Bulletin of the *Transilvania* University of Braşov • Vol 12(61), No. 1 - 2019
Series III: Mathematics, Informatics, Physics, 65-76
https://doi.org/10.31926/but.mif.2019.12.61.1.6

SOME CURVATURE PROPERTIES OF TRANS SASAKIAN MANIFOLDS ADMITTING A QUARTER-SYMMETRIC NON-METRIC CONNECTION

C. LALMALSAWMA*[1] and J. P. SINGH[2]

Abstract

In this paper we studied some properties of trans Sasakian manifold admitting a quarter-symmetric non-metric connection. We studied some symmetric properties with respect to quarter symmetric non-metric connection and we obtain results in locally symmetric, Ricci semi-symmetric and generalized recurrent trans Sasakian manifolds. We also obtained the necessary and sufficient condition for group manifolds.

2010 *Mathematics Subject Classification:* 53D15, 53C25.
Key words: Trans-Sasakian manifolds, quarter-symmetric non-metric connection, Locally symmetric manifolds, Ricci semi-symmetric manifolds, Generalized recurrent manifolds, group manifolds.

1 Introduction

Golab [12] introduced a quarter-symmetric linear connection on a differentiable manifold. A quarter-symmetric metric connection generalized a semi-symmetric connection which is introduced by Friedman and Schouten [11] in 1924. A quarter-symmetric connection is further studied by many authors such as: Barman [2], Prakasha and Vikas [15], Singh [17, 18, 19], Prasad and Haseeb [16], Dey et al. [8, 9] etc.

Golab [12] defined a quarter-symmetric connection in a Riemannian manifold as "a linear connection $\tilde{\nabla}$, whose torsion tensor $\tilde{T}$ of $\tilde{\nabla}$ satisfies

$$\tilde{T}(U,V) = \eta(V)\phi U - \eta(U)\phi V,$$

[1]* *Corresponding author*, Mathematics and Computer Science, *Mizoram University*, Aizawl, Mizoram, India, e-mail: sweezychawngthu@gmail.com
[2]Faculty of Mathematics and Computer Science, *Mizoram University*, Aizawl, Mizoram, India, e-mail: jpsmaths@gmail.com

where η is a 1-form and ϕ is a $(1,1)$ tensor field". If $\phi X = X$ for all $X \in \chi(M)$, then it reduces to a semi-symmetric connection. Moreover $\tilde{\nabla}$ is a metric connection if

$$(\tilde{\nabla}_X g)(Y,Z) = 0,$$

otherwise it is non-metric.

The paper is organized as follows: after the introduction and preliminaries, we obtain the relations between a quarter-symmetric connection and a Riemannian connection. In section 4 we obtain the condition for $\tilde{\nabla} = \nabla$. In section 5 we prove the necessary and sufficient condition for α-Sasakian manifold. The final two sections are dedicated to the study of generalized recurrent manifolds and group manifolds.

2 Preliminaries

Almost contact metric manifold is defined as [3] "an n-dimensional differentiable manifold M^n $(n = 2m+1)$ admitting a 1-form η, a $(1,1)$ tensor field ϕ, a vector field ζ and Riemannian metric g which satisfy

$$\phi^2 U = -U + \eta(U)\zeta, \qquad \eta(\zeta) = 1, \qquad \phi\zeta = 0, \qquad \eta(\phi U) = 0, \tag{1}$$

$$g(U,\zeta) = \eta(U), \qquad g(\phi U, V) = -g(U, \phi V), \tag{2}$$

$$g(\phi U, \phi V) = g(U,V) - \eta(U)\eta(V), \tag{3}$$

for all $U, V, X \in \chi(M)$".

The manifold M^n is trans-Sasakian Manifold [13] "if $\big(M^n \times \mathbb{R}, J, G\big)$ belong to the class ω_4 of the Hermitian manifolds, where G is the product metric on $\big(M^n \times \mathbb{R}\big)$ and J is the almost complex structure on $\big(M^n \times \mathbb{R}\big)$ defined by

$$J\big(U, f\frac{d}{dt}\big) = (\phi Z - f\zeta, \eta(U)\frac{d}{dt})$$

for any $U \in \chi(M)$. This may be stated by the relation [13]

$$(\nabla_X \phi)(U) = \alpha\{g(X,U)\zeta - \eta(U)X\} + \beta\big\{g(\phi X, U)\zeta - \eta(U)\phi X\big\}, \tag{4}$$

where α, β are smooth functions on M^n".

From the above relations we have [4]

$$\nabla_X \zeta = -\alpha\phi X + \beta\big\{X - \eta(X)\zeta\big\}, \tag{5}$$

$$(\nabla_X \eta)(U) = -\alpha g(\phi X, U) + \beta g(\phi X, \phi U). \tag{6}$$

Further in trans-Sasakian manifolds the curvature tensor and Ricci tensor satisfy [7]

$$R(U,V)\zeta = (\alpha^2-\beta^2)\{\eta(V)U-\eta(U)V\}+2\alpha\beta\{\eta(V)\phi U-\eta(U)\phi V\} \\ -(U\alpha)\phi V+(V\alpha)\phi U-(U\beta)\phi^2 V+(V\beta)\phi^2 U, \tag{7}$$

$$R(\zeta,V)Z = (\alpha^2-\beta^2)\{g(V,Z)\zeta-\eta(Z)V\}+2\alpha\beta\{g(\phi Z,V)\zeta+\eta(Z)\phi V\} \\ +(Z\alpha)\phi V+(Z\beta)\{V-\eta(V)\zeta\}+g(\phi Z,V)(grad\alpha)-g(\phi Z,\phi V)(grad\beta), \tag{8}$$

$$R(\zeta,V)\zeta = (\alpha^2-\beta^2-\zeta\beta)\{\eta(V)\zeta-V\}, \\ 2\alpha\beta+(\zeta\alpha)=0, \tag{9}$$

$$S(U,\zeta) = \{(n-1)(\alpha^2-\beta^2)-(\zeta\beta)\}\eta(U)-(n-2)(U\beta)-\big((\phi U)\alpha\big), \tag{10}$$

$$Q\zeta = \{(n-1)(\alpha^2-\beta^2)-(\zeta\beta)\}\zeta-(n-2)(grad\beta)+\phi(grad\alpha). \tag{11}$$

3 Quarter-symmetric non-metric connection in trans-Sasakian manifolds

We consider a linear connection $\tilde\nabla$ on a trans-Sasakian manifold which is given by

$$\tilde\nabla_X U = \nabla_X U+\eta(U)\phi(X). \tag{12}$$

Thus $\tilde\nabla$ is a quarter-symmetric conection on the manifold. Also using (12) we obtain

$$\big(\tilde\nabla_X g\big)(U,V) = -g(\phi X,V)\eta(U)-g(\phi X,U)\eta(V),$$

which shows that $\tilde\nabla$ is a non-metric connection.

The relation between the Riemannian curvature tensor R and the curvature tensor $\tilde R$ with respect to $\tilde\nabla$ is given by [14]

$$\tilde R(U,V)Z = R(U,V)Z+\alpha\{g(\phi V,Z)\phi U-g(\phi U,Z)\phi V+\eta(U)\eta(Z)V \\ -\eta(V)\eta(Z)U\}+\beta\{g(U,Z)\phi V-g(V,Z)\phi U+2\eta(Z)g(\phi U,V)\zeta\}. \tag{13}$$

From (13), it follows that

$$\tilde R(\zeta,V)Z = (\alpha^2-\beta^2)\{g(V,Z)\zeta-\eta(Z)V\}+2\alpha\beta\{g(\phi Z,V)\zeta+\eta(Z)\phi V\} \\ +(Z\alpha)\phi V+g(\phi Z,V)grad\alpha+(Z\beta)\{V-\eta(V)\zeta\} \\ -g(\phi Z,\phi V)grad\beta+\alpha\eta(Z)\{V-\eta(V)\zeta\}+\beta\eta(Z)\phi V, \tag{14}$$

$$\tilde{R}(U,V)\zeta = (\alpha^2-\beta^2-\alpha)\{\eta(V)U-\eta(U)V\} - (U\alpha)\phi V + (V\alpha)\phi U$$
$$+(2\alpha\beta\ -\beta)\{\eta(V)\phi U-\eta(U)\phi V\} - (U\beta)\phi^2 V + (V\beta)\phi^2 U + 2\beta g(\phi U,V)\zeta, \quad (15)$$

$$\tilde{R}(\zeta,V)\zeta = (\alpha^2-\beta^2-\alpha-\zeta\beta)\{\eta(V)\zeta - V\} + \beta\phi V. \quad (16)$$

Contracting (13) we obtain [14]

$$\tilde{S}(U,V) = S(U,V) + \alpha\{g(\phi U,\phi V) - (n-1)\eta(U)\eta(V)\} + \beta g(\phi U,V). \quad (17)$$

Again from (13), we have

$$\begin{aligned}\tilde{S}(U,\zeta) &= \{(n-1)(\alpha^2-\beta^2-\alpha) - \zeta\beta\}\eta(U) - (n-2)(U\beta) - \big((\phi U)\alpha\big)\\ &= \tilde{S}(\zeta,U), \end{aligned} \quad (18)$$

$$\tilde{S}(\zeta,\zeta) = (n-1)(\alpha^2-\beta^2-\alpha-\zeta\beta). \quad (19)$$

Also from (5) we obtain

$$\tilde{\nabla}_X\zeta = -(\alpha-1)\phi X + \beta\{X-\eta(X)\zeta\}. \quad (20)$$

Again from (5) we obtain

$$\tilde{r} = r, \quad (21)$$

where $\tilde{r} = \sum_{i=1}^{n} \tilde{S}(e_i,e_i)$ and $r = \sum_{i=1}^{n} S(e_i,e_i)$ are the scalar curvatures.

4 Locally symmetric trans-Sasakian manifolds

Definition 1. *[5]"A Riemannian manifold is said to be (locally) symmetric if the Riemannian curvature tensor R satisfies $\nabla R = 0$".*

Consider a trans-Sasakian manifold which is symmetric with respect to $\tilde{\nabla}$. Then we have

$$(\tilde{\nabla}_X\tilde{R})(U,V)Z = 0 \quad (22)$$

for all $X,U,V,Z \in \chi(M)$.

By properties of $\tilde{\nabla}$ we have

$$\begin{aligned}(\tilde{\nabla}_X\tilde{R})(U,V)Z = \tilde{\nabla}_X\tilde{R}(U,V)Z - \tilde{R}(\tilde{\nabla}_X U,V)Z\\ -\tilde{R}(U,\tilde{\nabla}_X V)Z - \tilde{R}(U,V)\tilde{\nabla}_X Z.\end{aligned} \quad (23)$$

Using (13) in (23) we obtain

$$
\begin{aligned}
(\tilde{\nabla}_X\tilde{R})(U,V)Z &= (\tilde{\nabla}_X R)(U,V)Z + (X\alpha)\big[g(\phi V,Z)\phi U - g(\phi U,Z)\phi V \\
&+\eta(U)\eta(Z)V - \eta(V)\eta(Z)U\big] + (X\beta)\big[g(U,Z)\phi V - g(V,Z)\phi U \\
&+2\eta(Z)g(\phi U,V)\zeta\big] + \alpha\Big[\big\{(\tilde{\nabla}_X g)(\phi V,Z) + g\big((\tilde{\nabla}_X\phi)V,Z\big)\big\}\phi U \\
&+g(\phi V,Z)(\tilde{\nabla}_X\phi)U - \big\{(\tilde{\nabla}_X g)(\phi U,Z) - g\big((\tilde{\nabla}_X\phi)U,Z\big)\big\}\phi V \\
&-g(\phi U,Z)(\tilde{\nabla}_X\phi)V + (\tilde{\nabla}_X\eta)(Z)\big\{\eta(U)V - \eta(V)U\big\} \\
+\eta(Z)\big\{(\tilde{\nabla}_X\eta)(U)V - (\tilde{\nabla}_X\eta)(V)U\big\}\Big] &+ \beta\Big[(\tilde{\nabla}_X g)(U,Z)\phi V - (\tilde{\nabla}_X g)(V,Z)\phi U \\
&+g(U,Z)(\tilde{\nabla}_X\phi)V - g(V,Z)(\tilde{\nabla}_X\phi)U + 2(\tilde{\nabla}_X\eta)(Z)g(\phi U,V)\zeta \\
&+2\eta(Z)\{(\tilde{\nabla}_X g)(\phi U,V) + g\big((\tilde{\nabla}_X\phi)U,V\big)\}\zeta\Big]. \qquad (24)
\end{aligned}
$$

Again using (12) in (24) we obtain

$$
\begin{aligned}
(\tilde{\nabla}_X\tilde{R})(U,V)Z &= (\nabla_X R)(U,V)Z + (X\alpha)\big[g(\phi V,Z)\phi U - g(\phi U,Z)\phi V \\
&+\eta(U)\eta(Z)V - \eta(V)\eta(Z)U\big] + (X\beta)\big[g(U,Z)\phi V - g(V,Z)\phi U \\
&+2\eta(Z)g(\phi U,V)\zeta\big] + \alpha\Big[\big\{(\tilde{\nabla}_X g)(\phi V,Z) + g\big((\tilde{\nabla}_X\phi)V,Z\big)\big\}\phi U \\
&+g(\phi V,Z)(\tilde{\nabla}_X\phi)U - \big\{(\tilde{\nabla}_X g)(\phi U,Z) - g\big((\tilde{\nabla}_X\phi)U,Z\big)\big\}\phi V \\
&-g(\phi U,Z)(\tilde{\nabla}_X\phi)V + (\tilde{\nabla}_X\eta)(Z)\big\{\eta(U)V - \eta(V)U\big\} \\
+\eta(Z)\big\{(\tilde{\nabla}_X\eta)(U)V - (\tilde{\nabla}_X\eta)(V)U\big\}\Big] &+ \beta\Big[(\tilde{\nabla}_X g)(U,Z)\phi V - (\tilde{\nabla}_X g)(V,Z)\phi U \\
&+g(U,Z)(\tilde{\nabla}_X\phi)V - g(V,Z)(\tilde{\nabla}_X\phi)U + 2(\tilde{\nabla}_X\eta)(Z)g(\phi U,V)\zeta \\
&+2\eta(Z)\{(\tilde{\nabla}_X g)(\phi U,V) + g\big((\tilde{\nabla}_X\phi)U,V\big)\}\zeta\Big] + \eta\big(R(U,V)Z\big)\phi X \\
&-\eta(U)R(\phi X,V)Z - \eta(V)R(U,\phi X)Z - \eta(Z)R(U,V)\phi X. \qquad (25)
\end{aligned}
$$

Combining (22) and (25)

$$
\begin{aligned}
&(\nabla_X R)(U,V)Z + (X\alpha)\big[g(\phi V,Z)\phi U - g(\phi U,Z)\phi V \\
&+\eta(U)\eta(Z)V - \eta(V)\eta(Z)U\big] + (X\beta)\big[g(U,Z)\phi V - g(V,Z)\phi U \\
&+2\eta(Z)g(\phi U,V)\zeta\big] + \alpha\Big[\big\{(\tilde{\nabla}_X g)(\phi V,Z) + g\big((\tilde{\nabla}_X\phi)V,Z\big)\big\}\phi U \\
&+g(\phi V,Z)(\tilde{\nabla}_X\phi)U - \big\{(\tilde{\nabla}_X g)(\phi U,Z) - g\big((\tilde{\nabla}_X\phi)U,Z\big)\big\}\phi V \\
&-g(\phi U,Z)(\tilde{\nabla}_X\phi)V + (\tilde{\nabla}_X\eta)(Z)\big\{\eta(U)V - \eta(V)U\big\} \\
&+\eta(Z)\big\{(\tilde{\nabla}_X\eta)(U)V - (\tilde{\nabla}_X\eta)(V)U\big\}\Big] + \beta\Big[(\tilde{\nabla}_X g)(U,Z)\phi V - (\tilde{\nabla}_X g)(V,Z)\phi U \\
&+g(U,Z)(\tilde{\nabla}_X\phi)V - g(V,Z)(\tilde{\nabla}_X\phi)U + 2(\tilde{\nabla}_X\eta)(Z)g(\phi U,V)\zeta \\
&+2\eta(Z)\{(\tilde{\nabla}_X g)(\phi U,V) + g\big((\tilde{\nabla}_X\phi)U,V\big)\}\zeta\Big] + \eta\big(R(U,V)Z\big)\phi X \\
&-\eta(U)R(\phi X,V)Z - \eta(V)R(U,\phi X)Z - \eta(Z)R(U,V)\phi X = 0. \qquad (26)
\end{aligned}
$$

Setting $X = U = Z = \zeta$ in (26) and using (9), (12), (16) and (20), we obtain the following equation

$$\big(2\alpha(\zeta\alpha) - 2\beta(\zeta\beta) - \zeta\alpha - \zeta(\zeta\beta)\big)\big(\eta(V)\zeta - V\big) + (\zeta\beta)\phi V = 0. \qquad (27)$$

If β is a non-zero constant, then (27) become

$$(2\alpha - 1)(\zeta\alpha)\big(\eta(V)\zeta - V\big) = 0,$$

or

$$2\alpha\beta(1 - 2\alpha)\big(\eta(V)\zeta - V\big) = 0.$$

From the above equation we have

$$\eta(V)\zeta - V = 0 \tag{28}$$

for all $V \in \chi(M)$, provided $\alpha \neq 0, \frac{1}{2}$.

Now from (28) we obtain

$$\phi V = 0 \tag{29}$$

for all $V \in \chi(M)$, provided $\alpha \neq 0, \frac{1}{2}$.

Using (28) and (29) in (25) we get

$$(\tilde{\nabla}_X \tilde{R})(U, V)Z = (\nabla_X R)(U, V)Z. \tag{30}$$

Theorem 1. *If β is a non-zero constant in a locally symmetric trans Sasakian manifold with respect to $\tilde{\nabla}$, then is locally symmetric with respect to ∇ provided $\alpha \neq 0, \frac{1}{2}$.*

Again using (29) in (12) we get

$$\tilde{\nabla}_X U = \nabla_X U, \tag{31}$$

provided $\alpha \neq 0, \frac{1}{2}$.

Theorem 2. *If a trans Sasakian manifold is locally symmetric with respect to $\tilde{\nabla}$, then $\tilde{\nabla} = \nabla$ if and only if β is a non-zero constant, provided $\alpha \neq 0, \frac{1}{2}$.*

5 Ricci semi-symmetric trans-Sasakian manifolds

Definition 2. *[20, 21]"A trans-Sasakian manifold is said to be Ricci semi-symmetric if the Riemannian curvature tensor R and Ricci tensor S satisfy $R.S = 0$".*

Let M^n be a Ricci semi-symmetric trans-Sasakian manifold with respect to $\tilde{\nabla}$, we have

$$\tilde{R}(U, V).\tilde{S}(Z, W) = 0 \tag{32}$$

for all $U, V, Z, W \in \chi(M)$.

From (33) we have

$$\tilde{S}(\tilde{R}(U, V)Z, W) + \tilde{S}(Z, \tilde{R}(U, V)W) = 0. \tag{33}$$

Using (13) in (33 we get

$$\tilde{S}(R(U,V)Z,W) + \tilde{S}(Z,R(U,V)W) + \alpha\{g(\phi V,Z)S(\phi U,W) - g(\phi U,Z)S(\phi V,W) + \eta(U)\eta(Z)S(V,W) - \eta(V)\eta(Z)S(U,W)\} + \beta\{g(U,Z)S(\phi V,W) - g(V,Z)S(\phi U,W) + 2\eta(Z)g(\phi U,V)S(\zeta,W)\} + \alpha\{g(\phi V,W)S(Z,\phi U) - g(\phi U,W)S(Z,\phi V) + \eta(U)\eta(W)S(Z,V) - \eta(V)\eta(W)S(Z,U)\} + \beta\{g(U,W)S(Z,\phi V) - g(V,W)S(Z,\phi U) + 2\eta(W)g(\phi U,V)S(Z,\zeta)\} = 0. \tag{34}$$

Setting $U = Z = W = \xi$ in (34) yields

$$(\alpha^2 - \beta^2 - \alpha - \zeta\beta)\big[(n-2)\big\{(V\beta) - (\zeta\beta)\eta(V)\big\} + \big((\phi V)\alpha\big)\big] - \beta\big[(n-2)\big((\phi V)\beta\big) - (V\alpha) + \eta(V)(\zeta\beta)\} = 0. \tag{35}$$

For $\beta = 0$ we have

$$(\alpha^2 - \alpha)\big((\phi V)\alpha\big) = 0,$$

or

$$\big((\phi V)\alpha\big) = 0 \tag{36}$$

for all $V \in M$, provided $\alpha \neq 0, 1$. That is if $\beta = 0$ then α is constant.

Conversely, we suppose that α is a non zero constant. From (9) we have

$$\beta = 0. \tag{37}$$

Thus we can state

Theorem 3. *In a Ricci semi-symmetric trans Sasakian manifold with respect to $\tilde{\nabla}$, $\beta = 0$ if and only if α is constant, provided $\alpha \neq 0, 1$.*

We know that a tran Sasakian manifold of type $(\alpha, 0)$ is α-Sasakian manifold. Thus we have

Corollary 1. *A Ricci semi-symmetric trans Sasakian manifold with respect to $\tilde{\nabla}$ is α-Sasakian manifold if and only if α is constant.*

6 Generalized recurrent trans-Sasakian manifolds

Definition 3. *[6]"A Riemannian manifold is said to be a generalized recurrent manifold if it satisfies*

$$(\tilde{\nabla}_X R)(U,V)Z = A(X)R(U,V)Z + B(X)\big[g(V,Z)U - g(U,Z)V\big]".$$

We consider a generalized recurrent trans Sasakian manifold with respect to $\tilde{\nabla}$ and we denote the manifold by $G\{(K_n)TS\}$

$$(\tilde{\nabla}_X \tilde{R})(U,V)Z = A(X)\tilde{R}(U,V)Z + B(X)\big[g(V,Z)U - g(U,Z)V\big]. \tag{38}$$

Contracting (38) we get

$$(\tilde{\nabla}_X \tilde{S})(V,Z) = A(X)\tilde{S}(V,Z) + B(X)(n-1)g(V,Z). \tag{39}$$

It is well known that

$$(\tilde{\nabla}_X \tilde{S})(V,Z) = \tilde{\nabla}_X \tilde{S}(V,Z) - \tilde{S}(\tilde{\nabla}_X V, Z) - \tilde{S}(V, \tilde{\nabla}_X Z). \tag{40}$$

Setting $V = Z = \zeta$ in the above equation and using (18), (19), (20) we get

$$\begin{aligned}(\tilde{\nabla}_X \tilde{S})(\zeta,\zeta) = (n-1)\{2\alpha(X\alpha) - 2\beta(X\beta) - X\alpha - X(\zeta\beta)\}\\ +2(n-2)\beta\{(X\beta) - \eta(X)(\zeta\beta)\} - 2(n-2)(\alpha-1)(\phi X)\beta\\ +2\beta\big((\phi X)\alpha\big) + 2(\alpha-1)\{(X\alpha) - \eta(X)(\zeta\alpha)\}.\end{aligned} \tag{41}$$

Again setting $V = Z = \zeta$ in the above equation and using (41) and (19) we get

$$\begin{aligned}A(X)(n-1)(\alpha^2 - \beta^2 - \alpha - \zeta\beta) + B(X)(n-1)\\ = (n-1)\{2\alpha(X\alpha) - 2\beta(X\beta) - X\alpha - X(\zeta\beta)\}\\ +2(n-2)\beta\{(X\beta) - \eta(X)(\zeta\beta)\} - 2(n-2)(\alpha-1)(\phi X)\beta\\ +2\beta\big((\phi X)\alpha\big) + 2(\alpha-1)\{(X\alpha) - \eta(X)(\zeta\alpha)\}.\end{aligned} \tag{42}$$

Theorem 4. *In a $G\{(K_n)TS\}$ the relation between the 1-forms A and B is given by equation (42).*

Now contracting (39) over V and Z we get

$$dr(X) = rA(X) + n(n-1)B(X). \tag{43}$$

Using (42) from (43) to eliminate B we have

$$\begin{aligned}A(X)\big[(n-1)(\alpha^2 - \beta^2 - \alpha - \zeta\beta) - \frac{r}{n}\big] + \frac{dr(X)}{n}\\ = (n-1)\{2\alpha(X\alpha) - 2\beta(X\beta) - X\alpha - X(\zeta\beta)\}\\ +2(n-2)\beta\{(X\beta) - \eta(X)(\zeta\beta)\} - 2(n-2)(\alpha-1)(\phi X)\beta\\ +2\beta\big((\phi X)\alpha\big) + 2(\alpha-1)\{(X\alpha) - \eta(X)(\zeta\alpha)\}.\end{aligned} \tag{44}$$

Using (43) in (44) we get

$$\begin{aligned}B(X)\big[(n-1) - \frac{n(n-1)^2(\alpha^2 - \beta^2 - \alpha - \zeta\beta)}{r}\big]\\ = (n-1)\{2\alpha(X\alpha) - 2\beta(X\beta) - X\alpha - X(\zeta\beta)\}\\ +2(n-2)\beta\{(X\beta) - \eta(X)(\zeta\beta)\} - 2(n-2)(\alpha-1)(\phi X)\beta\\ +2\beta\big((\phi X)\alpha\big) + 2(\alpha-1)\{(X\alpha) - \eta(X)(\zeta\alpha)\}\beta\\ -\frac{dr(X)}{r}(n-1)(\alpha^2 - \beta^2 - \alpha - \zeta\beta).\end{aligned} \tag{45}$$

Theorem 5. *In a $G\{(K_n)TS\}$ the expression for A and B is given by equations (44). and (45) respectively*

Again contracting (39) over X and Z we get

$$\frac{1}{2}dr(V) = A(\tilde{Q}V) + (n-1)B(V), \tag{46}$$

where $\tilde{Q}$ is the Ricci operator with respect $\tilde{\nabla}$.

Replacing V by X in (46)

$$dr(X) = 2A(\tilde{Q}X) + 2(n-1)B(X). \tag{47}$$

Again from (47) from (43) we obtain

$$A(\tilde{Q}X) = \frac{r}{n}A(X) + \frac{n-2}{2n}dr(X). \tag{48}$$

If the scalar curvature is constant i.e $dr(X) = 0$, (48) become

$$A(\tilde{Q}X) = \frac{r}{n}A(X). \tag{49}$$

Theorem 6. *If in a $G\{(K_n)TS\}$ the scalar curvature is constant, then the 1-forms A satisfies (49).*

7 Group manifolds

Definition 4. *[10, 1]"A Riemannian manifold is a group manifold with respect to the quarter-symmetric connection if*

$$\tilde{R}(U,V)Z = 0,$$
$$(\tilde{\nabla}_X\tilde{T})(U,V) = 0 \tag{50}$$

for all $U, V, Z \in \chi(M)$".

Now we suppose that the curvature satisfies

$$\tilde{R}(U,V)Z = 0 \tag{51}$$

for all $U, V, Z \in \chi(M)$. From the above equation it is clear that

$$\tilde{S}(V,Z) = 0 \tag{52}$$

for all $V, Z \in \chi(M)$.

Setting $U = Z = \zeta$ in (51) we get

$$(\alpha^2 - \beta^2 - \alpha - \zeta\beta)\{\eta(V)\zeta - V\} + \beta\phi V = 0. \tag{53}$$

Again setting $V = Z = \zeta$ in (52) we get

$$(n-1)(\alpha^2 - \beta^2 - \alpha - \zeta\beta) = 0. \tag{54}$$

Since $n \geq 3$, the above equation becomes

$$(\alpha^2 - \beta^2 - \alpha - \zeta\beta) = 0. \tag{55}$$

Using (55) in (53) we get

$$\beta\phi V = 0 \tag{56}$$

for all $V \in \chi(M)$.

From (55) we have

$$\phi V = 0 \tag{57}$$

for all $V \in \chi(M)$, provided $\beta \neq 0$.

Using (12) we obtain the torsion tensor with respect to the quarter symmetric connection as

$$\tilde{T}(U, V) = \eta(V)\phi U - \eta(U)\phi V, \tag{58}$$

for all $V \in \chi(M)$.

Using (57) in (58) we obtain

$$(\tilde{\nabla}_X \tilde{T})(U, V) = 0 \tag{59}$$

for all $U, V \in \chi(M)$, provided $\beta \neq 0$. Thus we have

Theorem 7. *A trans Sasakian manifold is group manifold with respect to the quarter symmetric connection if and only if $\tilde{R}(U, V)Z = 0$, provided $\beta \neq 0$.*

By virtue of (12) and (57) we have

Corollary 2. *If a trans Sasakian manifold is a group manifold with respect to $\tilde{\nabla}$, then $\tilde{\nabla} = \nabla$, provided $\beta \neq 0$.*

Moreover we obtain

$$\tilde{R}(U, V)Z = R(U, V)Z \tag{60}$$

and

$$\tilde{T}(U, V) = T(U, V) \tag{61}$$

for all $U, V, Z \in \chi(M)$. Thus we have

Corollary 3. *If a trans Sasakian manifold is a group manifold with respect to $\tilde{\nabla}$, then it is a group manifold with respect to ∇, provided $\beta \neq 0$.*

References

[1] Agashe N. S. and Chafle M. R., *A semi-symmetric non-metric connection on a Riemannian manifold*, Indian Journal of Pure and Applied Mathematics **23(6)** (1992), 399-403.

[2] Barman A., *Weakly symmetric and weakly-Ricci symmetric LP-Sasakian Manifolds admitting a quarter-symmetric metric connection*, Novi Sad J. Math **45(2)** (2015), 143-153.

[3] Blair D. E., *Contact Manifolds in Riemannian geometry*, Lecture Notes in Mathematics, 509, Springer-Verlag, Berlin, 1976.

[4] Blair D. E. and Oubina J. A., *Conformal and related changes of metric on the product of two almost contact metric manifolds*, Publ. Mat. **34(1)** (1990), 199-207.

[5] Cartan E., *Sur une classe remarquable d'espaces de Riemannian*, Bull. Soc. Math. France **54(55)** (1926), 214-264.

[6] De U. C. and Guha N., *On generalized recurrent manifolds*, J. National Academy of Math., India **9** (1991), 85-92.

[7] De U. C. and Mukut M. T., *Ricci tensor in 3-dimensional trans-Sasakian manifolds*, Kyungpook mathematical journal **43(2)** (2003), 247-247.

[8] Dey S. and Bhattacharyya A., *Some Properties of Lorentzian α-Sasakian Manifolds with Respect to Quarter-symmetric Metric Connection*, Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica **54(2)** (2015), 21-40.

[9] Dey S., Pal B. and Bhattacharyya A., *Some classes of Lorentzian a-Sasakian manifolds with respect to quarter-symmetric metric connection*, Tbilisi Mathematical Journal **10(4)** (2017), 1-16.

[10] Eisenhart L. P., *Continuous groups of transformations*, Princeton University press, 1933.

[11] Friedmann A. and Schouten J.C., *Uber die Geometric der halbsymmetrischen Ubertragung*, Math. Zeitschr. **21** (1924), 211–223.

[12] Golab S., *On semi-symmetric and quarter-symmetric linear connections*, Tensor (N. S.) **29** (1975), 249-254.

[13] Oubina J. A., *New class of almost contact metric structures*, Publ. Math. Debrecen. **32** (1985), 187-193.

[14] Patra C. and Bhattacharyya A., *Trans-Sasakian manifold admitting quarter-symmetric non-metric connection*, Acta Universitatis Apulensis **36** (2013), 39-49.

[15] Prakasha D. G. and Vikas K., *On weak symmetries of Kenmotsu Manifolds with respect to quarter-symmetric metric connection*, Annales Mathematicae et Informaticae **45** (2015), 79-90.

[16] Prasad R. and Haseeb A., *On a Lorentzian Para-Sasakian manifolds with respect to the quarter-symmetric metric connection*, Novi Sad J. Math. **46(2)** (2016), 103-116.

[17] Singh J. P., *On a type of LP-Sasakian manifolds admitting a quarter symmetric non-metric connection*, The Mathematics Student **84** (2015), 57-67.

[18] Singh J. P. and Devi M. S., *On a type of quarter symmetric non -metric connection in an LP- Sasakian manifold*, Science and Technology Jour. **4(1)** (2015), 65-68.

[19] Singh J. P., *Some properties of LP-Sasakian manifolds admitting a quarter symmetric nonmetric connections*, Tomsui Oxford J. Inf. Math. Sc. **29(4)** (2014), 505-517.

[20] Prasad Z. I., *Structure theorems on Riemannian spaces satisfying* $R(X,Y).R = 0$. *I. The local version*, J. Differential Geom. **17** (1982), 531-582.

[21] Prasad Z. I., *Structure theorems on Riemannian spaces satisfying* $R(X,Y).R = 0$. *II. Global versions*, Geometriae Dedicata **19(1)** (1985), 65-108.