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STABILITY OF ESSENTIAL SPECTRA OF CLOSED OPERATORS UNDER T-COMPACT EQUIVALENCE AND APPLICATIONS

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Abstract

The main subject of this paper is to introduce and study the concept of T-compact equivalence of closed linear operators in Hilbert spaces. Many results are proved via this equivalence, especially the invariance of essential spectra of T-compact equivalent closed operators. The results obtained are used to describe some Fredholm essential spectra of transport operators.

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1 Introduction

The essential spectrums and their stability properties under additive perturbations in an appropriate class of operators, have been a research interest of many authors (see e.g. [8], [10]). In 1909, Weyl showed the stability of the essential spectrum of a self-adjoint operator under a compact perturbation in Hilbert space [12]. Schechter extends this result to bounded Fredholm operators in Banach space. Von Neumann Theorem asserts that two bounded self-adjoint operators on a Hilbert space are unitarily equivalent modulocompacts. Note that the question of unitary equivalence modulo the compacts is precisely that of the unitary equivalence in the algebra of Calkin.
In 1984, Brigitte Mercier introduced the notion of compact equivalence between two closed, densely defined linear operators in a Hilbert space and consequently

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that of essentially normal unbounded operators [7]. She generalized Weyl's theorem and also the results of Brown, Douglas and Fillmore.
In this work, we generalize some results of Mercier [7] and Labrousse-Mercier [6], where we introduce the notion of weak and strong T-compact equivalence of closed densely defined linear operators in a Hilbert space via a bounded invertible positive operator T and apply this generalization to perturbation of essential spectra of not self-adjoint operators. In analogy with Weyl's theorem, one would like that the essential spectrums to be invariant under arbitrary compact perturbations. The compact equivalence is not suitable in this direction and the situation is considerably more complicated, because it is possible for the unperturbed operator to have only a discrete spectrum while the point spectrum of the perturbed operator is the whole complex plane, and some operators have point eigenvalues which are not isolated and are carried into the resolvent under a compact perturbation. However, there are applications in which one would like to know that certain types of singularities do not appear after compact perturbations, even though such singularities lie outside the essential spectrum. This motivates several other possible definitions of the essential spectrum (for an arbitrary operator) as the largest subset of the spectrum remaining invariant under arbitrary compact perturbations.

The paper is organized as follows. We give in the second section some preliminary results required in the sequel about perturbation theory for Fredholm operators and the associated essential spectrums. In the third section, we illustrate the concept of T-compact equivalence in Hilbert spaces between closed densely defined linear perators where we introduce in this equivalence a bounded invertible operator T. The T-compact equivalence generalizes the weak and strong compact equivalence concepts introduced by Labrousse and Mercier in [6]. In the last section we apply the main results to study the stability of certain essential spectra of transport operators .

2 Preliminaries

In this section we present briefly some notations, definitions and theorems which are used throughout this work. Let H be a complex Hilbert space. Let $\mathcal{C}(H)$ and $\mathcal{B}(H)$ denote the set of closed linear operators with dense domain in H and bounded linear operators with domain H, respectively. $\mathcal{K}(H)$ is the set of compact elements of $\mathcal{B}(H)$. The domain, null space, range and the graph of $A \in \mathcal{C}(H)$ will be denoted by $D(A)$, $N(A)$, $\mathcal{R}(A)$ and $G(A)$, respectively. The identity operator on H is denoted by I.

Definition 2.1. $A \in \mathcal{C}(H)$ is said to be a Fredholm operator (we denote $A \in \mathcal{F}(H)$) if:

1) $\mathcal{R}(A)$ is closed,

2) $0 < \alpha(A) = dim N(A) < \infty$ and $\beta(A) = codim \mathcal{R}(A) < \infty$.

Then, the index of A ($ind(A)$) is defined by:

$$ind(A) = \alpha(A) - \beta(A).$$

It is well known that if $A \in \mathcal{B}(H) \cap \mathcal{F}(H)$ and $K \in \mathcal{K}(H)$, then $B = A + K \in \mathcal{F}(H)$ and $ind(B) = ind(A)$.

Definition 2.2. Let $A, B \in \mathcal{C}(H)$ and $P_{G(A)}$ (resp. $P_{G(B)}$) the orthogonal projection of $H \oplus H$ onto $G(A)$ (resp. $G(B)$). Then we define:

$$\delta(A,B) = \|(I - P_{G(B)})P_{G(A)}\|,$$

and

$$g(A,B) = \|P_{G(A)} - P_{G(B)}\|.$$

Proposition 2.3. *([3, 5]) Let $A, B \in \mathcal{C}(H)$.*

$$\delta(A,B) = \delta(A^*, B^*)$$

$$g(A,B) = \max\{\delta(A,B), \delta(B,A)\} = g(A^*, B^*).$$

Furthermore,

$$g(A,B) < 1 \Longrightarrow \begin{cases} G(A) \cap G(B)^{\perp} = G(A)^{\perp} \cap G(B) = \{(0,0)\} \\ G(A) + G(B)^{\perp} = G(A)^{\perp} + G(B) = H \oplus H \end{cases}.$$

Definition 2.4. The set of upper semi-Fredholm operators in H is defined by

$$\mathcal{F}_+(H) = \{A \in \mathcal{C}(H) : \alpha(A) < \infty \text{ and } \mathcal{R}(A) \text{ is closed in } H\},$$

the set of lower semi-Fredholm operators in H is defined by

$$\mathcal{F}_-(H) = \{A \in \mathcal{C}(H) : \beta(A) < \infty \text{ and } \mathcal{R}(A) \text{ is closed in } H\},$$

the set of semi-Fredholm operators in H is defined by

$$\mathcal{F}_\pm(H) = \mathcal{F}_+(H) \cup \mathcal{F}_-(H),$$

the set of Fredholm operators in H is defined by

$$\mathcal{F}(H) = \mathcal{F}_+(H) \cap \mathcal{F}_-(H),$$

The classes of semi-Fredholm operators lead to the definition of the upper semi-Fredholm spectrum of $A \in \mathcal{C}(H)$ by:

$$\sigma_{uf}(A) := \{\lambda \in \mathbb{C} : \lambda I - A \notin \mathcal{F}_+(H)\},$$

and the lower semi-Fredholm spectrum of A by:

$$\sigma_{lf}(A) := \{\lambda \in \mathbb{C} : \lambda I - A \notin \mathcal{F}_-(H)\}.$$

The semi-Fredholm spectrum is defined by

$$\sigma_{sf}(A) = \{\lambda \in \mathbb{C} : \lambda I - A \notin \mathcal{F}_\pm(H)\},$$

while the Fredholm spectrum is defined by

$$\sigma_{ef}(A) := \{\lambda \in \mathbb{C} : \lambda I - A \notin \mathcal{F}(H)\}.$$

Clearly, $\sigma_{sf}(A) = \sigma_{uf}(A) \cap \sigma_{lf}(A)$ and $\sigma_{ef}(A) = \sigma_{uf}(A) \cup \sigma_{lf}(A)$.

We shall distinguish the following classes of operators:
the set of upper semi-Weyl operators is defined by

$$\mathcal{W}_+(H) := \{A \in \mathcal{F}_+(H) : \operatorname{ind}(A) \leq 0\},$$

the set of lower semi-Weyl operators is defined by

$$\mathcal{W}_-(H) := \{A \in \mathcal{F}_-(H) : \operatorname{ind}(A) \geq 0\},$$

and the set of Weyl operators is defined by

$$\mathcal{W}(H) := \mathcal{W}_+(H) \cap \mathcal{W}_-(H) = \{A \in \mathcal{F}(H) : \operatorname{ind}(A) = 0\}.$$

The various classes of operators defined above are associated with the following essential spectra:
The upper semi-Weyl spectrum is defined by:

$$\sigma_{uw}(A) := \{\lambda \in \mathbb{C} : \lambda I - A \notin \mathcal{W}_+(H)\}.$$

The lower semi-Weyl spectrum is defined by:

$$\sigma_{lw}(A) := \{\lambda \in \mathbb{C} : \lambda I - A \notin \mathcal{W}_-(H)\}.$$

The Weyl spectrum is defined by:

$$\sigma_{ew}(A) := \{\lambda \in \mathbb{C} : \lambda I - A \notin \mathcal{W}(H)\} = \sigma_{uw}(A) \cup \sigma_{lw}(A).$$

In this paper, we must recall some properties of the operator $R_A = (I+A^*A)^{-1}$ for $A \in \mathcal{C}(H)$, introduced by [2, 5] and widely used in the paper [6] and recently with a general version in [4].

Lemma 2.5. *Let $A \in \mathcal{C}(H)$. Then we have the following statements:*

i) A^*A is self-adjoint and $D(A^*A)$ is a core for A (the closure of $A_{|D(A^*A)}$ is A).

ii) $I + A^*A$ is bijective from $D(A^*A)$ onto H, so that $(I + A^*A)^{-1} \in \mathcal{B}(H)$ with $0 \leq (I + A^*A)^{-1} \leq I$.

iii) The closure of $(I + AA^*)^{-1}A$ is $A(I + A^*A)^{-1}$, $A(I + A^*A)^{-1} \in \mathcal{B}(H)$ and $\left\|A(I + A^*A)^{-1}\right\| \leq 1$.

We denote $\mathcal{B}^+(H)$ the set of all bounded, positive definite and invertible operators on H.

Lemma 2.6. *Let $A \in \mathcal{C}(H)$ and $T \in \mathcal{B}^+(H)$. Then $R_A^T = (T + A^*A)^{-1}$ and AR_A^T are everywhere defined transformations on H and bounded, $\|R_A^T\| \leq \|T\|^{-1}$ and $\|AR_A^T\| \leq \|T\|^{-1/2}$.*

Proof. Note that:

$$\begin{aligned} T + A^*A &= T^{1/2}\left[I + T^{-1/2}A^*AT^{-1/2}\right]T^{1/2} \\ &= T^{1/2}\left[I + (AT^{-1/2})^*(AT^{-1/2})\right]T^{1/2} \end{aligned}$$

Since $AT^{-1/2} \in C(H)$, the Lemma 2.5 applied with $AT^{-1/2}$ instead of A, shows that $\left[I + (AT^{-1/2})^*(AT^{-1/2})\right]$ is bijective from $T^{1/2}(D(A^*A))$ onto H,

$$\begin{aligned} \left[I + (AT^{-1/2})^*(AT^{-1/2})\right]^{-1} &\in \mathcal{B}(H), \\ AT^{-1/2}\left[I + (AT^{-1/2})^*(AT^{-1/2})\right]^{-1} &\in \mathcal{B}(H). \end{aligned}$$

with
$0 \leq \left[I + (AT^{-1/2})^*(AT^{-1/2})\right]^{-1} \leq I$; $\|AT^{-1/2}(I + (AT^{-1/2})^*(AT^{-1/2}))^{-1}\| \leq 1$.
Hence,

$$R_A^T = (T + A^*A)^{-1} = T^{-1/2}\left[I + (AT^{-1/2})^*(AT^{-1/2})\right]^{-1}T^{-1/2} \in \mathcal{B}(H),$$

$$AT^{-1/2}\left[I + (AT^{-1/2})^*(AT^{-1/2})\right]^{-1} = AR_A^T T^{1/2} \in \mathcal{B}(H),$$

$$\|R_A^T\| \leq \|T^{-1}\| \left\|\left[I + (AT^{-1/2})^*(AT^{-1/2})\right]^{-1}\right\| \leq \|T^{-1}\|,$$

$$\|AR_A^T\| = \left\|AR_A^T T^{1/2}T^{-1/2}\right\| \leq \left\|AR_A^T T^{1/2}\right\| \left\|T^{-1/2}\right\| \leq \frac{1}{\sqrt{\|T\|}}.$$

□

In the following, T is a fixed element in $\mathcal{B}^+(H)$ and $A_T = AT^{-1/2}$ where $A \in \mathcal{C}(H)$. So, $D(A_T) = T^{1/2}(D(A))$ and $A_T \in \mathcal{C}(H)$.

It is important to note that when working with the A_T block, the commutation condition between operators A and T is steadily avoided. Indeed, case A commuting with T constitutes an elegant generalization of the results of Labrousse and Mercier [6] and remains also a very particular case of our work.

Remark 2.7. By virtue of Lemma 2.6, we deduce that $S_{A_T} = \sqrt{R_{A_T}}$ is a positive and symmetric square root of R_{A_T}.

Proposition 2.8. *([2, 9]) Let $A \in \mathcal{C}(H)$, then*

1)

$$A_T^*A_TR_{A_T} = I - R_{A_T}. \tag{2.1}$$

2) If $x \in D(A_T)$,

$$R_{A_T^*}A_Tx = A_TR_{A_T}x, \tag{2.2}$$

thus,

$$(A_TR_{A_T})^* = A_T^*R_{A_T^*}, \tag{2.3}$$

and for all $x \in H$

$$\|(\frac{1}{2}I - R_{A_T})x\|^2 + \|A_TR_{A_T}x\|^2 = \frac{1}{4}\|x\|^2, \tag{2.4}$$

so

$$\|R_{A_T}\| \leq 1 \text{ and } \|A_TR_{A_T}\| \leq \frac{1}{2}.$$

3) $\mathcal{R}(S_{A_T}) = D(A_T)$ *and if* $x \in D(A_T)$ *we have* $S_{A_T^*}A_Tx = A_TS_{A_T}x$.

$$(A_TS_{A_T})^* = A_T^*S_{A_T^*}, \tag{2.5}$$

and for all $x \in H$,

$$\|S_{A_T}x\|^2 + \|A_TS_{A_T}x\|^2 = \|x\|^2$$

so

$$\|S_{A_T}\| \leq 1, \quad \|A_TS_{A_T}\| \leq 1.$$

Proposition 2.9. *Let* $A \in \mathcal{C}(H)$, *then the orthogonal projection of* $H \oplus H$ *onto the graph of* A_T *is given by:*

$$P_{G(A_T)} = \begin{pmatrix} T^{1/2}R_A^TT^{1/2} & A_T^*T^{1/2}R^{T^{-1}}_{(A_T^*T^{-1/2})}T^{1/2} \\ AR_A^TT^{1/2} & I - T^{1/2}R^{T^{-1}}_{(A_T^*T^{-1/2})}T^{1/2} \end{pmatrix}.$$

Therefore, if $A, B \in \mathcal{C}(H)$, *then*

$$(I - P_{G(B_T)})P_{G(A_T)} = M_{B_T}\begin{pmatrix} 0 & 0 \\ B_TS_{B_T}S_{A_T} - S_{B_T^*}A_TS_{A_T} & 0 \end{pmatrix}M_{A_T} \tag{2.6}$$

where $M_{C_T} = \begin{pmatrix} S_{C_T} & C_T^*S_{C_T} \\ C_TS_{C_T} & -S_{C_T^*} \end{pmatrix}$, *for* $C = A$ *or* B.

Proof. We know from [2, 11] that:

$$P_{G(A_T)} = \begin{pmatrix} R_{A_T} & A_T^*R_{A_T^*} \\ A_TR_{A_T} & I - R_{A_T^*} \end{pmatrix}.$$

So, we deduce the result since:

$$R_{A_T^*} = T^{1/2}R^{T^{-1}}_{(A_T^*T^{-1/2})}T^{1/2},$$

and

$$A_T^*T^{1/2}R^{T^{-1}}_{(A_T^*T^{-1/2})}T^{1/2} = R_{A_T}A_T^*.$$

□

3 T-compact equivalence

In this section, we further extend the notion of compact equivalence introduced in [7] to that of T-compact equivalence by replacing in the definition of R_A the identity operator by a bounded positive definite and invertible operator on H.

3.1 Weak T-compact equivalence

Definition 3.1. Let $A, B \in \mathcal{C}(H)$. We say that A and B are weakly T-compact equivalent (and we write $A \underset{T}{\sim} B$) if $\left(P_{G(A_T)} - P_{G(B_T)}\right) \in \mathcal{K}(H \oplus H)$.

Remark 3.2. 1) Lemmas $III.1.2$ and $III.1.3$ of [7] show that T-compact equivalence of operators coincides on $\mathcal{B}(H)$ with the usual relation $A \sim B$ if $A - B \in \mathcal{K}(H)$.

2) $\underset{T}{\sim}$ is an equivalence relation on $\mathcal{C}(H)$.

3) We deduce from Proposition 2.9 that if $A \underset{T}{\sim} B$ then $R_A^T - R_B^T \in \mathcal{K}(H)$. Indeed, $R_A^T - R_B^T = T^{-1/2}(R_{A_T} - R_{B_T})T^{-1/2}$, so $R_A^T - R_B^T \in \mathcal{K}(H)$ as soon as $R_{A_T} - R_{B_T} \in \mathcal{K}(H)$.

By virtue of Lemma 2.6, Proposition 2.8, formula (2.6) and Propositions 1.5 and 2.7 of [6], it is easy to show the following results:

Theorem 3.3. *Let $A, B \in \mathcal{C}(H)$ and $C \in \mathcal{B}(H)$. Then:*

1)

$$A \underset{T}{\sim} B \Leftrightarrow A^* \underset{T}{\sim} B^*.$$

2)

$$A \underset{T}{\sim} B \Rightarrow \lambda A \underset{T}{\sim} \lambda B, \text{ for all } \lambda \in \mathbb{C}. \tag{3.1}$$

3)

$$A \underset{T}{\sim} B \Leftrightarrow \begin{cases} A_T S_{A_T} S_{B_T} - S_{A_T^*} B_T S_{B_T} \in \mathcal{K}(H) \\ B_T S_{B_T} S_{A_T} - S_{B_T^*} A_T S_{A_T} \in \mathcal{K}(H) \end{cases} \tag{3.2}$$

4)

$$A \underset{T}{\sim} B \Rightarrow A + CT^{1/2} \underset{T}{\sim} B + CT^{1/2}.$$

5) If $A \underset{T}{\sim} B$,

$$I + A_T^* B_T, \quad I + B_T A_T^*, \quad I + B_T^* A_T, \quad I + A_T B_T^* \in \mathcal{F}(H). \tag{3.3}$$

$$\begin{aligned} (I + A_T^* B_T)^* &= I + B_T^* A_T, & (3.4)\\ (I + B_T A_T^*)^* &= I + A_T B_T^*. & (3.5) \end{aligned}$$

$$\begin{aligned} ind(I + A_T^* B_T) &= ind(I + B_T A_T^*) = -ind(I + B_T^* A_T) & (3.6)\\ &= -ind(I + A_T B_T^*). \end{aligned}$$

Remark 3.4. If $A, B \in \mathcal{C}(H)$ and $g(A_T, B_T) = \|P_{G(A_T)} - P_{G(B_T)}\| < 1$, then (3.3), (3.4) and (3.6) are satisfied and $ind(I + A_T^* B_T) = 0$.

Indeed, by using Proposition *2.3*, $g(A_T, B_T) = \|P_{G(A_T)} - P_{G(B_T)}\| < 1$ implies that:

$$G(A_T) \oplus G(B_T)^\perp = G(A_T)^\perp \oplus G(B_T) = H \oplus H.$$

Hence, $\dim N(I + A_T^* B_T) = \dim N(I + B_T A_{T^*}) = \dim N(I + A_T B_T^*) =$
$= \dim N(I + B_T^* A_T) = 0$.

In general, the condition $ind(I + A_T B_T^*) = 0$ is not satisfied even if $A \underset{T}{\sim} B$, for $A, B \in \mathcal{C}(H)$. Therefore, in the following definition we introduce the strong T-compact equivalence principle.

3.2 Strong T-compact equivalence

Definition 3.5. Let $A, B \in \mathcal{C}(H)$. We say A and B are strongly T-compact equivalent (and we write $A \underset{T}{\approx} B$) if $A \underset{T}{\sim} B$ and $ind(I + A_T B_T^*) = 0$.

Theorem 3.6. *Let $A, B \in \mathcal{F}(H)$, then:*

$$A \underset{T}{\approx} B \Rightarrow ind(A_T) = ind(B_T).$$

Proof. From (3.1) we have $\lambda A \underset{T}{\sim} \lambda B$, we deduce from (3.3) that $I + |\lambda^2| A_T B_T^* \in \mathcal{F}(H)$. It follows that $I + A_T B_T^*$ and $A_T B_T^*$ are homotopic, in the sense that there exists a continuous application F from $[0, 1]$ to $\mathcal{F}(H)$ equipped with the gap metric g, such that $F(0) = A_T B_T^*$ and $F(1) = I + A_T B_T^*$.
By virtue of [Theorem 4.1, [2]], $ind(A_T B_T^*) = ind(I + A_T B_T^*) = 0$. Thus, $ind(A_T) = ind(B_T)$. □

It is clear that $\underset{T}{\approx}$ is an equivalence relation. On the other hand due to Theorem 3.3, we have the following result.

Corollary 3.7. *Let $A, B \in \mathcal{C}(H)$ and $\lambda \in \mathbb{C}$, then:*

$$A \underset{T}{\approx} B \Rightarrow \begin{cases} A^* \underset{T}{\approx} B^* \\ \lambda A \underset{T}{\approx} \lambda B \end{cases} \tag{3.7}$$

Now, we show the stability of some versions of the essential spectrum under weak and strong T-compact additive perturbations. Precisely, we have the following main result:

Theorem 3.8. *Let $A, B \in \mathcal{C}(H)$. Then*

$$A \underset{T}{\sim} B \Rightarrow \sigma_i(A_T) = \sigma_i(B_T), \quad i = lf, \quad uf, \quad sf, \quad ef.$$

and

$$A \underset{T}{\approx} B \Rightarrow \sigma_i(A_T) = \sigma_i(B_T), \quad i = ew, \quad uw, \quad lw. \tag{3.8}$$

Proof. Taking $C = -\lambda I$, for $\lambda \in \mathbb{C}$, in the implication (4) of Theorem 3.3, we obtain:

$$A - \lambda T^{1/2} \underset{T}{\sim} B - \lambda T^{1/2} \Leftrightarrow P_{G(A_T-\lambda I)} - P_{G(B_T-\lambda I)} \in \mathcal{K}(H).$$

Then, due to [Proposition 2.6, [6]], we have $\sigma_{ef}(A_T) = \sigma_{ef}(B_T)$. Consequently, we have $\sigma_i(A_T) = \sigma_i(B_T)$ for $i = lf, \quad uf, \quad sf$.
On the other hand, if $A \underset{T}{\approx} B$ then $A \underset{T}{\sim} B$ and $ind(I + A_T B_T^*) = 0$. According to [Proposition 3.6, [6]] and Theorem 3.6, we have:

$$A \underset{T}{\sim} B \Longrightarrow \begin{cases} ind(I + (B_T - \lambda I)(A_T^* - \lambda I)) = ind(I + B_T A_T^*), \text{ for all } \lambda \in \mathbb{C}, \\ ind(B_T - \lambda I) = ind(A_T - \lambda I) + ind(I + B_T A_T^*), \text{ for all } \lambda \notin \sigma_{ef}(A) \\ \hfill = \sigma_{ef}(B) \end{cases}$$

Hence, $ind(A_T - \lambda I) = ind(B_T - \lambda I)$ for all $\lambda \notin \sigma_{ef}(A) = \sigma_{ef}(B)$. Then:

$$\sigma_i(A_T) = \sigma_i(B_T), \quad i = ew, \quad uw, \quad lw.$$

□

4 Application

In this section, we will apply the stability results obtained to essential spectra of transport operators on L^2-spaces.
Let

$$X_2 = L^2((-a,a) \times (-1,1), dx\, d\xi), \ a > 0.$$

We consider the transport operator:

$$A_H = \mathcal{T}_H + K_1 + K_2$$

where

$$\mathcal{T}_H \psi(x,\xi) = -\xi \frac{\partial \psi}{\partial x}(x,\xi) - \sigma(\xi)\psi(x,\xi) + \int_{-1}^{1} \kappa(x,\xi,\xi')\psi(x,\xi')\, d\xi'$$

with the boundary conditions

$$\psi^i = H(\psi^o)$$

where H is bounded linear operator defined on suitable boundary spaces and $\sigma(.) \in L^\infty(-1,1)$. Here $x \in (-a,a)$ and $\xi \in (-1,1)$ and $\psi(x,\xi)$ represents the angular density of particles (for instance gas molecules, photons, or neutrons) in a homogeneous slab of thickness $2a$. The functions $\sigma(.)$ and $\kappa(.,.,.)$ are called, respectively, the collision frequency and the scattering kernel. K_1 is the collision operator given by:

$$\begin{aligned} K_1 : X_2 &\longrightarrow X_2 \\ \psi &\longrightarrow \int_{-1}^{1} \kappa(x,\xi,\xi')\psi(x,\xi')\, d\xi' \end{aligned}.$$

where $\kappa(.,.,.)$ is a measurable function form $[-a,a]\times[-1,1]\times[-1,1]$ to $\mathbb{R}$. Observe that the operator K_1 acts only on the variable ξ, so x may be viewed merely as a parameter in $[-a,a]$. Hence we may consider

$$K_1 : [-a,a] \longrightarrow K(x) \in \mathcal{Z} = \mathcal{B}(L^2([-1,1],d\xi)).$$

and $K_2 \in \mathcal{K}(X_2)$.

Definition 4.1. A collision operator K is said to be regular if it satisfies the following assumptions:

i) $\{x \in [-a,a] : K(x) \in \mathcal{O}\}$ is measurable if $\mathcal{O} \subset \mathcal{Z}$ is open.

ii) There exists a compact subset $E \subset \mathcal{Z}$ such that $K(x) \in E$ a .e. on $[-a,a]$.

iii) $K(x) \in \mathcal{K}(L^2([-1,1],d\xi))$ a .e. on $[-a,a]$.

4.1 Compact perturbations of transport operator

The following theorem traits the Fredholm perturbations for transport operator A_H.

Theorem 4.2. *Suppose that the collision operator K_1 is regular, then:*

$$\sigma_i(A_H) = \sigma_i(\mathcal{T}_H), \quad i = lf,\ uf,\ sf,\ ef,\ ew,\ uw,\ lw.$$

Proof. Since K_1 is a regular operator and $K_2 \in \mathcal{K}(X_2)$ then $K_1 + K_2 \sim K_1$. Hence $\mathcal{T}_H + K_1 + K_2 \underset{T}{\sim} \mathcal{T}_H + K_1$, then due to Theorem 3.8 and [1], we have:

$$\sigma_i(A_H) = \sigma_i(\mathcal{T}_H), \quad i = lf,\ uf,\ sf,\ ef.$$

On the other hand,

$$\begin{aligned}
I &= (I + (\mathcal{T}_H + K_1 + K_2)(\mathcal{T}_H + K_1 + K_2)^*)R_{(\mathcal{T}_H+K_1+K_2)^*} \\
&= R_{(\mathcal{T}_H+K_1+K_2)^*} \\
&\quad + (\mathcal{T}_H + K_1 + K_2)S_{(\mathcal{T}_H+K_1+K_2)}(\mathcal{T}_H + K_1 + K_2)^* S_{(\mathcal{T}_H+K_1+K_2)^*} \\
&= R_{(\mathcal{T}_H+K_1+K_2)^*} + K_2 S_{(\mathcal{T}_H+K_1+K_2)}(\mathcal{T}_H + K_1 + K_2)^* S_{(\mathcal{T}_H+K_1+K_2)^*} \\
&+ (\mathcal{T}_H + K_1)S_{(\mathcal{T}_H+K_1+K_2)}(\mathcal{T}_H + K_1 + K_2)^* S_{(\mathcal{T}_H+K_1+K_2)^*} \\
&= K_2 S_{(\mathcal{T}_H+K_1+K_2)}(\mathcal{T}_H + K_1 + K_2)^* S_{(\mathcal{T}_H+K_1+K_2)^*} \\
&+ (I + (\mathcal{T}_H + K_1)(\mathcal{T}_H + K_1 + K_2)^*)R_{(\mathcal{T}_H+K_1+K_2)^*}.
\end{aligned}$$

Since, $K_2 S_{(\mathcal{T}_H+K_1+K_2)}(\mathcal{T}_H + K_1 + K_2)^* S_{(\mathcal{T}_H+K_1+K_2)^*} \in \mathcal{K}(H)$, then:

$$ind\left((I + (\mathcal{T}_H + K_1)(\mathcal{T}_H + K_1 + K_2)^*)R_{(\mathcal{T}_H+K_1+K_2)^*}\right) = 0.$$

Furthermore, $R^{-1}_{(\mathcal{T}_H+K_1+K_2)^*} \in \mathcal{F}(H)$, then:

$$ind(I + (\mathcal{T}_H + K_1)(\mathcal{T}_H + K_1 + K_2)^*) = 0.$$

So, by virtue of (3.6) and (3.8), we have:

$$\sigma_i(A_H) = \sigma_i(\mathcal{T}_H), \quad i = \ ew,\ uw,\ lw.$$

□

4.2 Quasi-nilpotent perturbations of transport operator

Now, we consider the transport operator

$$A_H = \mathcal{T}_H + K$$

where K is the bounded operator given by

$$\begin{cases} K : X_p \longrightarrow X_p \\ \qquad \psi \longrightarrow \displaystyle\int_{-1}^{\xi} \kappa(x,\xi,\xi')\psi(x,\xi')\,d\xi' \end{cases}$$

and κ satisfies the following assumptions:

$$(\mathbf{H}) \begin{cases} \kappa(.,.,.) \text{ is a measurable function form } [-a,a]\times[-1,1]\times[-1,1] \text{ to } \mathbb{R} \text{ and} \\ |\kappa(x,\xi,\xi')| \le c < \infty, \ a.e. \end{cases}$$

Lemma 4.3. *([1]) If κ satisfies* $(\mathbf{H})$ *then, for any integer $n \ge 1$*

$$\|K^n\| \le \frac{2^{n+3/2}}{n!} c^n.$$

Theorem 4.4. *Suppose that collision operator K_1 satisfies* $(\mathbf{H})$ *on X_2 and $K_1\mathcal{T}_H - \mathcal{T}_H K_1 \in \mathcal{PF}(X_2)$, then:*

$$\sigma_i(A_H) = \sigma_i(\mathcal{T}_H), \qquad i = lf,\ uf,\ sf,\ ef,\ ew,\ uw,\ lw.$$

where $\mathcal{PF}(X_2) = \{J \in \mathcal{L}(X_2),\ A + J \in \mathcal{F}(X_2) \ \text{ for } \ A \in \mathcal{F}(X_2)\}$.

Proof. Due to [Theorem 25, [1]] and identically to proof of Theorem 4.2, we have the result. □

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