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RATIONAL APPROXIMATION OF THE HEAD EQUATION IN UNBOUNDED DOMAIN

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Abstract

In this paper, a Galerkin-type approximation using induced rational functions of Chebyshev polynomials is proposed and analyzed in order to determine the solution of the heat equation over a whole $\mathbb{R}$.
We have shown by numerical tests that these new rational functions are very well adapted to approximations of PDEs in unbounded domain.

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1 Introduction

How to accurately and efficiently solve partial differential equations on unbounded domains is a very important topic because many problems in science and engineering are defined on unbounded domains. Yet, it is a very difficult subject since the domain's unboundedness introduces considerable theoretical elements that are not present in bounded domains.

The spectral approximations of Legendre or Chebyshev for partial differential equations on bounded domains have achieved great success and popularity in recent years. Nevertheless, the spectral approximations for PDEs on unbounded domains received only small attention.

Recently, a number of different spectral methods have been proposed for problems in unbounded domains. The first approach consists of using the spectral approximations associated with existing orthogonal systems like the Laguerre and Hermite polynomials. A second approach is to use an appropriate application to reformulate the original problems in unbounded domains into singular problems in the domains bounded, and then use an appropriate Jacobi approximation to deal with singular problems.

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Another class of spectral methods is based on rational approximations, like Christov in [10], Wang and Guo in [5] and Boyd in [9, 8], who proposed some spectral methods in infinite intervals, using a certain system of orthogonal rational functions.
The content of this article is organised as follows:
The second section is devoted to some reminders of the theoretical tools necessary for the understanding and demonstration of the different results established for the heat equation. We will recall the new basis of rational functions induced by Chebyshev's polynomials by *Ben-Yu Guo, Wang Zhong-Qing* [5], their various properties as well as some results of inclusions and inverse inequalities useful for error estimation results and convergence of the approximate solution of the problem that will be studied later.
In the third section, we will develop some results, for the approximation of the solution of the heat equation on $\mathbb{R}$, using the family of rational functions constructed in the previous section. We will show among other things the convergence and stability of the solution and write in the end the approximate problem in the form of a linear system that is easy to solve.
In order to show the effectiveness of the spectral approximation proposed, an implementation of the method is given using an explicit finite difference scheme in the fourth section. The numerical tests will be illustrated by representative curves.
It is to be noted that the techniques presented in this work will be useful for studying other linear or non-linear partial differential equations in fluid dynamics, quantum mechanics and many other fields.

2 Necessary tools for the modified rational Chebyshev approximation

2.1 Building a new family of rational functions from Tchebyshev polynomials

Let $\Omega = \{x \ -\infty < x < +\infty\}$ and $w(x)$ be the weight function. We define, for every positive m, the weighted Sobolev space by

$$H_w^m(\Omega) = \{v \,|\, \partial_x^k \in L_w^2(\Omega), \quad 0 \leq k \leq m\}.$$

The scalar product, the semi standard and the norm are respectively denoted by

$$\|v\|_{m,w} = (v,v)_{m,w}^{\frac{1}{2}}, \quad (u,v)_w = \sum_{k=0}^{m} (\partial_x^k u, \partial_x^k v)_w, \quad |v|_{m,w} = \|\partial_x^m v\|_w.$$

For all $r > 0$, we denote by $H^r(\Omega)$ the interpolation space.

Let $T_n(y)$ be the Chebyshev polynomial of the first kind of degree n. We consider the basis of modified Chebyshev rational functions built in [5] and defined by

$$R_n(x) = \frac{1}{\sqrt{x^2+1}} T_n\left(\frac{x}{\sqrt{x^2+1}}\right), \quad n = 0, 1, \cdots$$

For $y = \dfrac{x}{\sqrt{x^2+1}}$, it's clear that

$$\frac{\partial y}{\partial x} = (x^2+1)^{-\frac{3}{2}}, \quad \frac{\partial x}{\partial y} = (1-y^2)^{-\frac{3}{2}}.$$

Properties

Property 1 From the properties of Chebyshev polynomials and the variable change above, we show that $R_n(x)$ is the proper function of the Sturm-Liouville problem.

$$\sqrt{1+x^2}\partial_x[(x^2+1)\partial_x(\sqrt{x^2+1}R_n(x))]+n^2R_n(x) = 0, \quad n = 0, 1, 2, \cdots \quad (1)$$

Property 2 Modified Chebyshev's Rational Functions verify the Relationship

$$R_{n+1}(x) = \frac{2x}{\sqrt{x^2+1}} R_n(x) - R_{n-1}(x), \quad n \geq 1 \quad (2)$$

Property 3 We have

$$\begin{aligned} R_n(x) &= \frac{1}{2(n+1)}((x^2+1)^{\frac{1}{2}} x R_{n+1}(x) + (x^2+1)^{\frac{3}{2}} \partial_x R_{n+1}(x)) \\ &\quad - \frac{1}{2(n-1)}((x^2+1)^{\frac{1}{2}} x R_{n-1}(x) + (x^2+1)^{\frac{3}{2}} \partial_x R_{n-1}(x)) \quad (3) \end{aligned}$$

This property results from the recurrence relation verified by the derivative of the following Chebyshev polynomials

$$\frac{T'_{n+1}(y)}{n+1} - \frac{T'_{n-1}(y)}{n-1} = 2T_n(y); \quad n \geq 2.$$

Property 4 From the orthogonality of Chebyshev polynomials, we have

$$\int_\Omega R_n(x) R_m(x) dx = \frac{\pi}{2} C_n \delta_{n,m} \quad (4)$$

where $\delta_{n,m}$ is the function of Kronecker and $C_0 = 2$. $C_n = 1$, $\forall n \geq 1$. So, for all $v \in L^2(\Omega)$, we can write

$$v(x) = \sum_{n=0}^{\infty} v_n \ R_n(x) \qquad \text{où} \quad v_n = \frac{2}{\pi c_n} \int_\Omega v(x) R_n(x) dx.$$

Property 5 Put

$$w_1(x) = x^2 + 1.$$

By virtue of (1) and (4), the set $\{\partial_x((x^2+1)^{\frac{1}{2}} R_m(x))\}$ is an orthogonal system in space $L^2_{w_1}(\Omega)$. In addition, we have

$$\int_\Omega \partial_x((x^2+1)^{\frac{1}{2}} R_n(x)) \partial_x((x^2+1)^{\frac{1}{2}} R_m(x)) w_1(x) dx = \frac{\pi}{2} n^2 C_n \delta_{n,m}. \quad (5)$$

2.2 Some inverse inequalities

Let N be a positive integer and consider

$$\mathcal{R}_N = [\{R_0, R_1, \dots, R_N\}].$$

To note by C a positive constant independent of all functions and N. In this section, we will recall some inequalities [5], without demonstration, which will be used later for the heat equation.

Theorem 2.1. *For all $\phi \in \mathcal{R}_N$ and $1 \leq p \leq q \leq \infty$, we have*

$$\|\phi\|_{L^q} \leq C N^{\lambda(p,q)(\frac{1}{p}-\frac{1}{q})} \|\phi\|_{L^p}.$$

where $\lambda(p,q) = p-1$ for $p \geq 2$ and $\lambda(p,q) = 1$ otherwise.

Theorem 2.2. *For all $\phi \in \mathcal{R}_N$ and $r \geq 0$*

$$|\phi|_r \leq C N^r \|(x^2+1)^{-\frac{r}{2}} \phi\|.$$

2.3 Modified rational Chebyshev approximation

In this section, we consider some orthogonal projections. We start with the orthogonal projection P_N defined from $L^2(\Omega)$ in $\mathcal{R}_N$ by

$$(P_N v - v, \phi) = 0\,; \quad \forall \phi \in \mathcal{R}_N \quad \text{and} \quad v \in L^2(\Omega).$$

To derive approximation results, we introduce the operator A defined by

$$Av(x) = -(x^2+1)^{\frac{1}{2}} \partial_x \Big[(x^2+1)\partial_x((x^2+1)^{\frac{1}{2}} v(x))\Big].$$

for any integer $r \geq 0$

$$H^r_{A_q}(\Omega) = \{v \,|\, v \text{ is measurable on } \Omega \text{ and } \|v\|_{r,A_q} < \infty\}.$$

where

$$\|v\|_{r,A_0} = \|A^{\frac{1}{2}} v\| \quad \text{and for all} \quad q \geq 1$$

$$\|v\|_{r,A_q} = \|(x^2+1)\partial_x (x^2+1)^{\frac{1}{2}} v\|_{r-1,A_{q-1}}.$$

For all real $r > 0$, we define the space $H^r_{A_q}(\Omega)$ and its norm $\|v\|_{r,A_q}$ by interpolation.

Theorem 2.3. *For all $v \in H^r_{A_0}(\Omega)$ and $r \geq 0$, we have*

$$\|P_N v - v\| \leq C N^{-r} \|v\|_{r,A_0}.$$

We now consider P_N^m the orthogonal projection defined on $H^m(\Omega)$ in $\mathcal{R}_N$ by

$$(P_N^m v - v, \phi) = 0; \quad \forall \phi \in \mathcal{R}_N \quad \text{and} \quad v \in H^m(\Omega).$$

Theorem 2.4. *For all $v \in H^r_{A_m}(\Omega)$ and $0 \leq m \leq r$*

$$\|P_N^m v - v\|_m \leq C N^{m-r} \|v\|_{r,A_m}.$$

Theorem 2.5. *For all $v \in H^m_{A_m}(\Omega)$ and $\mu < m - \frac{1}{2}$*

$$\|P_N^m v\|_{\mu,\infty} \leq C \|v\|_{m,A_m}.$$

3 Rational approximation for the heat equation

Now, we shall consider as an example the heat equation over the whole real line, given by

$$\begin{cases} \partial_t U(t,x) - \partial_x^2 U(t,x) = f(t,x), & x \in \Omega, \qquad 0 \le t \le T \\ \lim\limits_{|x|\to+\infty} U(t,x) = \lim\limits_{|x|\to+\infty} \partial_x U(t,x) = 0, & 0 \le t \le T \\ U(0,x) = U_0(x), & x \in \Omega \end{cases} \tag{6}$$

Multiplying the equation of problem (6) by $v \in \mathscr{D}(\Omega)$ and integrating by parts over Ω, we obtain

$$\int_\Omega \partial_t U(t,x)v(x)dx - \int_\Omega \partial_x^2 U(t,x)v(x)dx = \int_\Omega f(t,x)v(x)dx, \qquad \forall v \in H_0^1(\Omega).$$

Hence, the weak formulation of (6) is
Find $U(t,x) \in H_0^1(\Omega)$ such as

$$\begin{cases} (\partial_t U(t), v) + (\partial_x U(t), \partial_x v) = (f(t), v), \qquad \forall v \in H_0^1(\Omega), \qquad 0 \le t \le T, \\ U(0) = U_0 \end{cases} \tag{7}$$

The rational spectral scheme of Chebyshev modified for (7) is
Find $u_N(t,x) \in \mathcal{R}_N$ such as

$$\begin{cases} (\partial_t u_N(t), \phi) + (\partial_x u_N(t), \partial_x \phi) = (f(t), \phi), \qquad \forall \phi \in \mathcal{R}_N, \qquad 0 \le t \le T, \\ u_N(0) = P_N U_o \end{cases} \tag{8}$$

3.1 Error estimation and convergence

After having established the variational formulation of the problem, let us now examine the existence and uniqueness of the solution. Let us first note that the convergence of the approximation of the solution is assured because a method of modified rational approximation Chebyshev has been used. That is to say, we have used an approximate problem of finite dimension, which will have a unique solution. The stability for this remains to be shown. We establish the following result:

Theorem 3.1. *Let U and u_N be problem solutions (6) and (8) respectively. Set*

$$U_N = P_N^2 U \qquad \text{and} \qquad \tilde{U}_N = u_N - U_N.$$

Then, there is a positive constant C_1 which depends only on $L^2(0,T,W^{1,\infty}(\Omega) \cap H^1(0,T,H_{A_2}^2(\Omega)))$, such as

$$\|\tilde{U}_N(t)\|^2 \le C \int_0^t \|\tilde{U}_N(s)\|^2 ds + C_1(U) N^{4-2r}.$$

Proof – The equation (6) is still written

$$\begin{cases} (\partial_t[U(t)+U_N(t)-U_N(t)],\phi)+(\partial_x U(t)+\partial_x U_N(t)-\partial_x U_N(t),\partial_x\phi) \\ =(f(t),\phi),\ \ \forall\phi\in\mathcal{R}_N,\ \ 0\le t\le T \\ U_N(0)=P_N^2U_o \end{cases}$$

or

$$\begin{cases} (\partial_t U_N(t),\phi)+(\partial_x U_N(t),\partial_x\phi)+\sum_{j=1}^{2}G_j(t,\phi)=(f(t),\phi) \\ \forall\phi\in\mathcal{R}_N, \qquad 0\le t\le T \\ U_N(0)=P_N^2U_o \end{cases} \tag{9}$$

such as

$$G_1(t,\phi)=(\partial_t U(t)-\partial_t U_N(t),\phi).$$

$$G_2(t,\phi)=(\partial_x U(t)-\partial_x U_N(t),\partial_x\phi).$$

Subtract (9) from (8) gives

$$(\partial_t(u_N(t)-U_N(t)),\phi)+(\partial_x u_N(t)-\partial_x U_N(t),\partial_x\phi)-\sum_{j=1}^{2}G_j(t,\phi)=0$$

In other words, for $U_N=P_N^2U$ and $\tilde{U}_N=u_N-U_N$ gives

$$\begin{cases} (\partial_t\tilde{U}_N(t),\phi)+(\partial_x\tilde{U}_N(t),\partial_x\phi)=\sum_{j=1}^{2}G_j(t,\phi) \\ \forall\phi\in\mathcal{R}_N, \qquad 0\le t\le T \\ \tilde{U}_N(0)=P_NU_0-P_N^2U_o \end{cases} \tag{10}$$

If we take $\phi=\tilde{U}_N(t)$, we get

$$(\partial_t\tilde{U}_N(t),\tilde{U}_N(t))+(\partial_x\tilde{U}_N(t),\partial_x\tilde{U}_N(t))=\sum_{j=1}^{2}G_j(t,\tilde{U}_N(t)).$$

where

$$(\partial_x\tilde{U}_N(t),\partial_x\tilde{U}_N(t))=\int_\Omega(\partial_x\tilde{U}_N(t,x))^2dx\ge 0.$$

So

$$(\partial_t\tilde{U}_N(t),\tilde{U}_N(t))\le\sum_{j=1}^{2}|G_j(t,\tilde{U}_N(t))|$$

i.e.

$$\frac{d}{dt}\|\tilde{U}_N(t)\|^2\le 2\sum_{j=1}^{2}|G_j(t,\tilde{U}_N(t))|. \tag{11}$$

Now let's move on to estimating terms on the right side. According to Theorem2.4, we have

$$\begin{aligned}|G_1(t,\tilde{U}_N(t))| &= |\int_\Omega [\partial_t U(t,x) - \partial_t U_N(t,x)]\tilde{U}_N(t,x)dx| \\ &\leq \|\tilde{U}_N(t)\|^2 + \|\partial_t U(t) - \partial_t U_N(t)\|^2 \\ &\leq \|\tilde{U}_N(t)\|^2 + CN^{4-2r}\|\partial_t U(t)\|^2_{r,A_2},\end{aligned}$$

and

$$\begin{aligned}|G_2(t,\tilde{U}_N(t))| &= |(\partial_x U(t) - \partial_x U_N(t), \partial_x \tilde{U}_N(t))| \\ &= |\int_\Omega [\partial_x U(t,x) - \partial_x U_N(t,x)]\partial_x \tilde{U}_N(t,x)dx|\end{aligned}$$

By integration by parts, we have

$$\begin{aligned}&\int_\Omega [\partial_x U(t,x) - \partial_x U_N(t,x)]\partial_x \tilde{U}_N(t,x) \\ &\qquad = -\int_\Omega [\partial_x^2 U(t,x) - \partial_x^2 U_N(t,x)]\tilde{U}_N(t,x).\end{aligned}$$

Hence, according to Theorem2.4

$$\begin{aligned}|G_2(t,\tilde{U}_N(t))| &= |\int_\Omega [\partial_x^2 U(t,x) - \partial_x^2 U_N(t,x)]\tilde{U}_N(t,x)dx| \\ &\leq \|\tilde{U}_N(t)\|^2 + \|\partial_x^2 U(t) - \partial_x^2 U_N(t)\|^2 \\ &\leq \|\tilde{U}_N(t)\|^2 + CN^{4-2r}\|U(t)\|^2_{r,A_2},\end{aligned}$$

Furthermore

$$\begin{aligned}\|\tilde{U}_N(0)\|^2 &\leq \|U_0 - P_N U_0\|^2 + \|U_0 - P_N^2 U_0\|^2 \\ &\leq CN^{4-2r}\|U_0\|_{r,A_2^2}.\end{aligned}$$

Substituting in (11), we obtain

$$\frac{d}{dt}\|\tilde{U}_N(t)\|^2 \leq 2\|\tilde{U}_N(t)\|^2 + CN^{4-2r}[\|\partial_t U(t)\|^2_{r,A_2} + \|U(t)\|^2_{r,A_2}].$$

By integrating of 0 to t

$$\begin{aligned}\int_0^t \frac{d}{ds}\|\tilde{U}_N(s)\|^2 ds &= \|\tilde{U}_N(t)\|^2 - \|\tilde{U}_N(0)\|^2 \\ &\leq \|\tilde{U}_N(t)\|^2 - CN^{4-2r}\|U_0\|^2_{r,A_2}\end{aligned}$$

i.e.

$$\begin{aligned}\int_0^t \frac{d}{ds}\|\tilde{U}_N(s)\|^2 ds &= \|\tilde{U}_N(t)\|^2 - \|\tilde{U}_N(0)\|^2 \\ &\leq 2\int_0^t \|\tilde{U}_N(s)\|^2 ds \\ &\quad + CN^{4-2r}\int_0^t [\|\partial_s U(s)\|^2_{r,A_2} + \|U(s)\|^2_{r,A_2}]ds\end{aligned}$$

Which implies

$$\begin{aligned}\|\tilde{U}_N(t)\|^2 &\leq 2\int_0^t \|\tilde{U}_N(s)\|^2 ds + CN^{4-2r} \\ &\quad \times \int_0^t [\|\partial_s U(s)\|^2_{r,A_2} + \|U(s)\|^2_{r,A_2}] ds + \|\tilde{U}_N(0)\|^2 \\ &\leq 2\int_0^t \|\tilde{U}_N(s)\|^2 ds + CN^{4-2r} \\ &\quad \times \Big[\int_0^t [\|\partial_s U(s)\|^2_{r,A_2} + \|U(s)\|^2_{r,A_2}] ds + \|U_0\|^2_{r,A_2}\Big].\end{aligned}$$

Finally

$$\|\tilde{U}_N(t)\|^2 \leq 2\int_0^t \|\tilde{U}_N(s)\|^2 ds + C_1(U)CN^{4-2r}.$$

where $C_1(U)$ is a positive constant that depends only on $L^2(0,T,W^{1,\infty}(\Omega)\cap H^1(0,T,H^2_{A_2}(\Omega)))$. ■

Using Gronwall's lemma and Theorem 2.3, we obtain

$$\begin{aligned}\|\tilde{U}(t)\|^2 &\leq c_1(U)N^{4-2r}e^{2t} \\ &\leq c_2(U)N^{4-2r}, \quad \forall t \in [0,T].\end{aligned}$$

We conclude

Theorem 3.2. *For all* $r \geq 2$, $u \in L^2(0,T,W^{1,\infty}(\Omega) \cap H^1(0,T,H^r_{A_2}(\Omega)))$, *hence for all* $0 \leq t \leq T$

$$\|U - u_N\| \leq c_2(U)N^{2-r}.$$

$c_2(U)$ *is a positive constant dependent only on* $\|U\|_{L^2(0,T,W^{1,\infty}(\Omega)\cap H^1(0,T,H^2_{A_2}(\Omega)))}$.

4 Implementation and numerical result

In order to solve problem (6) using the modified Chebyshev rational function basis introduced in section 2, we begin by replacing in formulation (8) the approximation of the solution

$$u_N(t) = \sum_{j=0}^{N} V_j^N(t)R_j.$$

This gives the problem

Find V_j^N for all $j = 0, \ldots, N$, such as

$$(\partial_t(\sum_{j=0}^{N} V_j^N(t)R_j), \phi) + (\partial_x(\sum_{j=0}^{N} V_j^N(t)R_j), \partial_x\phi) = (f(t), \phi), \qquad \forall \phi \in \mathcal{R}_N$$

In particular for $\phi = R_i$, the equation is written as follows

$$(\partial_t(\sum_{j=0}^{N} V_j^N(t)R_j), R_i) + (\partial_x(\sum_{j=0}^{N} V_j^N(t)R_j), \partial_x R_i) = (f(t), R_i), \qquad \forall i = \overline{0, N}$$

i.e

$$\sum_{j=0}^{N}(R_j, R_i)\partial_t V_j^N(t) + \sum_{j=0}^{N}(\partial_x R_j, \partial_x R_i)V_j^N(t) = (f(t), R_i), \qquad \forall i = \overline{0, N}$$

Using an explicit schema, we get

$$\sum_{j=0}^{N}(R_j, R_i)(\partial_t V_j^N(t))^n + \sum_{j=0}^{N}(\partial_x R_j, \partial_x R_i)(V_j^N(t))^n = (f(t_n), R_i), \qquad \forall i = \overline{0, N},$$

such that

$$(\partial_t V_j^N(t))^n = \frac{V_j^{n+1} - V_j^n}{\Delta t}$$

The schema is then written

$$\sum_{j=0}^{N}(R_j, R_i)\Big(\frac{V_j^{n+1} - V_j^n}{\Delta t}\Big) + \sum_{j=0}^{N}(\partial_x R_j, \partial_x R_i)V_j^n = (f(t_n), R_i), \qquad \forall R_i \in \mathfrak{R}_N.$$

Or

$$\sum_{j=0}^{N}(R_j, R_i)(V_j^{n+1} - V_j^n) + \Delta t\sum_{j=0}^{N}(\partial_x R_j, \partial_x R_i)V_j^n = \Delta t(f(t_n), R_i), \qquad \forall R_i \in \mathfrak{R}_N.$$

As

$$(R_j, R_i) = \frac{\pi}{2}C_i\delta_{ij},$$

By replacing in the equation, we get

$$\frac{\pi}{2}C_iV_j^{n+1} - \frac{\pi}{2}C_iV_j^n + \Delta t\sum_{j=0}^{N}(\partial_x R_j, \partial_x R_i)V_j^n = \Delta t(f(t_n), R_i)$$

i.e.

$$\frac{\pi}{2}C_iV_j^{n+1} = -\Delta t\Big(\sum_{j=0}^{N}(\partial_x R_j, \partial_x R_i) - \frac{\pi}{2\Delta t}C_i\Big)V_j^n + \Delta t(f(t_n), R_i).$$

Either in the following matrix form

$$\frac{\pi}{2}C_iV_j^{n+1} = -\Delta tAV_j^n + \Delta tF.$$

where

$$A_{ij} = \Big(\sum_{j=0}^{N}(\partial_x R_j, \partial_x R_i) - \frac{\pi}{2\Delta t}C_i\Big), \quad \text{and} \quad F_i = (f(t_n), R_i).$$

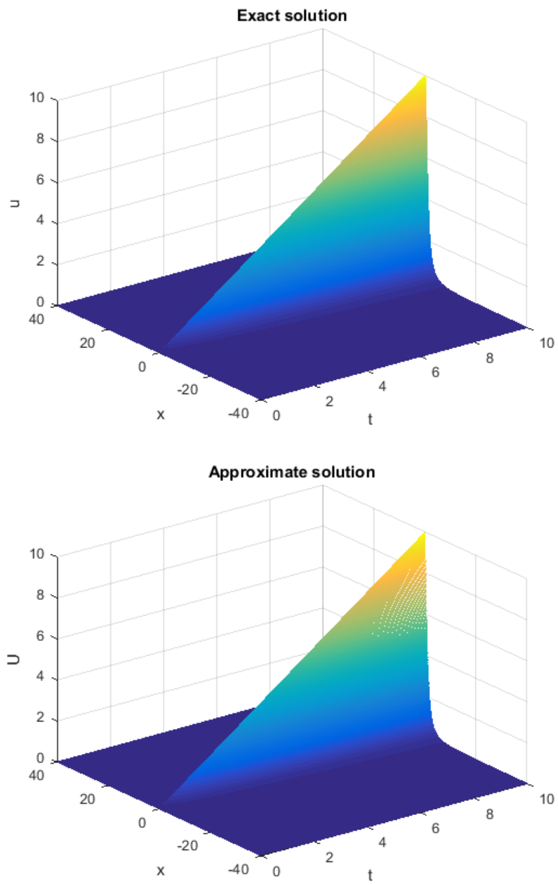

Figure 1: Representation of the solution of the heat equation and its approximation.

Now, for the representation of the numerical solution, consider the problem (6) for

$$f(t,x) = \frac{(x^2+1)^2 - 2t(3x^2-1)}{(x^2+1)^3}.$$

it admits the solution

$$u(t,x) = \frac{t}{x^2+1}.$$

In order to compare the exact solution with the approximation, we take $N = 15$. The representative results in Figure 1 are then obtained.

In Figure 2, we have the representation of the relative error of the solution approximation at time $t = 4$, and in Figure 3, we have the representation of the relative errors for different values of N.

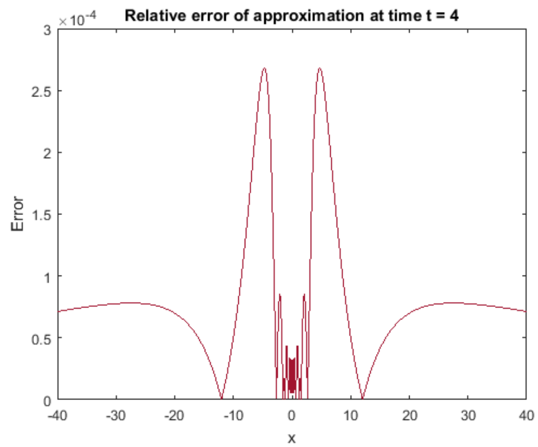

Figure 2: Representation of the relative error of the solution approximation for the heat equation

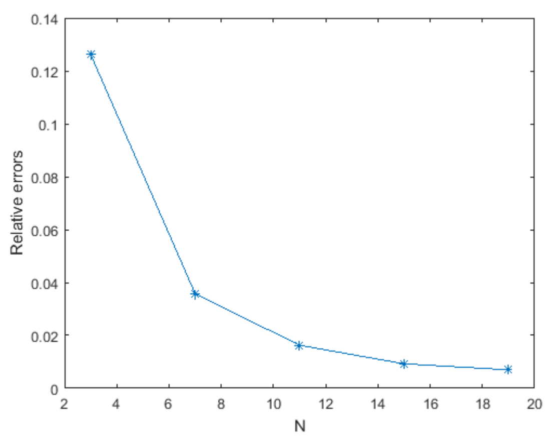

Figure 3: Convergence rates of the rational spectral approximation

Conclusion

The main objective, in this paper, is to obtain an approximation of the solution of the heat equation, in unbounded domain, via a new family of rational functions deduced from Chebyshev polynomials. Convergence stability and error estimation have been established through some numerical analysis results. The numerical tests given at the end corroborate the effectiveness of the method used.

References

[1] Allaire, G., *Analyse numérique et optimisation.* Edition de l'école polytechnique, France, 2012.

[2] Ben, G. Y., *Error estimation of Hermite spectral method for nonlinear partial differential equations.* Math. Comp. **69** (1999), 1067-1078.

[3] Ben, G. Y., *Jacobi approximation in certain Hilbert space and their applications to singular differential equations.* J. Math. Anal. Appl. **243** (2000), 373-408.

[4] Ben, G. Y. and Shen, J., *On spectral approximations using modified Legendre rational function: application to Korteweg-de Vries equation on the half line.* Indiana J. of Math. **50** (2001), 181-204.

[5] Ben, G. Y. and Wang, Z. Q., *Modified Chebyshev rational spectral method for the whole line*, Wilmington, NC, USA, (2003), 365-374.

[6] Ben, G. Y. and Xu, C. L., *Hermite pseudospectral method for nonlinear partial differential equations.* RARIO M2AN **24** (2000), 859-872.

[7] Bernardi, Ch. and Maday, Y., *Approximations spectrales de problèmes aux limites elliptiques*, Mathématiques & Applications (Berlin) [Mathematics & Applications], Springer-Verlag, Paris, 1992.

[8] Boyd, J. P., *Orthogonal rational functions on a semi-infinite interval.* J. Comp. Phys. **70** (1987), 63-88.

[9] Boyd, J. P. *Spectral method using rational basis functions on an infinite interval.* J. Comp. Phys. **69**, (1987), 112-142.

[10] Christov, C. I. *A complete orthogonal system of functions in* $L_2(-\infty, +\infty)$ *space.* SIAM J. Appl. Math. **42** (1982), 1337-1344.

[11] Funaro, D. and Kavian, O., *Approximation of some diffusion evolution equations in unbounded domains by Hermite functions.* Math. Comp. **57** (1991), 597-619.

[12] Guo, Ben, Y., *Spectral Methods and Their Application*, Spectral theory (Mathematics), World Scientific Publishing Co. Pte. Ltd., Singapore, 1998.

[13] Hardy, G. H. Littlewood, J. E. and Polay, G., *Inequalities*, Cambridge Univ. Press: Cambridge, UK, 1952.

[14] Wang, Z. Q. and Ben, G. Y., *A rational approximation and its applications to nonlinear partial differential equations on the whole line.* J. Math. Anal. Appl. **274** (2002), 374-403.