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SOME GRÜSS TYPE INEQUALITIES INVOLVING GENERALIZED FRACTIONAL INTEGRAL OPERATOR

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Abstract

The analogous version of Grüss inequalities has been established using the generalized hypergeometric function fractional integral operators. The results are generalizations of Grüss type inequalities in fractional integral operators. Our main deduction will break into results noted for appropriate changes of fractional integral parameter and degree of fractional operator.

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1 Introduction

Following [8], the well known Grüss inequality, is defined as follows (see also, [7], [13], p. 296):
"Let f and g be two continuous functions defined on $[a, b]$, such that $m \leq f(x) \leq M$ and $p \leq g(x) \leq P$ for each $t \in [a, b]$, where m, p, M, P are given real constants, then

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx\right| \leq \frac{1}{4}(M-m)(P-p), \quad (1)$$

where 1/4 is a finest likely constant."

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Using fractional integral operators, several developments of the classical inequalities, including (1), are studied by many authors, see [1, 2, 3, 4, 5, 9, 10, 14, 18] and references therein. In this direction, Dahmani *et al.* [6] established a generalization of Grüss inequality by means of Riemann Liouville fractional integral operators. Moreover, Kalla and Rao [10] also investigated certain new versions of Grüss type inequality associated with the Saigo's fractional integral operators. We try to generalize inequality (1), by making use of fractional integral operator of Saigo-Maeda type. So, our prime intention in this paper is to provide analogous versions of Grüss inequality by means of generalized hypergeometric function fractional order integral operators. The results are generalizations of Grüss inequality in fractional integral operators.

2 Basic definitions

Now, we use the following definitions and related details.

Definition 1. *Considering $t \geq 0$, a real valued function $f(t)$ is said to be in the space C_μ ($\mu \in R$), if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1 \in C[0,\infty)$ and $C[0,\infty)$ is the set of continuous functions in the interval $[0,\infty)$.*

Definition 2. *Two functions f and g are said to be synchronous on the interval $[0,\infty)$, if*

$$H(\tau,\rho) = (f(\tau) - f(\rho))\,(g(\tau) - g(\rho)) \geq 0, \quad (\tau, \rho \in (0,\infty)). \tag{2}$$

Consequently, we can write

$$f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau). \tag{3}$$

Definition 3 ([17])**.** *Let $\alpha, \alpha', \beta, \beta', \gamma \in \Re$ and $\gamma > 0$, then the Saigo and Maeda fractional integral operator, is defined in the following form:*

$$\left(I_t^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3\left(\alpha,\alpha',\beta,\beta',\gamma, 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)dt. \tag{4}$$

The Appell function $F_3(.)$ appearing as a kernel for the above operator, is defined as

$$F_3(\alpha,\alpha',\beta,\beta',\gamma,x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!} \tag{5}$$

$$\left[max.\left(|x|, |y|\right) < 1\right],$$

where the pochhammer symbol $(\alpha)_m$ for $(m \in N)$, is denoted as under:

$$(\alpha)_m = \alpha(\alpha+1)(\alpha+2)...(\alpha+m-1).$$

Definition 4. *Suppose* $\alpha, \alpha', \beta, \beta', \gamma \in \Re$, *such that*

$$\gamma > max. \left\{0, \left(\alpha + \alpha' + \beta - 1\right), \left(\alpha + \alpha' - 1\right) , \left(\alpha' + \beta - 1\right)\right\}$$

and

$$\beta' > max. \left\{-1, \left(\alpha' - 1\right)\right\}$$

then we define a fractional integral operator, associated with the Appell function, as follows:

$$\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{\Gamma(1+\gamma-\alpha-\alpha')\Gamma(1+\gamma-\alpha'-\beta)\Gamma(1+\beta')}{\Gamma(1+\gamma-\alpha-\alpha'-\beta)\Gamma(1+\beta'-\alpha')} x^{\alpha+\alpha'-\gamma}$$

$$\times \left(I_t^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x). \tag{6}$$

where $I_t^{\alpha,\alpha',\beta,\beta',\gamma}$ *is the Saigo-Maeda fractional integral of order* γ.

3 Main results

The following Lemmas are required to establish our main results.

Lemma 1 ([16], p.394, eq.(4.18)). *Suppose* $\alpha, \alpha', \beta, \beta', \gamma \in \Re(\gamma > 0)$ *and* $\rho > max. \{0, (\alpha + \alpha' + \beta - 1), (\alpha' - \beta')\}$, *then the subsequent image formula holds:*

$$\left(I_t^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\right)(x) = x^{\rho-\alpha-\alpha'+\gamma-1}$$

$$\times \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)\Gamma(\rho+\beta')}. \tag{7}$$

Lemma 2. *For* $\alpha, \alpha', \beta, \beta', \gamma, \rho \in \Re$; $\rho > max.\{0, -(\gamma+\alpha+\alpha'+\beta), -(\alpha'-\beta')\} - 1, \gamma > max.\{0, \alpha+\alpha', \alpha'+\beta\alpha+\alpha'+\beta\}, \beta' < 1, \beta'-\alpha' > -1$, *we have*

$$\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho}\right)(x) = \frac{\Gamma(1+\gamma-\alpha-\alpha')\Gamma(1+\gamma-\alpha'-\beta)\Gamma(1+\beta')}{\Gamma(\rho+1+\beta')\Gamma(\rho+1+\gamma-\alpha-\alpha')}$$

$$\times \frac{\Gamma(\rho+1)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+1+\beta'-\alpha')}{\Gamma(\rho+1+\gamma-\alpha'-\beta)\Gamma(1+\gamma-\alpha-\alpha'-\beta)\Gamma(1+\beta'-\alpha')} x^{\rho}. \tag{8}$$

Proof. Using relation (6), the left hand side of (8) can be written as

$$\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho}\right)(x) = \frac{\Gamma(1+\gamma-\alpha-\alpha')\Gamma(1+\gamma-\alpha'-\beta)\Gamma(1+\beta')}{\Gamma(1+\gamma-\alpha-\alpha'-\beta)\Gamma(1+\beta'-\alpha')}$$

$$\times x^{\alpha+\alpha'-\gamma} \left(I_t^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho}\right)(x). \tag{9}$$

On using (7), the above relation easily arrives at (8). □

Particularly, for $\rho = 0$, we have

$$\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} t^0\right)(x) = 1.$$

or

$$\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} K\right)(x) = K, \tag{10}$$

where K is any constant.

The important result contained in the above lemmas will be required to establish our main results:

Theorem 1. *If $\alpha, \alpha', \beta, \beta', \gamma \in \Re$ such that $\gamma > max.\{\alpha, \alpha', \beta, \beta'\} > 0$, then the following inequality holds*

$$F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{t}{x}; 1 - \frac{x}{t}\right) > 0, \tag{11}$$

provided $-1 < \left(1 - \frac{t}{x}\right) < 0$ and $0 < \left(1 - \frac{x}{t}\right) < \frac{1}{2}$. Also, if $f(x) > 0$, then

$$\left(I_t^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) > 0.$$

Proof. We consider the left hand side of the above inequality, say L and use definition (5) to write

$$L = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{\left(1-\frac{t}{x}\right)^m}{m!} \frac{\left(1-\frac{x}{t}\right)^n}{n!}.$$

Or

$$L = \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{(\gamma)_m} \frac{\left(1-\frac{t}{x}\right)^m}{m!} \sum_{n=0}^{\infty} \frac{(\alpha')_n(\beta')_n}{(\gamma+m)_n} \frac{\left(1-\frac{x}{t}\right)^n}{n!}$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{(\gamma)_m} \frac{\left(1-\frac{t}{x}\right)^m}{m!} \left(1 + \frac{(\alpha')(\beta')}{(\gamma+m)} \frac{\left(1-\frac{x}{t}\right)}{1!} + \dots\right.$$

$$\left. + \frac{(\alpha')_{n-1}(\beta')_{n-1}}{(\gamma+m)_{n-1}} \frac{\left(1-\frac{x}{t}\right)^{n-1}}{(n-1)!} + \frac{(\alpha')_n(\beta')_n}{(\gamma+m)_n} \frac{\left(1-\frac{x}{t}\right)^n}{n!} + \dots\right).$$

In the above expression, all terms except first term are positive due to the condition imposed with (11), then we can write

$$\left(1 + \frac{(\alpha')(\beta')}{(\gamma)} \frac{\left(1-\frac{x}{t}\right)}{1!} + \dots + \frac{(\alpha')_{n-1}(\beta')_{n-1}}{(\gamma)_{n-1}} \frac{\left(1-\frac{x}{t}\right)^{n-1}}{(n-1)!}\right.$$

$$\left. + \frac{(\alpha')_n(\beta')_n}{(\gamma)_n} \frac{\left(1-\frac{x}{t}\right)^n}{n!} + \dots\right) > 0.$$

Hence,

$$L=\Bigg(1+\frac{(\alpha')(\beta')}{(\gamma)}\frac{\left(1-\frac{x}{t}\right)}{1!}+...+\frac{(\alpha')_{n-1}(\beta')_{n-1}}{(\gamma)_{n-1}}\frac{\left(1-\frac{x}{t}\right)^{n-1}}{(n-1)!}$$
$$+\frac{(\alpha')_{n}(\beta')_{n}}{(\gamma)_{n}}\frac{\left(1-\frac{x}{t}\right)^{n}}{n!}+...\Bigg)+\frac{(\alpha)(\beta)}{(\gamma)}\frac{\left(1-\frac{t}{x}\right)}{1!}\Bigg(1+\frac{(\alpha')(\beta')}{(\gamma+1)}\frac{\left(1-\frac{x}{t}\right)}{1!}+...$$
$$+\frac{(\alpha')_{n-1}(\beta')_{n-1}}{(\gamma+1)_{n-1}}\frac{\left(1-\frac{x}{t}\right)^{n-1}}{(n-1)!}+\frac{(\alpha')_{n}(\beta')_{n}}{(\gamma+1)_{n}}\frac{\left(1-\frac{x}{t}\right)^{n}}{n!}+...\Bigg)+...$$
$$+\frac{(\alpha)_{m-1}(\beta)_{m-1}}{(\gamma)_{m-1}}\frac{\left(1-\frac{t}{x}\right)^{m-1}}{(m-1)!}\Bigg(1+\frac{(\alpha')(\beta')}{(\gamma+m-1)}\frac{\left(1-\frac{x}{t}\right)}{1!}+...$$
$$+\frac{(\alpha')_{n-1}(\beta')_{n-1}}{(\gamma+m-1)_{n-1}}\frac{\left(1-\frac{x}{t}\right)^{n-1}}{(n-1)!}+\frac{(\alpha')_{n}(\beta')_{n}}{(\gamma+m-1)_{n}}\frac{\left(1-\frac{x}{t}\right)^{n}}{n!}+...\Bigg)$$
$$+\frac{(\alpha)_{m}(\beta)_{m}}{(\gamma)_{m}}\frac{\left(1-\frac{t}{x}\right)^{m}}{m!}\Bigg(1+\frac{(\alpha')(\beta')}{(\gamma+m)}\frac{\left(1-\frac{x}{t}\right)}{1!}+...$$
$$+\frac{(\alpha')_{n-1}(\beta')_{n-1}}{(\gamma+m)_{n-1}}\frac{\left(1-\frac{x}{t}\right)^{n-1}}{(n-1)!}+\frac{(\alpha')_{n}(\beta')_{n}}{(\gamma+m)_{n}}\frac{\left(1-\frac{x}{t}\right)^{n}}{n!}+...\Bigg)+...$$
$$=T_1+T_2+...+T_{m-1}+T_m+...(\text{Say})$$

Comparing between first and second term, we get

$$\Bigg(1+\frac{(\alpha')(\beta')}{(\gamma)}\frac{\left(1-\frac{x}{t}\right)}{1!}+...+\frac{(\alpha')_{n-1}(\beta')_{n-1}}{(\gamma)_{n-1}}\frac{\left(1-\frac{x}{t}\right)^{n-1}}{(n-1)!}$$
$$+\frac{(\alpha')_{n}(\beta')_{n}}{(\gamma)_{n}}\frac{\left(1-\frac{x}{t}\right)^{n}}{n!}+...\Bigg)>\frac{(\alpha)(\beta)}{(\gamma)}\frac{\left(1-\frac{t}{x}\right)}{1!}\Bigg(1+\frac{(\alpha')(\beta')}{(\gamma+1)}\frac{\left(1-\frac{x}{t}\right)}{1!}+...$$
$$+\frac{(\alpha')_{n-1}(\beta')_{n-1}}{(\gamma+1)_{n-1}}\frac{\left(1-\frac{x}{t}\right)^{n-1}}{(n-1)!}+\frac{(\alpha')_{n}(\beta')_{n}}{(\gamma+1)_{n}}\frac{\left(1-\frac{x}{t}\right)^{n}}{n!}+...\Bigg).$$

Here, we observe that $T_1>T_2>0$, as $\left(1-\frac{x}{t}\right)$ and $\alpha,\alpha',\beta,\beta',\gamma$ are positive but $\left(1-\frac{t}{x}\right)$ is negative, in the given range mentioned in the theorem. Similarly $T_{m-1}>T_m>0$. Thus, if m is an odd number then $T_{m-1}>T_m$. Similarly if m is an even number, then $T_{m-2}>T_{m-1}$ and $T_m>T_{m+1}$.
Hence,

$$F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right)>0.$$

Thereupon, we can easily say

$$\left(I_t^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)=\frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x(x-t)^{\gamma-1}t^{-\alpha'}F_3\left(\alpha,\alpha',\beta,\beta',\gamma,1-\frac{t}{x},1-\frac{x}{t}\right)f(t)dt,$$

is positive, if $f(x)>0$. □

Theorem 2. *Let h be an integrable function on $[0,\infty)$ and satisfying the condition $m \leq h(x) \leq M$ $(m, M \in \Re)$, then for all $x \in [0,\infty)$ and $\alpha, \alpha', \beta, \beta', \gamma \in \Re$ we have*

$$\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} h^2\right)(x) - \left(\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} h\right)(x)\right)^2$$
$$= \left(M - \left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} h\right)(x)\right)\left(\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} h\right)(x) - m\right)(M-h)(h-m)(x). \tag{12}$$

provided $(\alpha, \alpha', \beta, \beta', \gamma) > 0$.

Proof. Let h be an integrable function on $[0,\infty)$ such that $m, M \in \Re$ with $m \leq h(x) \leq M$, then for any $u, v \in [0,\infty)$ we can write the following

$$(M - h(u))(h(v) - m) + (M - h(v))(h(u) - m)$$
$$-(M - h(u))(h(u) - m) - (M - h(v))(h(v) - m)$$
$$= h^2(u) + h^2(v) - 2h(u)h(v). \tag{13}$$

Now, on multiplying the above relation (13) by the factor

$$\frac{\Gamma(1+\gamma-\alpha-\alpha')\Gamma(1+\gamma-\alpha'-\beta)\Gamma(1+\beta')}{\Gamma(1+\gamma-\alpha-\alpha'-\beta)\Gamma(1+\beta'-\alpha')}$$
$$\times \frac{u^{-\alpha'}}{\Gamma(\gamma)}(x-u)^{\gamma-1} x^{\alpha'-\gamma} F_3\left(\alpha, \alpha', \beta, \beta'\gamma, 1-\frac{u}{x}, 1-\frac{x}{u}\right)$$
$$(u \in (0,x), x > 0),$$

and integrating with respect to u from 0 to x, and by applying Definition 4, we obtain

$$\left(M - \left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} h\right)(x)\right)(h(v) - m) + (M - h(v))\left(\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} h\right)(x) - m\right)$$
$$-\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma}(M-h)(h-m)\right)(x) - (M - h(v))(h(v) - m)$$
$$= \left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} h^2\right)(x) + h^2(v) - 2\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} h\right)(x)\ h(v). \tag{14}$$

Again, on multiplying equation (14) by

$$\frac{\Gamma(1+\gamma-\alpha-\alpha')\ \Gamma(1+\gamma-\alpha'-\beta)\Gamma(1+\beta')}{\Gamma(1+\gamma-\alpha-\alpha'-\beta)\Gamma(1+\beta'-\alpha')}$$
$$\times \frac{v^{-\alpha'}}{\Gamma(\gamma)}(x-v)^{\gamma-1} x^{\alpha'-\gamma} F_3\left(\alpha, \alpha', \beta, \beta'\gamma, 1-\frac{v}{x}, 1-\frac{x}{v}\right)$$
$$(v \in (0,x); x > 0),$$

and integrating with respect to v from 0 to x, and then employing Definition 4, we obtain

$$\left(M - \left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} h\right)(x)\right)\left(\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} h\right)(x) - m\right)$$

$$+\left(M-\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma}h\right)(x)\right)\left(\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma}h\right)(x)-m\right)$$

$$-\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma}(M-h)(h-m)\right)(x)-\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma}(M-h)(h-m)\right)(x)$$

$$=\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma}h^2\right)(x)+\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma}h^2\right)(x)-2\left(\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma}h\right)(x)\right)^2, \quad (15)$$

which on further simplification arrive at the desired result (12). □

Theorem 3. *Let f and g be two functions defined and integrable on $[0,\infty]$ with $f,g\in C_\mu$ and M,m,P,p are real constant, such that*

$$m\le f(x)\le M \quad and \quad p\le g(x)\le P,$$

then

$$\left|\left(S^{\alpha,\alpha',\beta,\beta',\gamma}fg\right)(x)-\left(S^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)\left(S^{\alpha,\alpha',\beta,\beta',\gamma}g\right)(x)\right|$$

$$\le\frac{1}{4}(P-p)(M-m) \qquad (\forall x>[0,\infty)) \quad (16)$$

where $\alpha,\alpha',\beta,\beta',\gamma\in\Re$ and $\alpha,\alpha',\beta,\beta',\gamma>0$.

Proof. We define a function as

$$H(\tau,\rho)=(f(\tau)-f(\rho))(g(\tau)-g(\rho))\,; \quad \tau,\rho\in(0,x),\ x>0. \quad (17)$$

Now, on multiplying the above equation (17) by

$$\frac{1}{(\Gamma(\gamma))^2}(\tau\rho)^{-\alpha'}(x-\tau)^{\gamma-1}(x-\rho)^{\gamma-1}F_3\left(\alpha,\alpha',\beta,\beta',\gamma,1-\frac{\tau}{x},1-\frac{x}{\tau}\right)$$

$$\times F_3\left(\alpha,\alpha',\beta,\beta',\gamma,1-\frac{\rho}{x},1-\frac{x}{\rho}\right),$$

and integrating with respect to τ and ρ respectively from 0 to x, and further multiplying by

$$\left[\frac{\Gamma(1+\gamma-\alpha-\alpha')\Gamma(1+\gamma-\alpha'-\beta)\Gamma(1+\beta')}{\Gamma(1+\gamma-\alpha-\alpha'-\beta)\Gamma(1+\beta'-\alpha')}x^{\alpha'-\gamma}\right]^2,$$

and then using result (6), (4) and (10), we obtain

$$\left[\frac{\Gamma(1+\gamma-\alpha-\alpha')\Gamma(1+\gamma-\alpha'-\beta)\Gamma(1+\beta')}{\Gamma(1+\gamma-\alpha-\alpha'-\beta)\Gamma(1+\beta'-\alpha')}x^{\alpha'-\gamma}\right]^2\frac{1}{(\Gamma(\gamma))^2}$$

$$\times\int_0^x\int_0^x(\tau\rho)^{-\alpha'}(x-\tau)^{\gamma-1}(x-\rho)^{\gamma-1}F_3\left(\alpha,\alpha',\beta,\beta',\gamma,1-\frac{\tau}{x},1-\frac{x}{\tau}\right)$$

$$\times F_3\left(\alpha,\alpha',\beta,\beta',\gamma,1-\frac{\rho}{x},1-\frac{x}{\rho}\right)H(\tau,\rho)d\tau d\rho$$

$$=2\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma}fg\right)(x)-2\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma}g\right)(x). \quad (18)$$

Applying Cauchy Schwarz inequality, we can write

$$\begin{aligned}
&\left(\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} fg\right)(x) - \left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x)\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} g\right)(x)\right)^2 \\
&\le \left(\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} f^2\right)(x) - \left(\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x)\right)^2\right) \\
&\times \left(\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} g^2\right)(x) - \left(\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} g\right)(x)\right)^2\right). \qquad (19)
\end{aligned}$$

Since $(M - f(x))(f(x) - m) \ge 0$ and $(P - f(x))(f(x) - p) \ge 0$, therefore we have

$$\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma}(M - f)(f - m)\right)(x) \ge 0, \quad \left(S_t^{\alpha,\alpha',\beta,\beta',\gamma}(P - f)(f - p)\right)(x) \ge 0. \qquad (20)$$

Thus, using Theorem 2, we get

$$\begin{aligned}
&\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} f^2\right)(x) - \left(\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x)\right)^2 \\
&= \left(M - \left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x)\right)\left(\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) - m\right), \qquad (21) \\
&\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} g^2\right)(x) - \left(\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} g\right)(x)\right)^2 \\
&= \left(P - \left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} g\right)(x)\right)\left(\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} g\right)(x) - p\right), \qquad (22)
\end{aligned}$$

By using relations (21) and (22), inequality (19) reduces to the subsequent form

$$\begin{aligned}
&\left(\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} fg\right)(x) - \left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x)\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} g\right)(x)\right)^2 \\
&\le \left(M - \left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} h\right)(x)\right)\left(\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} h\right)(x) - m\right) \\
&\times \left(P - \left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} g\right)(x)\right)\left(\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} g\right)(x) - p\right). \qquad (23)
\end{aligned}$$

Further, on using the elementary inequality $(a+b)^2 \ge 4ab,\ \ a, b \in \mathbb{R}$, we can easily get

$$4\left(M - \left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} h\right)(x)\right)\left(\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} h\right)(x) - m\right) \le (M - m)^2,$$

and similarly,

$$4\left(P - \left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} h\right)(x)\right)\left(\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} h\right)(x) - p\right) \le (P - p)^2.$$

Applying these inequalities in (23), we obtain the desired result. □

Theorem 4. *Let f and g be two synchronous functions on $[0,\infty)$ and let $v, w : [0,\infty) \to [0,\infty)$. Then for all $t > 0$,*

$$\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} vfg\right)(x)\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} w\right)(x)$$

$$+\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} wfg\right)(x)\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} v\right)(x)$$

$$\geq \left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} wg\right)(x)\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} vf\right)(x)$$

$$+\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} wf\right)(x)\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} vg\right)(x). \tag{24}$$

Proof. Using Definition 2, we have

$$f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau).$$

On multiplying both sides of above relation by

$$\frac{1}{\Gamma(\gamma)}(\tau)^{-\alpha'}(x-\tau)^{\gamma-1}F_3\left(\alpha,\alpha',\beta,\beta',\gamma,1-\frac{\tau}{x},1-\frac{x}{\tau}\right)v(\tau),$$

and integrating with respect to τ from 0 to x, and then further multiply by

$$\frac{\Gamma(1+\gamma-\alpha-\alpha')\Gamma(1+\gamma-\alpha'-\beta)\Gamma(1+\beta')}{\Gamma(1+\gamma-\alpha-\alpha'-\beta)\Gamma(1+\beta'-\alpha')}x^{\alpha'-\gamma},$$

we get

$$\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} vfg\right)(x) + f(\rho)g(\rho)$$

$$\geq g(\rho)\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} vf\right)(x) + f(\rho)\left(S_t^{\alpha,\alpha',\beta,\beta',\gamma} vg\right)(x). \tag{25}$$

Again, multiply both sides of the above inequality by term below

$$\frac{1}{\Gamma(\gamma)}(\rho)^{-\alpha'}(x-\rho)^{\gamma-1}F_3\left(\alpha,\alpha',\beta,\beta',\gamma,1-\frac{\rho}{x},1-\frac{x}{\rho}\right)w(\rho),$$

and integrating with respect to ρ from 0 to x, and then multiplying by

$$\frac{\Gamma(1+\gamma-\alpha-\alpha')\Gamma(1+\gamma-\alpha'-\beta)\Gamma(1+\beta')}{\Gamma(1+\gamma-\alpha-\alpha'-\beta)\Gamma(1+\beta'-\alpha')}x^{\alpha'-\gamma},$$

we arrive at the desired result. $\square$

4 Concluding remarks

We now briefly consider some consequences of the derived results in the previous section. Operator (4) would reduce immediately to the extensively investigated Saigo, Erdélyi-Kober, and Riemann-Liouville type fractional integral operators, respectively, given by the following relationships (see [11, 15]).

$$\left(I_t^{\alpha,0,\beta,\beta',\gamma}f\right)(x)=\left(I_t^{\gamma,\alpha-\gamma,-\beta}f\right)(x)=\frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x(x-t)_2^{\gamma-1}F_1(\alpha,\beta;\gamma;1-\frac{t}{x})f(t)dt, \tag{26}$$

$$(\gamma>0,\alpha,\beta\in\Re).$$

$$\left(I_t^{\gamma,0,\beta,\beta',\gamma}f\right)(x)=\left(I_t^{\gamma,-\beta}f\right)(x)=\frac{x^{-\gamma+\beta}}{\Gamma(\gamma)}\int_0^x(x-t)^{\gamma-1}t^{-\beta}f(t)dt, \tag{27}$$

$$(\gamma>0,\beta\in\Re).$$

$$\left(I_t^{0,0,\beta,\beta',\gamma}f\right)(x)=\left(I_t^{\gamma}f\right)(x)=\frac{1}{\Gamma(\gamma)}\int_0^x(x-t)^{\gamma-1}f(t)dt,\qquad(\gamma>0). \tag{28}$$

We acquire the special cases of the operator $S_t^{\alpha,\alpha',\beta,\beta',\gamma}()$ as follows by setting $\alpha'=0$; $\alpha'=0$ and $\alpha=\gamma$; $\alpha=\alpha'=0$ in (6), then instantly Definition 4 would reduce to operators involving Saigo, Erdélyi-Kober, and Riemann-Liouville type fractional integral operators, respectively, as follows:

$$\left(S_t^{\gamma,\alpha-\gamma,-\beta}f\right)(x)=\frac{\Gamma(1-(\alpha-\gamma))\Gamma(1+\gamma-\beta)}{\Gamma(1+\gamma-\beta-\alpha)}x^{\alpha-\gamma}\left(I_t^{\gamma,\alpha-\gamma,-\beta}f\right)(x), \tag{29}$$

$$\left(S_t^{\gamma,-\beta}f\right)(x)=\frac{\Gamma(1+\gamma-\beta)}{\Gamma(1-\beta)}\left(I_t^{\gamma,-\beta}f\right)(x), \tag{30}$$

$$\left(S_t^{\gamma}f\right)(x)=\Gamma(1+\gamma)\left(I_t^{\gamma}f\right)(x), \tag{31}$$

where the operators $\left(I_t^{\gamma,\alpha-\gamma,-\beta}\right),\left(I_t^{\gamma,-\beta}\right),\left(I_t^{\gamma}\right)$ are given by (26), (27), (28) respectively.
For instance, if we use $\alpha'=0$ and make use of functional relation (26), Lemma 2 and Theorem 2 provide the already known results due to Kalla and Rao [10].

We conclude the present investigations with the remark that, the results obtained are very suitable in stemming various fractional integral inequalities involving such relatively much familiar fractional integral operators.

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