

SOME FIXED POINT RESULTS FOR A MODIFIED F -CONTRACTIONS VIA A NEW TYPE OF (α, β) -CYCLIC ADMISSIBLE MAPPINGS IN METRIC SPACES

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Abstract

The aim of this paper is to define the new type of mappings which is called modified Suzuki-Berinde F -contraction mapping in the frame work of metric spaces. Fixed point theorems for such mappings in complete metric spaces are established. Furthermore, we present examples to support our main results, using this examples, we establish that our main results is a generalization of the fixed point result of Wardowski [Fixed points of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory and Appl.*, 94, (2012)], Piri and Kumam [Some fixed point theorems concerning F -contraction in complete metric spaces, *Fixed Point Theory and Appl.*, 210, (2014)] and a host of others in the literature.

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1 Introduction and Preliminaries

The theory of fixed point plays an important role in nonlinear functional analysis and known to be very useful in establishing the existence and uniqueness theorems for nonlinear differential and integral equations. Banach [2] in 1922 proved the well celebrated Banach contraction principle in the frame work of metric spaces. The importance of the Banach contraction principle cannot be over emphasized in the study of fixed point theory and its applications. Due to its importance and fruitful applications, researchers in this area generalize the concept by considering classes of nonlinear mappings and spaces which are more general than contraction mappings and metric spaces respectively (see [1, 13, 11, 21, 19] and the references

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therein). For example, Khan et al. [12] introduced the concept of alternating distance function, which is defined as follows: A function $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is called an alternating distance function if the following conditions are satisfied:

1. $\psi(0) = 0$,
2. ψ is monotonically nondecreasing,
3. ψ is continuous.

They established the following result.

Theorem 1.1. *Let (X, d) be a complete metric space, ψ be an alternating distance function and $T : X \rightarrow X$ be a self mapping which satisfies the following condition*

$$\psi(d(Tx, Ty)) \leq \delta\psi(d(x, y))$$

for all $x, y \in X$, where $\delta \in (0, 1)$. Then, T has a unique fixed point.

Remark 1.2. Clearly, if we take $\psi(x) = x$, for all $x \in X$, we obtain the Banach contraction mapping.

More so, Berinde [5, 6] introduced and studied a class of contractive mappings. In particular, he gave the following definition:

Definition 1.3. Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is said to be a generalized almost contraction if there exist $\delta \in [0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

for all $x, y \in X$.

Furthermore, in 2008, Suzuki [22] introduced a class of mappings satisfying condition (C) which is also known as Suzuki-type generalized nonexpansive mapping and he proved some fixed point theorems for this class of mappings.

Definition 1.4. Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is said to satisfy condition (C) if for all $x, y \in X$,

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y).$$

Theorem 1.5. *Let (X, d) be a compact metric space and $T : X \rightarrow X$ be a mapping satisfying condition (C) for all $x, y \in X$. Then T has a unique fixed point.*

In 2012, Wardowski [25] introduced the notion of F -contractions. This class of mappings is defined as follows:

Definition 1.6. Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is said to be an F -contraction if there exists $\tau > 0$ such that for all $x, y \in X$;

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)), \quad (1)$$

where $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a mapping satisfying the following conditions:

- (F_1) F is strictly increasing;
- (F_2) for all sequences $\{\alpha_n\} \subseteq \mathbb{R}^+$, $\lim_{n \rightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$;
- (F_3) there exists $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$.

He also established the following result:

Theorem 1.7. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be an F -contraction. Then T has a unique fixed point $x^* \in X$ and for each $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to x^* .*

Remark 1.8. [25] If we suppose that $F(t) = \ln t$, an F -contraction mapping becomes the Banach contraction mapping.

In [15], Piri et al. used the continuity condition instead of condition (F_3) and proved the following result:

Theorem 1.9. *Let X be a complete metric space and $T : X \rightarrow X$ be a selfmap of X . Assume that there exists $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$,*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ is continuous strictly increasing and $\inf F = -\infty$. Then T has a unique fixed point $z \in X$, and for every $x \in X$, the sequence $\{T^n x\}$ converges to z .

Secelean in [18] proved the following lemma.

Lemma 1.10. [18] *Let $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ be an increasing mapping and $\{\alpha_n\}$ be a sequence of positive integers. Then the following assertion hold:*

1. if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$ then $\lim_{n \rightarrow \infty} \alpha_n = 0$;
2. if $\inf F = -\infty$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$ then $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$.

Furthermore, the authors in [18] replaced the condition F_2 in the definition of F -contraction with the following condition.

(F_*) $\inf F = -\infty$

or, also by

(F_{**}) there exists a sequence $\{\alpha_n\}$ of positive real numbers such that

$$\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty.$$

We denote by \mathcal{F} the family of all functions $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ which satisfy conditions

(F'_1) F is strictly increasing,

(F'_2) $\inf F = -\infty$,

or, also by,

(F'_3) there exists a sequence $\{\alpha_n\}$ of positive real numbers such that

$$\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty,$$

(F'_4) F is continuous on $(0, \infty)$.

Samet et al. [20] introduced the notion of α -admissible mapping and obtain some fixed point results for this class of mappings.

Definition 1.11. [20] Let $\alpha : X \times X \rightarrow [0, \infty)$ be a function. We say that a self mapping $T : X \rightarrow X$ is α -admissible if for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 1.12. [11] Let $T : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$ be mappings. We say that T is a triangular α -admissible if

1. T is α -admissible and
2. $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1$ for all $x, y, z \in X$.

Theorem 1.13. [20] Let (X, d) be a complete metric space and $T : X \rightarrow X$ be an α -admissible mapping. Suppose that the following conditions hold:

1. for all $x, y \in X$, we have $\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$, where $\psi : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$;
2. there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
3. either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, then $\alpha(x_n, x) \geq 1$.

Then T has a fixed point.

In 2016, Chandok et al. [7] introduced another class of mappings, called the TAC-contractive and established some fixed point results in the frame work of complete metric spaces.

Definition 1.14. Let $T : X \rightarrow X$ be a mapping and let $\alpha, \beta : X \rightarrow \mathbb{R}^+$ be two functions. Then T is called a cyclic (α, β) -admissible mapping, if

1. $\alpha(x) \geq 1$ for some $x \in X$ implies that $\beta(Tx) \geq 1$,
2. $\beta(x) \geq 1$ for some $x \in X$ implies that $\alpha(Tx) \geq 1$.

Definition 1.15. Let (X, d) be a metric space and let $\alpha, \beta : X \rightarrow [0, \infty)$ be two mappings. We say that T is a TAC-contractive mapping, if for all $x, y \in X$,

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(d(Tx, Ty)) \leq f(\psi(d(x, y)), \phi(d(x, y))),$$

where ψ is a continuous and nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$, ϕ is continuous with $\lim_{n \rightarrow \infty} \phi(t_n) = 0 \Rightarrow \lim_{n \rightarrow \infty} t_n = 0$ and $f : [0, \infty)^2 \rightarrow \mathbb{R}$ is continuous, $f(a, t) \leq a$ and $f(a, t) = a \Rightarrow a = 0$ or $t = 0$ for all $s, t \in [0, \infty)$.

Theorem 1.16. Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be a cyclic (α, β) -admissible mapping. Suppose that T is a TAC contraction mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1, \beta(x_0) \geq 1$ and either of the following conditions hold:

1. T is continuous,
2. if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq 1$, for all $n \geq 0$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, then $\beta(x) \geq 1$.

In addition, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in F(T)$ (where $F(T)$ denotes the set of fixed points of T), then T has a unique fixed point.

Lemma 1.17. [4] Suppose that (X, d) is a metric space and $\{x_n\}$ is a sequence in X such that $d(x_n, x_{n+1}) \rightarrow 0$ as $n \rightarrow \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exists an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$, $d(x_{m_k}, x_{n_{k-1}}) < \epsilon$ and

1. $\lim_{k \rightarrow \infty} d(x_{m_k}, x_{n_k}) = \epsilon$,
2. $\lim_{k \rightarrow \infty} d(x_{n_k}, x_{m_{k+1}}) = \epsilon$,
3. $\lim_{k \rightarrow \infty} d(x_{m_{k-1}}, x_{n_k}) = \epsilon$,
4. $\lim_{k \rightarrow \infty} d(x_{n_k}, x_{m_{k+1}}) = \epsilon$.

Inspired by the work of Wardowski [25], Piri et al. [15], Samet et al. [20] and Chandok et al. [7], we introduce the concept of (α, β) -cyclic admissible mapping and modified Suzuki-Berinde F -contraction mappings in the frame work of metric spaces. In addition, we establish the existence and uniqueness theorems of fixed points for such mappings in the frame work of complete metric spaces and present some examples to support our main results.

2 Main Result

In this section, we introduce the concept of (α, β) -cyclic admissible mapping, modified Suzuki-Berinde F -contraction and modified F -contraction mappings in the frame work of complete metric spaces and prove the existence and uniqueness theorems of fixed points for such mappings.

Definition 2.1. Let X be a nonempty set, $T : X \rightarrow X$ be a mapping and $\alpha, \beta : X \times X \rightarrow \mathbb{R}^+$ be two functions. We say that T is an (α, β) -cyclic admissible mapping, if for all $x, y \in X$

1. $\alpha(x, y) \geq 1 \Rightarrow \beta(Tx, Ty) \geq 1$,
2. $\beta(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$.

Remark 2.2. Clearly, if $\beta(x, y) = \alpha(x, y)$, we obtain Definition 1.11.

Lemma 2.3. Let X be a nonempty set and $T : X \rightarrow X$ be an (α, β) -cyclic admissible mapping. Suppose that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$. Define the sequence $x_{n+1} = Tx_n$, then $\alpha(x_m, x_{m+1}) \geq 1$ implies that $\beta(x_n, x_{n+1}) \geq 1$ and $\beta(x_m, x_{m+1}) \geq 1$ implies that $\alpha(x_n, x_{n+1}) \geq 1$, for all $n, m \in \mathbb{N} \cup \{0\}$ with $m < n$.

Proof. Using the fact that T is an (α, β) -cyclic admissible mapping and our hypothesis, we have that there exists $x_0 \in X$ such that

$$\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1 \Rightarrow \beta(Tx_0, Tx_1) = \beta(x_1, x_2) \geq 1$$

and

$$\beta(x_1, x_2) \geq 1 \Rightarrow \alpha(Tx_1, Tx_2) = \alpha(x_2, x_3) \geq 1.$$

Continuing this way, we obtain that

$$\alpha(x_{2n}, x_{2n+1}) \geq 1 \text{ and } \beta(x_{2n+1}, x_{2n+2}) \geq 1, \forall n \in \mathbb{N}.$$

Using similar approach, we obtain

$$\beta(x_{2n}, x_{2n+1}) \geq 1 \text{ and } \alpha(x_{2n+1}, x_{2n+2}) \geq 1, \forall n \in \mathbb{N}.$$

In similar sense, we obtain the same result for all $m \in \mathbb{N}$. That is

$$\alpha(x_{2m}, x_{2m+1}) \geq 1 \text{ and } \beta(x_{2m+1}, x_{2m+2}) \geq 1$$

and

$$\beta(x_{2m}, x_{2m+1}) \geq 1 \text{ and } \alpha(x_{2m+1}, x_{2m+2}) \geq 1, \forall m \in \mathbb{N}.$$

In addition, since

$$\alpha(x_m, x_{m+1}) \geq 1 \Rightarrow \beta(x_{m+1}, x_{m+2}) \geq 1 \Rightarrow \alpha(x_{m+2}, x_{m+3}) \geq 1 \cdots$$

with $m < n$, we deduce that

$$\alpha(x_m, x_{m+1}) \geq 1 \Rightarrow \beta(x_n, x_{n+1}) \geq 1.$$

Using similar approach, we have that

$$\beta(x_m, x_{m+1}) \geq 1 \Rightarrow \alpha(x_n, x_{n+1}) \geq 1.$$

□

Definition 2.4. Let (X, d) be a metric space, $\alpha, \beta : X \times X \rightarrow [0, \infty)$ be two functions and T be a self map on X . The mapping T is said to be a modified Suzuki-Berinde F -contraction mapping, if there exists $F \in \mathcal{F}, \tau > 0$ and $L \geq 0$ such that for all $x, y \in X$ with $Tx \neq Ty$

$$\begin{aligned} \alpha(x, Tx)\beta(y, Ty) \geq 1 \text{ and } \frac{1}{2}d(x, Tx) \leq d(x, y) \\ \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \end{aligned} \tag{2}$$

Example 2.5. Let $X = [0, \infty)$ and $d : X \times X \rightarrow [0, \infty)$ be defined as $d(x, y) = |x - y|$ for all $x, y \in X$. It is clear that (X, d) is a metric space. We defined $T : X \rightarrow X$ by

$$Tx = \begin{cases} \frac{x}{12} & \text{if } x \in [0, 1] \\ 9x & \text{if } x \in (1, \infty), \end{cases}$$

$\alpha, \beta : X \times X \rightarrow [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1] \\ 0 & \text{if } x, y \in (1, \infty), \end{cases}$$

$$\beta(x, y) = \begin{cases} 2 & \text{if } x, y \in [0, 1] \\ 0 & \text{if } x, y \in (1, \infty), \end{cases}$$

and $F(t) = \frac{-1}{t} + t$. Then T is a modified Suzuki-Berinde F -contraction but not an F -contraction as defined by Wardowski [25] and Piri et al. [15].

Proof. It is easy to see that for any $x, y \in [0, 1]$, we have that $\alpha(x, Tx) = 1$ and $\beta(y, Ty) = 2$ as such we have that $\alpha(x, Tx)\beta(y, Ty) > 1$. Since $\alpha(x, Tx)\beta(y, Ty) > 1$ if $x, y \in [0, 1]$, we need to show that T is a modified Suzuki-Berinde F -contraction mapping for any $x, y \in [0, 1]$ with $\frac{1}{2}d(x, Tx) \leq d(x, y)$. Let $x, y \in [0, 1]$ and without loss of generality we suppose that $x \leq y$. We then have that $\frac{1}{2}d(x, Tx) = \frac{1}{2}|x - \frac{x}{12}| = \frac{11x}{24}$. Thus for $\frac{1}{2}d(x, Tx) \leq d(x, y)$, we must have that $\frac{35x}{24} \leq y$. Observe that, for $\tau = 1$ and $L \geq 0$, it is easy to see that

$$\begin{aligned} \tau + F(d(Tx, Ty)) &= 1 + F(|\frac{y}{12} - \frac{x}{12}|) \\ &= 1 + F(\frac{1}{12}|y - x|) = 1 + \frac{|y - x|}{12} - \frac{12}{|y - x|} \\ &\leq |y - x| - \frac{1}{|y - x|} + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \\ &= F(d(x, y)) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \end{aligned}$$

Thus, T is a modified Suzuki-Berinde F contraction.

However to show that T is not an F -contraction as defined by Wardowski [25] and Piri et al. [15]. Suppose $x = 0$ and $y = 2$. Observe that

$$d(Tx, Ty) = 18 > 0,$$

but

$$\begin{aligned} \tau + F(d(Tx, Ty)) &= 1 + F(|0 - 18|) = 1 + 18 - \frac{1}{18} = 19 - \frac{1}{18} > 2 - \frac{1}{2} \\ &= F(d(x, y)). \end{aligned}$$

In addition, we have that

$$\frac{1}{2}d(x, Tx) = 0 < 2 = d(x, y),$$

but

$$\begin{aligned} \tau + F(d(Tx, Ty)) &= 1 + F(|0 - 18|) = 1 + 18 - \frac{1}{18} = 19 - \frac{1}{18} > 2 - \frac{1}{2} \\ &= F(d(x, y)). \end{aligned}$$

□

Theorem 2.6. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a modified Suzuki-Berinde F -contraction mapping. Suppose the following conditions hold:*

1. T is a (α, β) -cyclic admissible mapping,
2. there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$,
3. T is continuous.

Then T has a fixed point.

Proof. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If we suppose that $x_{n+1} = x_n$, we obtain the desired result. Now, suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is a (α, β) -cyclic admissible mapping and $\alpha(x_0, x_1) \geq 1$, we have $\beta(Tx_0, Tx_1) = \beta(x_1, x_2) \geq 1$ and this implies that $\alpha(x_2, x_3) = \alpha(Tx_1, Tx_2) \geq 1$, continuing the process, we have

$$\alpha(x_{2k}, x_{2k+1}) \geq 1 \quad \text{and} \quad \beta(x_{2k+1}, x_{2k+2}) \geq 1 \quad \forall k \in \mathbb{N} \cup \{0\}. \quad (3)$$

Using similar argument, we have that

$$\beta(x_{2k}, x_{2k+1}) \geq 1 \quad \text{and} \quad \alpha(x_{2k+1}, x_{2k+2}) \geq 1 \quad \forall k \in \mathbb{N} \cup \{0\}. \quad (4)$$

It follows from (3) and (4) that $\alpha(x_n, x_{n+1}) \geq 1$ and $\beta(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\alpha(x_n, x_{n+1})\beta(x_{n+1}, x_{n+2}) \geq 1$ and $\frac{1}{2}d(x_n, Tx_n) = \frac{1}{2}d(x_n, x_{n+1}) < d(x_n, x_{n+1})$, we obtain from (2)

$$\begin{aligned} \tau + F(d(x_{n+1}, x_{n+2})) &= \tau + F(d(Tx_n, Tx_{n+1})) \\ &\leq F(d(x_n, x_{n+1})) \\ &\quad + L \min\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1})\} \\ &= F(d(x_n, x_{n+1})) + L \cdot 0 \\ &= F(d(x_n, x_{n+1})), \end{aligned} \quad (5)$$

which implies that

$$F(d(x_{n+1}, x_{n+2})) \leq F(d(x_n, x_{n+1})) - \tau.$$

Using similar approach, it is easy to see that

$$F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) - \tau.$$

Thus by inductively, we obtain

$$F(d(x_n, x_{n+1})) \leq F(d(x_0, x_1)) - n\tau, \quad \forall n \in \mathbb{N} \cup \{0\}. \quad (6)$$

Since $F \in \mathcal{F}$, taking limit as $n \rightarrow \infty$ in (6), we have

$$\lim_{n \rightarrow \infty} F(d(x_n, x_{n+1})) = -\infty. \quad (7)$$

It follows from (F'_3) and Lemma 1.10 that

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \quad (8)$$

In what follows, we now show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence, then by Lemma 1.17, there exists an $\epsilon > 0$ and sequences of positive integers $\{x_{n_k}\}$ and $\{x_{m_k}\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$. For each $k > 0$, corresponding to m_k , we can choose n_k to be the smallest positive integer such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$, $d(x_{m_k}, x_{n_{k-1}}) < \epsilon$ and (1)–(4) of Lemma 1.17 hold. Since $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$, using Lemma 2.3, we obtain that $\alpha(x_{m_k}, x_{m_{k+1}})\beta(x_{n_k}, x_{n_{k+1}}) \geq 1$ and we can choose $n_0 \in \mathbb{N} \cup \{0\}$ such that

$$\frac{1}{2}d(x_{m_k}, Tx_{m_k}) < \frac{\epsilon}{2} < \epsilon \leq d(x_{m_k}, x_{n_k}).$$

Hence, for all $k \geq n_0$, we have

$$\begin{aligned} \tau + F(d(x_{m_{k+1}}, x_{n_{k+1}})) &\leq \tau + F(d(Tx_{m_k}, Tx_{n_k})) \\ &\leq F(d(x_{m_k}, x_{n_k})) \\ &\quad + L \min\{d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_{k+1}}), d(x_{m_k}, x_{n_{k+1}}), d(x_{n_k}, x_{m_{k+1}})\}. \end{aligned} \quad (9)$$

Using Lemma 1.17, (F'_4) and (8), we have that

$$\begin{aligned} \tau + F(\epsilon) &= \lim_{k \rightarrow \infty} [\tau + F(d(Tx_{m_k}, Tx_{n_k}))] \\ &\leq \lim_{k \rightarrow \infty} [F(d(x_{m_k}, x_{n_k})) \\ &\quad + L \min\{d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_{k+1}}), d(x_{m_k}, x_{n_{k+1}}), d(x_{n_k}, x_{m_{k+1}})\}] \\ &\leq F(\epsilon). \end{aligned}$$

That is

$$\tau + F(\epsilon) \leq F(\epsilon)$$

which is a contradiction. We therefore have that $\{x_n\}$ is Cauchy. Since (X, d) is complete, it follows that there exists $x \in X$ such that $\lim_{n \rightarrow \infty} x_n = x$. Since T is continuous, we have that

$$x = \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} Tx_n = T \lim_{n \rightarrow \infty} x_n = Tx.$$

Thus, T has a fixed point. \square

Theorem 2.7. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a modified Suzuki-Berinde F -contraction mapping. Suppose the following conditions hold:*

1. T is a (α, β) -cyclic admissible mapping,
2. there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$,
3. if for any sequence $\{x_n\}$ in X such that $x_n \rightarrow x$ as $n \rightarrow \infty$, then $\beta(x, Tx) \geq 1$ and $\alpha(x, Tx) \geq 1$.

Then T has a fixed point.

Proof. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. In Theorem 2.6, we have established that $\{x_n\}$ is Cauchy. Now suppose hypothesis (3) holds. We now establish that T has a fixed point.

Claim: We claim that

$$\frac{1}{2}d(x_n, x_{n+1}) < d(x_n, x)$$

or

$$\frac{1}{2}d(x_{n+1}, x_{n+2}) < d(x_{n+1}, x).$$

Proof of claim.

Suppose on the contrary that there exists $m \in \mathbb{N} \cup \{0\}$, such that

$$\begin{aligned} \frac{1}{2}d(x_m, x_{m+1}) &\geq d(x_m, x) \\ \frac{1}{2}d(x_{m+1}, x_{m+2}) &\geq d(x_{m+1}, x). \end{aligned} \quad (10)$$

Now observe that

$$\begin{aligned} 2d(x_m, x) &\leq d(x_m, x_{m+1}) \\ &\leq d(x_m, x) + d(x, x_{m+1}), \end{aligned} \quad (11)$$

which implies that $d(x_m, x) \leq d(x, x_{m+1})$.

It follows from (10) and (11), that

$$d(x_m, x) \leq d(x, x_{m+1}) \leq \frac{1}{2}d(x_{m+1}, x_{m+2}). \quad (12)$$

Since $\alpha(x_m, x_{m+1})\beta(x_{m+1}, x_{m+2}) \geq 1$ and $\frac{1}{2}d(x_m, x_{m+1}) < d(x_m, x_{m+1})$, we have that

$$\begin{aligned} \tau + F(d(x_{m+1}, x_{m+2})) &= \tau + F(d(Tx_m, Tx_{m+1})) \\ &\leq F(d(x_m, x_{m+1})) \\ &\quad + L \min\{d(x_m, x_{m+1}), d(x_{m+1}, x_{m+2}), d(x_m, x_{m+2}), d(x_{m+1}, x_{m+1})\} \\ &= F(d(x_m, x_{m+1})). \end{aligned} \quad (13)$$

It follows that

$$\tau + F(d(x_{m+1}, x_{m+2})) \leq F(d(x_m, x_{m+1})). \quad (14)$$

Using the fact that F is strictly increasing, we have that

$$d(x_{m+1}, x_{m+2}) < d(x_m, x_{m+1}).$$

Using this fact, (12) and (10), we have

$$\begin{aligned} d(x_{m+1}, x_{m+2}) &< d(x_m, x_{m+1}) \\ &\leq d(x_m, x) + d(x, x_{m+1}) \\ &\leq \frac{1}{2}d(x_{m+1}, x_{m+2}) + \frac{1}{2}d(x_{m+1}, x_{m+2}) \\ &= d(x_{m+1}, x_{m+2}), \end{aligned} \quad (15)$$

which is a contradiction. Thus we must have that

$$\frac{1}{2}d(x_n, x_{n+1}) < d(x_n, x)$$

or

$$\frac{1}{2}d(x_{n+1}, x_{n+2}) < d(x_{n+1}, x).$$

Hence, we have

$$\begin{aligned} \tau + F(d(x_{n+1}, Tx)) &= \tau + F(d(Tx_n, Tx)) \\ &\leq F(d(x_n, x)) \\ &\quad + L \min\{d(x_n, x_{n+1}), d(x, Tx), d(x_n, Tx), d(x, Tx_n)\}. \end{aligned}$$

Using the fact that $F \in \mathcal{F}$ and Lemma 1.10, we have that

$$\lim_{n \rightarrow \infty} F(d(Tx_n, Tx)) = -\infty$$

and so

$$\lim_{n \rightarrow \infty} d(Tx_n, Tx) = 0.$$

Now, observe that

$$d(x, Tx) = \lim_{n \rightarrow \infty} d(x_{n+1}, Tx) = \lim_{n \rightarrow \infty} d(Tx_n, Tx) = 0.$$

Clearly, we have that

$$d(x, Tx) = 0 \Rightarrow x = Tx.$$

Hence, T has a fixed point. \square

We present the an example to support the above results.

Example 2.8. Let $X = [0, \infty)$ and $d : X \times X \rightarrow [0, \infty)$ be defined as $d(x, y) = |x - y|$ for all $x, y \in X$. It is clear that (X, d) is a complete metric space. We defined $T : X \rightarrow X$ by

$$Tx = \begin{cases} \frac{x}{7} & \text{if } x \in [0, 2) \\ 2x - \frac{35}{4} & \text{if } x \in [2, \infty), \end{cases}$$

$\alpha, \beta : X \times X \rightarrow [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2 & \text{if } x, y \in [0, 1] \\ 0 & \text{if } x, y \in (1, \infty), \end{cases}$$

$$\beta(x, y) = \begin{cases} 1.5 & \text{if } x, y \in [0, 1] \\ 0 & \text{if } x, y \in (1, \infty), \end{cases}$$

and $F(t) = \frac{-1}{t} + t$.

Proof. It is easy to see that for any $x, y \in [0, 1]$, we have that $\alpha(x, y) > 1, \beta(x, y) > 1$, where $Tx = \frac{x}{7}$ and $Ty = \frac{y}{7}$ are in $[0, 1]$, as such we have that $\alpha(x, y) = 2 > 1 \Rightarrow \beta(Tx, Ty) = 1.5 > 1$ and $\beta(x, y) = 1.5 > 1 \Rightarrow \alpha(Tx, Ty) = 2 > 1$. Therefore, T is (α, β) -cyclic admissible mapping. It is easy to see that for any $x_0 \in [0, 1]$, we have that $\alpha(x_0, T_0) > 1$ and $\beta(x_0, Tx_0) > 1$.

We now establish that T is a modified Suzuki-Berinde F -contraction. Since $\alpha(x, Tx)\beta(y, Ty) > 1$ if $x, y \in [0, 1]$, we need to show that T is a modified Suzuki-Berinde F -contraction mapping for any $x, y \in [0, 1]$ with $\frac{1}{2}d(x, Tx) \leq d(x, y)$. Let $x, y \in [0, 1]$ and without loss of generality we suppose that $x \leq y$. We then have that $\frac{1}{2}d(x, Tx) = \frac{1}{2}|x - \frac{x}{7}| = \frac{6x}{14}$. Thus for $\frac{1}{2}d(x, Tx) \leq d(x, y)$, we must have that $\frac{20x}{14} \leq y$. Observe that, for $\tau = 1$ and $L \geq 0$, it is easy to see that

$$\begin{aligned} \tau + F(d(Tx, Ty)) &= 1 + F(|\frac{y}{7} - \frac{x}{7}|) \\ &= 1 + F(\frac{1}{7}|y - x|) = 1 + \frac{|y - x|}{7} - \frac{7}{|y - x|} \\ &\leq |y - x| - \frac{1}{|y - x|} \\ &\quad + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\ &\leq F(d(x, y)) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \end{aligned}$$

Thus T is a modified Suzuki-Berinde F contraction. It is easy to see that T satisfy all the hypothesis of Theorem 2.6, Theorem 2.7, and $x = 0$ and $x = \frac{35}{4}$ are the two fixed points of T . \square

Remark 2.9. As established in the above example, T has two fixed points $x = 0$ and $x = \frac{35}{4}$. Thus, for the uniqueness of fixed point of T , we need additional condition.

Theorem 2.10. *Suppose that the hypothesis of Theorem 2.7 holds and in addition suppose $\alpha(x, Tx) \geq 1$ and $\beta(y, Ty) \geq 1$ for all $x, y \in F(T)$, where $F(T)$ is the set of fixed point of T . Then T has a unique fixed point.*

Proof. Let $x, y \in F(T)$, that is $Tx = x$ and $Ty = y$ such that $x \neq y$. Since, $\alpha(x, Tx) \geq 1$ and $\beta(y, Ty) \geq 1$, we have $\alpha(x, Tx)\beta(y, Ty) \geq 1$ and $\frac{1}{2}d(x, Tx) = 0 \leq d(x, y)$, we obtain that

$$\begin{aligned} F(d(x, y)) &= F(d(Tx, Ty)) < \tau + F(d(Tx, Ty)) \\ &\leq F(d(x, y)) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \\ &\leq F(d(x, y)), \end{aligned}$$

which implies that

$$F(d(x, y)) < F(d(x, y)).$$

Clearly, we get a contradiction, thus, T has a unique fixed point. \square

We present the an example to support Theorem 2.10.

Example 2.11. Let $X = [0, \infty)$ and $d : X \times X \rightarrow [0, \infty)$ be defined as $d(x, y) = |x - y|$ for all $x, y \in X$. It is clear that (X, d) is a complete metric space. We defined $T : X \rightarrow X$ by

$$Tx = \begin{cases} \frac{x}{16} & \text{if } x \in [0, 1] \\ 2x + 5 & \text{if } x \in (1, \infty), \end{cases}$$

$\alpha, \beta : X \times X \rightarrow [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1.5 & \text{if } x, y \in [0, 1] \\ 0 & \text{if } x, y \in (1, \infty), \end{cases}$$

$$\beta(x, y) = \begin{cases} 2 & \text{if } x, y \in [0, 1] \\ 0 & \text{if } x, y \in (1, \infty), \end{cases}$$

and $F(t) = \frac{-1}{t} + t$.

Proof. It is easy to see that T is (α, β) -cyclic admissible mapping for any $x, y \in [0, 1]$. In addition for any $x_0 \in [0, 1]$, we have that $\alpha(x_0, Tx_0) > 1$ and $\beta(x_0, Tx_0) > 1$.

We now establish that T is a modified Suzuki-Berinde F -contraction. Since $\alpha(x, Tx)\beta(y, Ty) > 1$, if $x, y \in [0, 1]$, we need to show that T is a modified Suzuki-Berinde F -contraction mapping for any $x, y \in [0, 1]$ with $\frac{1}{2}d(x, Tx) \leq d(x, y)$. Let $x, y \in [0, 1]$ and without loss of generality we suppose that $x \leq y$. We then have

that $\frac{1}{2}d(x, Tx) = \frac{1}{2}|x - \frac{x}{16}| = \frac{15x}{32}$. Thus for $\frac{1}{2}d(x, Tx) \leq d(x, y)$, we must have that $\frac{47x}{32} \leq y$. Observe that, for $\tau = 1$ and $L \geq 0$, it is easy to see that

$$\begin{aligned} \tau + F(d(Tx, Ty)) &= 1 + F(|\frac{y}{16} - \frac{x}{16}|) \\ &= 1 + F(\frac{1}{16}|y - x|) = 1 + \frac{|y - x|}{16} - \frac{16}{|y - x|} \\ &\leq |y - x| - \frac{1}{|y - x|} \\ &\quad + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\ &\leq F(d(x, y)) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \end{aligned}$$

Thus T is a modified Suzuki-Berinde F contraction. Hence T satisfy all the hypothesis of Theorem 2.10. Clearly, we have $x = 0$ as the unique fixed points of T .

However, we observe that Theorem 1.7 and Theorem 1.9 are not applicable to the above example. To see this, let $x = 0$ and $y = 3$. Now, observe that

$$d(Tx, Ty) = 11 > 0,$$

but

$$\begin{aligned} \tau + F(d(Tx, Ty)) &= 1 + F(|0 - 11|) = 1 + 11 - \frac{1}{11} = 12 - \frac{1}{11} > 3 - \frac{1}{3} \\ &= F(d(x, y)). \end{aligned}$$

In addition, we have that

$$\frac{1}{2}d(x, Tx) = 0 < 3 = d(x, y),$$

but

$$\begin{aligned} \tau + F(d(Tx, Ty)) &= 1 + F(|0 - 11|) = 1 + 11 - \frac{1}{11} = 12 - \frac{1}{11} > 3 - \frac{1}{3} \\ &= F(d(x, y)). \end{aligned}$$

□

Corollary 2.12. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a mapping satisfying the following inequality*

$$\begin{aligned} \frac{1}{2}d(x, Tx) \leq d(x, y) \quad \text{and} \quad \alpha(x, Tx)\beta(y, Ty) \geq 1 \\ \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)). \end{aligned}$$

Suppose the following conditions hold:

1. T is a (α, β) -cyclic admissible mapping,

2. there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$,
3. T is continuous,
4. if for any sequence $\{x_n\}$ in X such that $x_n \rightarrow x$ as $n \rightarrow \infty$, then $\beta(x, Tx) \geq 1$ and $\alpha(x, Tx) \geq 1$.

Then T has a fixed point.

Corollary 2.13. Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a mapping satisfying the following inequality

$$\begin{aligned} \alpha(x, Tx)\beta(y, Ty) &\geq 1 \\ \Rightarrow \tau + F(d(Tx, Ty)) &\leq F(d(x, y)) \\ &\quad + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \end{aligned}$$

Suppose the following conditions hold:

1. T is a (α, β) -cyclic admissible mapping,
2. there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$,
3. T is continuous,
4. if for any sequence $\{x_n\}$ in X such that $x_n \rightarrow x$ as $n \rightarrow \infty$, then $\beta(x, Tx) \geq 1$ and $\alpha(x, Tx) \geq 1$.

Then T has a fixed point.

Corollary 2.14. Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a mapping satisfying the following inequality

$$\begin{aligned} \frac{1}{2}d(x, Tx) &\leq d(x, y) \\ \Rightarrow \tau + F(d(Tx, Ty)) &\leq F(d(x, y)) \\ &\quad + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \end{aligned}$$

Then T has a unique fixed point.

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