''_m(f, x_k) = 0, \quad k = 1, 2..., n.$$

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Further, in (1995) I.Joo and L.Sizli [5] studied the problem in the case when the fundamental points are the roots of Jacobi polynomials and considered the weight function w(x)=$(1-x)^{\frac{\alpha+1}{2}}(1+x)^{\frac{\beta+1}{2}}$ $(x\epsilon[-1,1];\alpha,\beta>-1)$. K.K. Mathur and R.B. Saxena [7] have extended the study of weighted (0,2) interpolation due to J.Balazs [2] and L. Sizli [8] to the case of weighted (0,1,3) interpolation on the zeros of Hermite polynomials. Later, M. Lenard initiated the study of interpolation [9],[10]. In paper [10] M. Lenard considered the function values are interpolated at the zeros of the polynomial $P_{n-1}^{(k+1)}(x)$ and the first derivative values are intepolated at the zeros of the polynomial $P_n^{(k)}(x)$ with hermite conditions on the interval $[-1,1]$.

The convergence of this interpolation process was studied by Xie [11] for k=0, if $f\in C^r[-1,1]$ for $x\in[-1,1]$, then

$$|f(x)-R_{2n+1}(x;f)|=O(n^{-r+1})w(f^{(r)};\frac{1}{n}). \tag{1}$$

Xie and Zhou [12] proved for k=0, if $f\in C^r[-1,1]$, $r\geq 2$, $for\ x\in[-1,1]$, then

$$|f'(x)-R'_{2n+1}(x;f)|=O(1)w(f^{(r)};\frac{1}{n})O(n^{-r+\frac{5}{2}}), \tag{2}$$

also stated the above property of convergence if $f\in C^2[-1,1]$, $f^2\in Lip\alpha$, $\alpha>\frac{1}{2}$,then $R'_{2n+1}(x;f)$ converges to $f'(x)$ uniformly on [-1,1] . For $k\geq 1$ Lenard [9] proved that if $f\in C^r[-1,1]\ for\ x\in[-1,1]$,then

$$|f(x)-R_m(x;f)|=O(n^{k-r+\frac{1}{2}})w(f^{(r)};\frac{1}{n}), \tag{3}$$

where $w(f^{(r)},.)$ denotes the modulus of continuity of the r^{th} derivative of the function $f(x)$.
The aim of this paper is to extend the study of (0;1) interpolation problem of M.Lenard [10] to the case (0,1;0) interpolation with Hermite-type boundary conditions on interval $[-1,1]$.
We have given the following problem.

Problem:

Let the set of knots be given by

$$-1=x_n^*<x_n<x_{n-1}^*<x_{n-1}<.....,<x_1^*<x_1<x_0^*=1,\quad n\geq 1, \tag{4}$$

where $\{x_i\}_{i=1}^n$ and $\{x_i^*\}_{i=1}^{n-1}$ are the roots of ultraspherical polynomials $P_n^{(k)}(x)$ and $P_{n-1}^{(k+1)}(x)$ respectively, on the knots (4) there exists a unique polynomial $R_m(x)$ of degree at most $m=3n+2k+1$ satisfying the interpolatory conditions.

$$R_m(x_i)=y_i,\quad (i=1,2,...,n), \tag{5}$$

$$R_m{}'(x_i) = y_i{}', \quad (i = 1, 2, ..., n), \tag{6}$$

$$R_m(x_i^*) = y_i^*, \quad (i = 1, 2, ..., n-1), \tag{7}$$

with (Hermite) boundary conditions.

$$R_m^{(l)}(1) = y_1^{(l)}, \quad (l = 0, 1, ..., k), \tag{8}$$

$$R_m^{(l)}(-1) = y_{-1}^{(l)}, \quad (l = 0, 1, ..., k+1), \tag{9}$$

where y_i ,$y_i{}'$, y_i^* , $y_1^{(l)}$ and $y_{-1}^{(l)}$ are arbitrary real numbers and k is a fixed non-negative integer.
In section 2, we gave some results of [3] and proved new results in section 3. The order of convergence and main theorem of convergence have been proved in section 4.

2 Preliminaries:

We shall use well known properties and results [3] of the Ultraspherical polynomials. Let $P_n^{(k)}(x)$=$P_n^{(k,k)}(x)$ $(k > -1, n \geq 0)$ denote the ultraspherical polynomial of degree n. we refer to [3] (4.2.1).

$$(1-x^2)P_n^{(k)''}(x) - 2x(k+1)P_n^{(k)'}(x) + n(n+2k+1)P_n^{(k)}(x) = 0, \tag{10}$$

refer to [3] (4.21.7)

$$P_n^{(k)'}(x) = \frac{n+2k+1}{2}P_{n-1}^{(k+1)}(x), \tag{11}$$

$$|P_n^{(k)}(x)| = O(n^k), \quad x \in [-1, 1], \tag{12}$$

$$(1-x^2)^{\frac{k}{2}+\frac{1}{4}}|P_n^{(k)}(x)| = O(\frac{1}{\sqrt{n}}). \tag{13}$$

The fundamental polynomials of Lagrange interpolation are given by:

$$l_j(x) = \frac{P_n^{(k)}(x)}{P_n^{(k)'}(x_j)(x-x_j)} \tag{14}$$

$$l_j^*(x) = \frac{P_{n-1}^{(k+1)}(x)}{P_{n-1}^{(k+1)'}(x_j^*)(x-x_j^*)}, \tag{15}$$

$$l_j(x) = \frac{P_n^{(k)}(x)}{P_n^{(k)'}(x_j)(x-x_j)} = \frac{\tilde{h}_n^{(k)}}{(1-x_j^2)[P_n^{(k)'}(x_j)]^2}\sum_{\nu=0}^{n-1}\frac{1}{h_\nu^{(k)}}P_\nu^{(k)}(x_j)P_\nu^{(k)}(x), \tag{16}$$

where

$$\tilde{h}_n^{(k)} = \frac{2^{2k}\Gamma\left(2(n+k+1)\right)}{\Gamma\left(n+1\right)\Gamma\left(n+2k+1\right)} \sim C_1, \tag{17}$$

$$h_\nu^{(k)} = \frac{2^{2k+1}}{2\nu+2k+1}\frac{\Gamma\left(2(\nu+k+1)\right)}{\Gamma\left(\nu+1\right)\Gamma\left(\nu+2k+1\right)}\begin{cases} \sim \frac{1}{\nu} & (\nu>0), \\ = C_2 & (\nu=0), \end{cases} \tag{18}$$

where the constants C_1, C_2 depend only on k.

If $x_1 > x_2 > .....,> x_n$ are the roots of $P_n^{(k)}(x)$, then the following relations hold [3].

$$(1-x_j^2) \sim \begin{cases} \frac{j^2}{n^2} & (x_j \geq 0), \\ \\ \frac{(n-j)^2}{n^2} & (x_j < 0), \end{cases} \tag{19}$$

$$|P_n^{(k)\prime}(x_j)| \sim \begin{cases} \frac{n^{k+2}}{j^{k+\frac{3}{2}}} & (x_j \geq 0), \\ \\ \frac{n^{k+2}}{(n-j)^{k+\frac{3}{2}}} & (x_j < 0). \end{cases} \tag{20}$$

3 Explicit Representation of Interpolatory polynomials:

We shall write $R_m(x)$ satisfying (5) - (9) as

$$R_m(x) = \sum_{j=1}^{n} A_j(x)y_j + \sum_{j=1}^{n} B_j(x){y_j}' + \sum_{j=1}^{n-1} C_j(x)y_j^* + \sum_{j=0}^{k} D_j(x)y_1^{(l)} + \sum_{j=0}^{k+1} E_j(x)y_{-1}^{(l)}, \tag{21}$$

where $A_j(x)$ and $C_j(x)$ are the fundamental polynomials of first kind and $B_j(x)$ is the fundamental polynomial of second kind. $D_j(x)$ and $E_j(x)$ are the fundamental polynomials which correspond to the boundary conditions each of degree $\leq 3n+2k+1$, uniquely determined by the following conditions,

For $j = 1, 2, ..., n$

$$\begin{cases} A_j(x_i) = \delta_{ji}, & (i = 1, 2, ..., n) \\ {A_j}'(x_i) = 0, & (i = 1, 2, ..., n) \\ A_j(x_i^*) = 0, & (i = 1, 2...., n-1) \\ {A_j}^l(1) = 0, & (l = 0, 1, ..., k) \\ {A_j}^l(-1) = 0, & (l = 0, 1, ..., k+1) \end{cases} \tag{22}$$

For $j = 1, 2, ..., n$

$$\begin{cases} B_j(x_i) = 0, & (i = 1, 2, ..., n) \\ {B_j}'(x_i) = \delta_{ji}, & (i = 1, 2, ..., n) \\ B_j(x_i^*) = 0, & (i = 1, 2...., n-1) \\ {B_j}^l(1) = 0, & (l = 0, 1, ..., k) \\ {B_j}^l(-1) = 0, & (l = 0, 1, ..., k+1) \end{cases} \tag{23}$$

For $j = 1, 2, ..., n-1$

$$\begin{cases} C_j(x_i) = 0, & (i = 1, 2, ..., n) \\ C_j{}'(x_i) = 0, & (i = 1, 2, ..., n) \\ C_j(x_i^*) = \delta_{ji}, & (i = 1, 2...., n-1) \\ C_j{}^l(1) = 0, & (l = 0, 1, ..., k) \\ C_j{}^l(-1) = 0, & (l = 0, 1, ..., k+1) \end{cases} \tag{24}$$

For $j = 0, 1, ..., k$

$$\begin{cases} D_j(x_i) = 0, & (i = 1, 2, ..., n) \\ D_j{}'(x_i) = 0, & (i = 1, 2, ..., n) \\ D_j(x_i^*) = 0, & (i = 1, 2...., n-1) \\ D_j{}^l(1) = \delta_{jl}, & (l = 0, 1, ..., k) \\ D_j{}^l(-1) = 0, & (l = 0, 1, ..., k+1) \end{cases} \tag{25}$$

For $j = 0, 1, ..., k+1$

$$\begin{cases} E_j(x_i) = 0, & (i = 1, 2, ..., n) \\ E_j{}'(x_i) = 0, & (i = 1, 2, ..., n) \\ E_j(x_i^*) = 0, & (i = 1, 2...., n-1) \\ E_j{}^l(1) = 0, & (l = 0, 1, ..., k) \\ E_j{}^l(-1) = \delta_{jl}, & (l = 0, 1, ..., k+1) \end{cases} \tag{26}$$

We proved the Explicit forms which are given in the following Lemmas.

Lemma 1. *The fundamental polynomial $C_j(x)$, for $j = 1, 2, ..., n-1$ satisfying the interpolatory conditions* (24) *is given by:*

$$C_j(x) = \frac{(1+x)(1-x^2)^{k+1}\{P_n^{(k)}(x)\}^2 l_j^*(x)}{(1+x_j^*)(1-x_j^{*2})^{k+1}\{P_n^{(k)}(x_j^*)\}^2}. \tag{27}$$

Lemma 2. *The fundamental polynomial $B_j(x)$, for $j = 1, 2, ..., n$ satisfying the interpolatory conditions* (23) *is given by:*

$$B_j(x) = \frac{(1+x)(1-x^2)^{k+1}P_n^{(k)}(x)P_n^{(k)'}(x)l_j(x)}{(1+x_j)(1-x_j{}^2)^{k+1}\{P_n^{(k)'}(x_j)\}^2}. \tag{28}$$

Lemma 3. *The fundamental polynomial $A_j(x)$, for $j = 1, 2, ..., n$ satisfying the interpolatory conditions* (22) *is given by:*

$$A_j(x) = \frac{(1+x)(1-x^2)^{k+1}P_n^{(k)'}(x)\{l_j(x)\}^2}{(1+x_j)(1-x_j^2)^{k+1}P_n^{(k)'}(x_j)} - \frac{\{1+2(1+x_j)l_j{}'(x_j)\}B_j(x)}{(1+x_j)}. \tag{29}$$

Lemma 4. *The fundamental polynomial $D_j(x)$, for $j = 0, 1, ..., k$ which corresponds to the boundary condition, satisfying the interpolatory conditions* (25) *is given by:*

$$D_j(x) =(1-x)^j(1+x)^{k+2}P_n^{(k)}(x)\{P_n^{(k)'}(x)\}^2 p_j(x) + (1+x)(1-x^2)^{k+1}P_n^{(k)'}(x)P_n^{(k)}(x) \times \{\frac{P_n^{(k)}(x)q_j(x) - P_n^{(k)'}(x)p_j(x)}{(1-x)^{k+1-j}}\}, \tag{30}$$

where degree $p_j(x) \leq k-j+1$ and degree $q_j(x) \leq k-j$.

Lemma 5. *The fundamental polynomial $E_j(x)$, for $j = 0, 1, ..., k+1$ which corresponds to the boundary condition, satisfying the interpolatory conditions* (26) *is given by:*
For $j = 0, 1..., k$

$$E_j(x) =(1-x)^{k+2}(1+x)^j P_n^{(k)}(x)\{P_n^{(k)'}(x)\}^2 \tilde{p_j}(x) + (1-x^2)^{k+1}P_n^{(k)}(x)P_n^{(k)'}(x) \times \{\frac{P_n^{(k)}(x)\tilde{q_j}(x) - (1-x)P_n^{(k)'}(x)\tilde{p_j}(x)}{(1+x)^{k+1-j}}\}, \tag{31}$$

where degree $\tilde{p_j}(x) \leq k-j+1$ and degree $\tilde{q_j}(x) \leq k-j+1$,
For $j = k+1$

$$E_{k+1}(x) = \frac{(1-x^2)^{k+1}\{P_n^{(k)}(x)\}^2 P_n^{(k)'}(x)}{(k+1)!2^{k+1}\{P_n^{(k)}(-1)\}^2 P_n^{(k)'}(-1)}. \tag{32}$$

Existence:

By Lemma 1 to Lemma 5, the polynomial $R_m(x)$ satisfies conditions (22)-(26)*, so there exists an interpolatory polynomial $R_m(x)$ of degree 3n+2k+1.*

4 Order of Convergence of the fundamental polynomials.

Theorem 1. *If $k > 0$, $n \geq 2$, for the first derivative of the first kind fundamental polynomials on [-1,1] holds*

$$\sum_{j=1}^{n-1}(1-x_j^{*2})|C_j'(x)| = O(n^{k+\frac{11}{2}}). \tag{33}$$

Proof. Differentiating (27) , we get

$$\sum_{j=1}^{n-1}(1-x_j^{*2})|C_j'(x)| = \eta_1 + \eta_2 + \eta_3,$$

where we use the decomposition (15) in η_1 for $l_j^*(x)$, we have

$$\eta_1 \leq \sum_{j=1}^{n-1} \frac{\{(1-x^2)^{k+1} + 2x(k+1)(1+x)(1-x^2)^k\}|P_n^{(k)}(x)|^2}{(1+x_j^*)(1-x_j^{*2})^{k+1}|P_n^{(k)}(x_j^*)|^2|P_{n-1}^{(k+1)'}(x_j^*)|^2} \times \tilde{h}_{n-1}^{(k+1)}$$
$$\times \{\gamma_1 + \sum_{\nu=1}^{n-2} \frac{1}{h_\nu^{(k+1)}} |P_\nu^{(k+1)}(x_j^*)||P_\nu^{(k+1)}(x)|\},$$

where γ_1 is a constant which is independent of n.
By using (19) and (20) , we get

$$\frac{1}{(1-x_j^{*2})^{k+1}|P_{n-1}^{(k+1)'}(x_j^*)|^2} = O(\frac{1}{n-1}), \tag{34}$$

using (12), (13) , (18), (19) , (20) and (34) , we obtain

$$\eta_1 = O(n^{k+\frac{7}{2}}).$$

Again using decomposition (15) in η_2 for $l_j^*(x)$, we have

$$\eta_2 \leq \sum_{j=1}^{n-1} \frac{2(1+x)(1-x^2)^{k+1}|P_n^{(k)}(x)||P_n^{(k)'}(x)|}{(1+x_j^*)(1-x_j^{*2})^{k+1}|P_n^{(k)}(x_j^*)|^2|P_{n-1}^{(k+1)'}(x_j^*)|^2} \times \tilde{h}_{n-1}^{(k+1)}$$
$$\times \{\gamma_2 + \sum_{\nu=1}^{n-2} \frac{1}{h_\nu^{(k+1)}} |P_\nu^{(k+1)}(x_j^*)||P_\nu^{(k+1)}(x)|\},$$

where γ_2 is a constant which is independent of n.
Using (12), (13), (18), (19), (20) and (34) , we get

$$\eta_2 = O(n^{k+\frac{11}{2}}).$$

Similarly using the above process, we can also find the order of η_3, so

$$\eta_3 = O(n^{k+\frac{11}{2}}).$$

Hence the theorem is proved. □

Theorem 2. *If $k > 0$, $n \geq 2$, for the first derivative of the second kind fundamental polynomials on [-1,1] holds*

$$\sum_{j=1}^{n} |B_j'(x)| = O(n^{k+\frac{7}{2}}). \tag{35}$$

Proof. Differentiating (28) , we get

$$\sum_{j=1}^{n}|B_j'(x)| = \zeta_1+\zeta_2+\zeta_3,$$

where we use the decomposition (16) in ζ_1 for $l_j(x)$, we get

$$\zeta_1 \le \sum_{j=1}^{n}\frac{\{(1-x^2)^{k+1}+2x(k+1)(1+x)(1-x^2)^k\}|P_n^{(k)}(x)||P_n^{(k)'}(x)|}{(1+x_j)\{(1-{x_j}^2)^{\frac{k}{2}+\frac{1}{2}}|P_n^{(k)'}(x_j)|\}^4}\times \tilde{h}_n^{(k)}$$
$$\times\{\gamma_3+\sum_{\nu=1}^{n-1}\frac{1}{h_\nu^{(k)}}(1-{x_j}^2)^k|P_\nu^{(k)}(x_j)|(1-x^2)^k|P_\nu^{(k)}(x)|,$$

where γ_3 is a constant which is independent of n.
By using (19) and (20), then it holds

$$\frac{1}{\{(1-{x_j}^2)^{\frac{k}{2}+\frac{1}{2}}|P_n^{(k)'}(x_j)|\}^4}=O(n)^{-2}, \tag{36}$$

using (11), (12) , (13) , (18), (19) and (36) , we have

$$\zeta_1 = O(n^{k+\frac{3}{2}}).$$

Using the decomposition (16) in ζ_2 for $l_j(x)$, we get

$$\zeta_2 \le \sum_{j=1}^{n}\frac{(1+x)(1-x^2)^{k+1}\{|P_n^{(k)'}(x)|^2+|P_n^{(k)}(x)||P_n^{(k)''}(x)|\ \}\times\tilde{h}_n^{(k)}}{(1+x_j)\{(1-{x_j}^2)^{\frac{k}{2}+\frac{1}{2}}|P_n^{(k)'}(x_j)|\}^4}$$
$$\times\{\gamma_4+\sum_{\nu=1}^{n-1}\frac{1}{h_\nu^{(k)}}(1-{x_j}^2)^k|P_\nu^{(k)}(x_j)|(1-x^2)^k|P_\nu^{(k)}(x)|,$$

where γ_4 is a constant which is independent of n,
by using (11) and (12), it holds

$$|P_n^{(k)''}(x)| = O(n^{k+4}), \tag{37}$$

using (11), (12), (13), (18), (19), (36) and (37) , we obtain

$$\zeta_2 = O(n^{k+\frac{7}{2}}).$$

Similarly using the above method, we can also determine the order of ζ_3. So, we have

$$\zeta_3 = O(n^{k+\frac{5}{2}}).$$

Hence the theorem is proved. □

Theorem 3. *If $k > 0$, $n \geq 2$, for the first derivative of the first kind fundamental polynomials on [-1,1] holds*

$$\sum_{j=1}^{n}(1-{x_j}^2)|A_j'(x)| = O(n^{k+\frac{11}{2}}). \tag{38}$$

Proof. Differentiating (29) , we get

$$\sum_{j=1}^{n}(1-{x_j}^2)|A_j'(x)| = \xi_1+\xi_2+\xi_3,$$

where, we use the decomposition (16) for $l_j(x)$, we have

$$\xi_1 \leq \sum_{j=1}^{n}\frac{\{(1-x^2)^{k+1}+2x(k+1)(1+x)(1-x^2)^k\}|P_n^{(k)'}(x)|}{(1+x_j)\{(1-{x_j}^2)^{\frac{k}{3}+\frac{1}{2}}|P_n^{(k)'}(x_j)|\}^5}\times\{\tilde{h}_n^{(k)}\}^2$$
$$\times\{\gamma_5+\sum_{\nu=1}^{n-1}\sum_{\nu=1}^{n-1}\frac{1}{\{h_\nu^{(k)}\}^2}(1-x_j^2)^{\frac{2k}{3}+\frac{1}{2}}|P_\nu^{(k)}(x_j)|^2(1-x^2)^k|P_\nu^{(k)}(x)|^2\},$$

where γ_5 is a constant which is independent of n.
Using (19) and (20), it holds

$$\frac{1}{\{(1-x_j^2)^{\frac{k}{3}+\frac{1}{2}}|P_n^{(k)'}(x_j)|\}^5} = O(n^{\frac{-5}{2}}), \tag{39}$$

by using (11), (12), (13), (18), (19) and (39) , we obtain

$$\xi_1 = O(n^{k+\frac{3}{2}}).$$

Using the decomposition (16) in ζ_1 for $l_j(x)$ and using (11) and (12), we get

$$\xi_2 \leq \sum_{j=1}^{n}\frac{(n+2k+1)^2(1+x)(1-x^2)^{k+1}|P_{n-2}^{(k+2)}(x)|}{4(1+x_j)\{(1-{x_j}^2)^{\frac{k}{3}+\frac{1}{2}}|P_n^{(k)'}(x_j)|\}^5}\times\{\tilde{h}_n^{(k)}\}^2$$
$$\times\{\gamma_6+\sum_{\nu=1}^{n-1}\sum_{\nu=1}^{n-1}\frac{1}{\{h_\nu^{(k)}\}^2}(1-x_j^2)^{\frac{2k}{3}+\frac{1}{2}}|P_\nu^{(k)}(x_j)|^2(1-x^2)^k|P_\nu^{(k)}(x)|^2\},$$

where γ_6 is a constant which is independent of n.
By using (12), (13), (18), (19) and (39), we get

$$\xi_2 = O(n^{k+\frac{5}{2}}).$$

By using the above procedure, we can also evaluate the order of ξ_3, then we obtain

$$\xi_3 = O(n^{k+\frac{7}{2}})$$

and

$$\xi_4 = \sum_{j=1}^{n}(1-x_j)\{1+2(1+x_j)|l_j{}'(x)|\}|B_j{}'(x)|,$$

using (11) and (12), it holds

$$l_j{}'(x_j) = \frac{P_n^{(k)''}(x_j)}{2P_n^{(k)'}(x_j)}$$

Futhermore,

$$|l_j{}'(x_j)| = O(n^2), \tag{40}$$

using (19), (35) and (40) , we have

$$\xi_4 = O(n^{k+\frac{11}{2}}). \tag{41}$$

Hence the theorem is proved. □

Main Theorem:

Let $k \geq 0$ be a fixed integer m=3n+2k+1 and let $\{x_i\}_{i=1}^{n}$ and $\{x_i^*\}_{i=1}^{n-1}$ be the roots of the Ultraspherical polynomials $P_n^{(k)}(x)$ and $P_{n-1}^{(k+1)}(x)$ respectively, if $f \in C^r[-1,1]$ $(r \geq k+1, n \geq 2r-k+2)$, then the interpolational polynomial

$$R_m(x;f) = \sum_{i=1}^{n} f(x_i)A_i(x) + \sum_{i=1}^{n} f'(x_i)B_i(x) + \sum_{i=1}^{n-1} f(x_i^*)C_i(x) + \sum_{j=0}^{k} f^{(j)}(1)D_j(x) + \sum_{j=0}^{k+1} f^{(j)}(-1)E_j(x) \tag{42}$$

satisfies (43) for $x \in [-1,1]$,

$$|f'(x) - R_m'(x;f)| = w(f^{(r)};\frac{1}{n})O(n^{k-r+\frac{11}{2}}), \tag{43}$$

where the fundamental polynomials $A_i(x)$, $B_i(x)$, $C_i(x)$, $D_j(x)$ and $E_j(x)$ are given in (27) - (32).

Proof. For k=0 we refer to (1), proved by Xie and Zhou [12] and we prove the case $k \geq 1$. Let $f \in C^r[-1,1]$, then by the theorem of Gopengauz [4] for every $m \geq 4r+5$ there exists a polynomial $p_m(x)$ of degree at most m such that for $j = 0, ..., r$

$$|f^{(j)}(x) - p_m^{(j)}(x)| \leq M_{r,j}(\frac{\sqrt{1-x^2}}{m})^{r-j}\ w(f^{(r)};\frac{\sqrt{1-x^2}}{m}), \tag{44}$$

where $w(f^{(r)};.)$ denotes the modulus of continuity of the function $f^{(r)}(x)$ and the constants $M_{r,j}$ depend only on r and j. Furthermore,

$$f^{(j)}(\pm1) = p_m^{(j)}(\pm1) \quad (j = 0, ...r).$$

By the uniqueness of the interpolational polynomials $R_m(x;f)$ it is clear that $R_m(x;p_m)$=$p_m(x)$.Hence for $x \in [-1,1]$

$$|f'(x) - R'_m(x;f)| \leq |f'(x) - p'_m(x)| + |R'_m(x;p_m) - R'_m(x;f)|$$
$$\leq |f'(x) - p'_m(x)| + \sum_{j=1}^{n} |f(x_j) - p_m(x_j)||A'_j(x)| + \sum_{j=1}^{n} |f'(x_j) - p'_m(x_j)||B'_j(x)| +$$
$$+ \sum_{j=1}^{n-1} |f(x_j^*) - p_m(x_j^*)||C'_j(x)|,$$

using (42) and (44) , applying the estimates (33), (35) and (38), we obtain

$$|f'(x) - R'_m(x;f)| = w(f^{(r)}; \frac{1}{n})O(n^{k-r+\frac{11}{2}}). \tag{45}$$

which is the proof of main theorem. □

By using, main theorem and (3) we can state the conclusion of the convergence theorem.

Conclusion:

Let $k \geq 0$ be a fixed integer, m=3n+2k+1 , $n \geq k+4$, let $\{x_i\}_{i=1}^{n}$ and $\{x_i^*\}_{i=1}^{n-1}$ be the roots of the ultraspherical polynomials $P_n^{(k)}(x)$ and $P_{n-1}^{(k+1)}(x)$ respectively. If $f \in C^{k+2}[-1,1]$, $f^{k+2} \in Lip\alpha$,$\alpha > \frac{1}{2}$, then $R_m(x;f)$ and $R'_m(x;f)$ uniformly converge to $f(x)$ and $f'(x)$, respectively on [-1,1] as $n \to \infty$.

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