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ON COMPACTNESS VIA bI-OPEN SETS IN IDEAL BITOPOLOGICAL SPACES

Santanu ACHARJEE[*1], **Ivan L. REILLY** [2] **and Diganta Jyoti SARMA**[3]

Abstract

Sarma introduced the concept of (i, j)-bI-compact spaces with respect to an ideal in a bitopological space. In this paper, our aim is to present several characterizations of these spaces. Also, the class of (i, j)-bI-countably compact spaces is introduced with some of its properties.

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1 Introduction

The concept of bitopological space was introduced by Kelly [10] in 1963. It is universally accepted that compactness is one of the most important notions of topology and it has a very significant role in the theory of topological spaces. The concept of ideal was introduced by Kuratowski [11] and further it was investigated by Vaidyanathswamy [16], Jankovic and Hamlett [9] and many others. With the concept of ideal, Newcomb [12] defined the notions of compactness and countable compactness in topological spaces. These notions have been investigated by Hamlett and Jankovic [7] and Hamlett et al. [8].

According to Swart [15], a cover $\mathcal{U}$ of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-open if $\mathcal{U} \subseteq \tau_1 \cup \tau_2$. If in addition, $\mathcal{U}$ contains at least one non-empty member of τ_1 and at least one non-empty member of τ_2; then it is called pairwise open [6]. If every pairwise open cover of (X, τ_1, τ_2) has a finite subcover then the space is called pairwise compact [6]. $A \subseteq X$ of (X, τ_1, τ_2) is said to be semi-open [5] if

[1*] *Corresponding author,* Economics and Computational Rationality Group, Department of Mathematics *Debraj Roy College*, Assam, e-mail: sacharjee326@gmail.com, santanuacharjee@rediffmail.com

[2] Department of Mathematics, *University of Auckland*, New Zealand, e-mail: i.reilly@auckland.ac.nz

[3] Department of Mathematics, *Central Institute of Technology*, Assam, e-mail: dj.sarma@cit.ac.in, djs_math@rediffmail.com

it is open in the upper bound topology of τ_1 and τ_2. A set $A \subseteq X$ of (X, τ_1, τ_2) is said to be semi-compact [5] if it is compact in the upper bound topology of τ_1 and τ_2; in other words A is semi-compact if and only if, given any covering of A by semi-open subsets of X, there exists a finite subcovering.

According to Swart [15], a cover $\mathcal{U}$ of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-open if $\mathcal{U} \subseteq \tau_1 \cup \tau_2$. If in addition, $\mathcal{U}$ contains at least one non-empty member of τ_1 and at least one non-empty member of τ_2; then it is called pairwise open [6]. If every pairwise open cover of (X, τ_1, τ_2) has a finite subcover then the space is called pairwise compact [6]. $A \subseteq X$ of (X, τ_1, τ_2) is said to be semi-open [5] if it is open in the upper bound topology of τ_1 and τ_2. A set $A \subseteq X$ of (X, τ_1, τ_2) is said to be semi-compact [5] if it is compact in the upper bound topology of τ_1 and τ_2; in other words A is semi-compact if and only if, given any covering of A by semi-open subsets of X, there exists a finite subcovering.

Reilly [13] defined (X, τ_1, τ_2) to be pairwise Lindelöf (respectively pairwise compact [6]) if each pairwise open cover of X has a countable (respectively finite) subcover. Cooke and Reilly [4] investigated the relationship between semi-compactness and pairwise compactness in a bitopological space.

Andrijevic [3] introduced the notion of b-open set in a topological space and this notion has been extended to bitopological space by Al-Hawary and Al-Omari [2]. Sarma [14] defined the notions of (i, j)-bI-open set, (i, j)-bI-continuous function and obtained several characterizations.

By (i, j), bitopologists often mean the pair of topologies (τ_i, τ_j) on various characterizations of a bitopological space (X, τ_1, τ_2), where $i, j \in \{1, 2\}, i \neq j$.

In this paper, we investigate various properties of (τ_1, τ_2)-bI-compactness in the bitopological category. The notion of (τ_1, τ_2)-bI-countably compactness is introduced and some properties are provided. The need to introduce these new notions can be understood by the fact that various generalized forms of compactness in advanced versions of general topology have been playing crucial roles in research related to theoretical physics, molecular topology, computer science, economics, biology, neuroscience, etc. Even after more than fifty years from its introduction by Kelly [10], it can be easily found that bitopological results have not been used widely by researchers for the betterment of human race. Thus, with the views of application oriented theoretical developments of bitopological space, we introduce (τ_1, τ_2)-bI-compactness and (τ_1, τ_2)-bI-countably compactness.

2 Preliminaries

In this paper, (X, τ_1, τ_2) denotes a bitopological space and (X, τ_1, τ_2, I) denotes an ideal bitopological space, on which no separation axioms are assumed.

According to Kuratowski [11], a collection $I \subseteq P(X)$, where $P(X)$ is the power set of a non-empty set X in a topological space (X, τ), is an ideal of X if it satisfies the following conditions:

(i) if $Q \in I$ and $R \subseteq Q$, then $R \in I$

(ii) if $Q \in I$ and $R \in I$, then $Q \cup R \in I$.

For any ideal topological space (X, τ, I), the operator $(.)^* : P(X) \to P(X)$ is the local function [8] of a subset Q of X with respect to the topology τ and ideal I; and it is defined as $Q^*(\tau, I) = \{a \in X : R \cap Q \notin I \text{ for every } R \in \tau(a)\}$, where $\tau(a) = \{U \in \tau : a \in U\}$. Simply, we write Q^* instead of $Q^*(\tau, I)$ if there is no chance of confusion. Kuratowski closure operator for a topology $\tau^*(I) = \{U \subseteq X : cl^*(X - U) = X - U\}$ [9] on X is defined by $cl^*(Q) = Q \cup Q^*$. Here, $\tau^*(I)$ is finer than τ.

Definition 2.1. *(2) A subset P of a bitopological space (X, τ_1, τ_2) is said to be (i, j)-b-open if $P \subseteq i - int(j - cl(P)) \cup j - cl(i - int(P))$, where $i, j \in \{1, 2\}, i \neq j$.*

The complement of (i, j)-b-open set is called (i, j)-b-closed set.

Definition 2.2. *(14) A subset P of an ideal bitopological space (X, τ_1, τ_2, I) is said to be (i, j)-bI-open if $P \subseteq i - int(j - cl^*(P)) \cup j - cl^*(i - int(P))$, where $i, j \in \{1, 2\}, i \neq j$.*

The complement of (i, j)-bI-open set is called (i, j)-bI-closed set.

Remark 2.1. *(14) Every (i, j)-bI-open set is (i, j)-b-open.*

Definition 2.3. *(14) A function $f : (X, \tau_1, \tau_2, I) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i, j)-bI-continuous if the inverse image of every σ_i-open set in Y is (i, j)-bI-open in X, where $i, j = 1, 2$ and $i \neq j$.*

Definition 2.4. *(14)A function $f : (X, \tau_1, \tau_2, I) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i, j)-bI-irresolute if the inverse image of every (i, j)-b-open set in Y is (i, j)-bI-open in X, where $i, j = 1, 2$ and $i \neq j$.*

Lemma 2.1. *(14) Let (X, τ_1, τ_2, I) be an ideal bitopological space and P, Q are subsets of X such that $Q \subset P$. Then, $Q^*(\tau_i|_P, I|_P) = Q^*(\tau_i, I) \cap P$, for $i \in \{1, 2\}$.*

Here, $\tau_i|_P$ is the relative topology on P and $I|_P = \{P \cap M : M \in I\}$ is an ideal on P.

Lemma 2.2. *(12) For any function $f : (X, \tau, I) \to (Y, \sigma)$, $f(I)$ is an ideal on Y.*

Lemma 2.3. *(12) If a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, I)$ is an injection, then $f^{-1}(I) = \{f^{-1}(P) : P \in I\}$ is an ideal on X.*

3 On (τ_1, τ_2)-bI-compactness

In this section, we introduce some new notions and study some properties.

Definition 3.1. *(14) An ideal bitopological space (X, τ_1, τ_2, I) is said to be (i, j)-I-compact, if for every cover $\{P_\mu : \mu \in \Lambda\}$ by τ_i-open sets of X, there exists a finite subset Λ_0 of Λ such that $X \setminus \cup\{P_\mu : \mu \in \Lambda_0\} \in I$, where $i, j = \{1, 2\}$ and $i \neq j$.*

In Definition 3.1 of [1], Acharjee and Tripathy have introduced some concepts of Lindelöfness in an ideal bitopological space.

Definition 3.2. *(14) An ideal bitopological space (X, τ_1, τ_2, I) is said to be (i, j)-bI-compact, if for every cover $\{P_\mu : \mu \in \Lambda\}$ by (i, j)-bI-open sets of X, there exists a finite subset Λ_0 of Λ such that $X \setminus \cup\{P_\mu : \mu \in \Lambda_0\} \in I$, where $i, j = \{1, 2\}$ and $i \neq j$.*

An example of a (τ_1, τ_2)-bI-compact space is given below.

Example 1. Let $X = \{p, q, r\}$, $\tau_1 = \{\emptyset, \{p\}, X\}$, $\tau_2 = \{\emptyset, \{q\}, X\}$ and $I = \{\emptyset, \{p\}\}$. Then, the set of (τ_1, τ_2)-bI-open sets of X is $\{\emptyset, \{p\}, \{q\}, \{p, q\}, \{q, r\}, X\}$ and (X, τ_1, τ_2, I) is (τ_1, τ_2)-bI-compact.

The proof of the following result is straight forward.

Theorem 3.1. *Let I and J be two ideals in a bitopological space (X, τ_1, τ_2) such that $J \subseteq I$. If the space (X, τ_1, τ_2, I) is (τ_1, τ_2)-bI-compact, then (X, τ_1, τ_2, I) is (τ_1, τ_2)-bJ-compact.*

Theorem 3.2. *For a space (X, τ_1, τ_2, I), the following are equivalent:*

(i) (X, τ_1, τ_2, I) is (τ_1, τ_2)-bI-compact,

(ii) $(X, \tau_1^*, \tau_2^*, I)$ is (τ_1^*, τ_2^*)-bI-compact.

Proof. (i) $\Rightarrow$ (ii) Suppose $\{P_\mu : \mu \in \Lambda\}$ is a cover of X by (τ_1^*, τ_2^*)-bI-open sets of X such that $P_\mu = Q_\mu \setminus J_\mu$, where Q_μ is (τ_1, τ_2)-bI-open set and $J_\mu \in I$. Then, $\{Q_\mu : \mu \in \Lambda\}$ is a cover of X by (τ_1, τ_2)-bI-open sets in X. Since, (X, τ_1, τ_2, I) is (τ_1, τ_2)-bI-compact, there exists a finite subset Λ_0 of Λ such that $X \setminus \cup\{Q_\mu : \mu \in \Lambda_0\} \in I$. Therefore, $X \setminus \cup\{P_\mu : \mu \in \Lambda_0\} \subseteq [X \setminus \cup\{Q_\mu : \mu \in \Lambda_0\}] \cup [\cup\{J_\mu : \mu \in \Lambda_0\}] \in I$. Thus, $(X, \tau_1^*, \tau_2^*, I)$ is (τ_1^*, τ_2^*)-bI-compact.

(ii) $\Rightarrow$ (i) It is obvious, since $\tau \subseteq \tau^*$.

□

Theorem 3.3. *For a space (X, τ_1, τ_2, I), the following are equivalent:*

(i) (X, τ_1, τ_2, I) is (τ_1, τ_2)-bI-compact,

(ii) if the collection $\{P_\mu : \mu \in \Lambda\}$ of (τ_1, τ_2)-bI-closed sets in X is such that $\cap\{P_\mu : \mu \in \Lambda\} = \emptyset$, then there exists a finite subset Λ_0 of Λ such that $\cap\{P_\mu : \mu \in \Lambda_0\} \in I$.

Proof. (i) $\Rightarrow$ (ii) Let $\{P_\mu : \mu \in \Lambda\}$ be the collection of (τ_1, τ_2)-bI-closed sets of X such that $\cap\{P_\mu : \mu \in \Lambda\} = \emptyset$. So, $\{X \setminus P_\mu : \mu \in \Lambda\}$ is a collection of (τ_1, τ_2)-bI-open sets of X. Thus, $\{X \setminus P_\mu : \mu \in \Lambda\}$ is a cover of X by (τ_1, τ_2)-bI-open sets of X. Since, (X, τ_1, τ_2, I) is (τ_1, τ_2)-bI-compact, there exists a finite subset Λ_0 of Λ such that $X \setminus \cup\{X \setminus P_\mu : \mu \in \Lambda_0\} \in I$. Therefore $\cap\{P_\mu : \mu \in \Lambda_0\} \in I$.

(ii) $\Rightarrow$ (i) Let $\{P_\mu : \mu \in \Lambda\}$ be a cover of X by (τ_1, τ_2)-bI-open sets of X. Then $\{X \setminus P_\mu : \mu \in \Lambda\}$ is a collection of (τ_1, τ_2)-bI-closed sets of X.

Since, $\cup\{P_\mu : \mu \in \Lambda\} = X$; so, we have that $\cap\{X \setminus P_\mu : \mu \in \Lambda\} = \emptyset$. By (ii), there exists a finite subset Λ_0 of Λ such that $\cap\{X \setminus P_\mu : \mu \in \Lambda_0\} \in I$. Thus, $X \setminus \cup\{P_\mu : \mu \in \Lambda_0\} \in I$. Hence, (X, τ_1, τ_2, I) is (τ_1, τ_2)-bI-compact.

□

Theorem 3.4. *If $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2, f(I))$ is a (τ_1, τ_2)-bI-irresolute surjection and X is (τ_1, τ_2)-bI-compact, then $(Y, \sigma_1, \sigma_2, f(I))$ is (σ_1, σ_2)-$bf(I)$-compact.*

Proof. Let $\{P_\mu : \mu \in \Lambda\}$ be a cover of Y by (σ_1, σ_2)-$bf(I)$-open sets of Y. Since, f is (τ_1, τ_2)-bI-irresolute, therefore $\{f^{-1}(P_\mu) : \mu \in \Lambda\}$ is a cover of X by (τ_1, τ_2)-bI-open sets of X.

By hypothesis, X is (τ_1, τ_2)-bI-compact, so there exists a finite subset Λ_0 of Λ such that $X \setminus \cup\{f^{-1}(P_\mu) : \mu \in \Lambda_0\} \in I$. Thus, $Y \setminus \cup\{P_\mu : \mu \in \Lambda_0\} \in f(I)$. Hence, $(Y, \sigma_1, \sigma_2, f(I))$ is (σ_1, σ_2)-$bf(I)$-compact.

□

Definition 3.3. *A function $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2, f(I))$ is called (τ_1, τ_2)-strongly-bI-open if $f(P)$ is (σ_1, σ_2)-$bf(I)$-open in Y, for every (τ_1, τ_2)-bI-open set P in X.*

The proof of the next result parallels that of Theorem 3.4.

Theorem 3.5. *Let $f : (X, \tau_1, \tau_2, f^{-1}(I)) \to (Y, \sigma_1, \sigma_2, I)$ be a (τ_1, τ_2)-strongly-bI-open bijection. If Y is (σ_1, σ_2)-bI-compact, then $(X, \tau_1, \tau_2, f^{-1}(I))$ is (τ_1, τ_2)-$bf^{-1}(I)$-compact.*

Theorem 3.6. *If a space (X, τ_1, τ_2, I) is (τ_1, τ_2)-bI-compact and $\tau_i \cap I = \{\emptyset\}$ for $i \in \{1, 2\}$, then there exists a finite collection $\{P_\mu : \mu \in \Lambda_0\}$ of (τ_1, τ_2)-bI-open sets of X such that $X = \tau_i - cl(\cup\{P_\mu : \mu \in \Lambda_0\})$ for $i \in \{1, 2\}$.*

Proof. Suppose that for any finite collection $\{P_\mu : \mu \in \Lambda_0\}$ of (τ_1, τ_2)-bI-open sets of X, $X \neq \tau_i - cl(\cup\{P_\mu : \mu \in \Lambda_0\})$ for $i \in \{1, 2\}$.

There exists $x \in X \setminus \tau_i - cl(\cup\{P_\mu : \mu \in \Lambda_0\})$. Hence, there exits $U \in \tau_i$ containing x such that $U \cap (\cup\{P_\mu : \mu \in \Lambda_0\}) = \emptyset$. Therefore, $U \subseteq X \setminus (\cup\{P_\mu : \mu \in$

$\Lambda_0\}) = L \in I$ and hence, $U \in \tau_i \cap I$. This is a contradiction. Hence, the result holds.

□

Definition 3.4. *A subset Q of an ideal bitopological space (X, τ_1, τ_2, I) is said to be (τ_1, τ_2)-bI-compact relative to X if for each cover $\{P_\mu : \mu \in \Lambda\}$ of Q by (τ_1, τ_2)-bI-open sets of X, there exists a finite subset Λ_0 of Λ such that $Q \setminus \cup\{P_\mu : \mu \in \Lambda_0\} \in I$.*

We state the following theorem without proof.

Theorem 3.7. *If Q and R are two (τ_1, τ_2)-bI-compact subsets relative to an ideal bitopological space (X, τ_1, τ_2, I), then $Q \cup R$ is (τ_1, τ_2)-bI-compact relative to X.*

Theorem 3.8. *A subset Q of an ideal bitopological space (X, τ_1, τ_2, I) is (τ_1, τ_2)-bI-compact relative to X if and only if $(Q, \tau_1|_Q, \tau_2|_Q, I|_Q)$ is $(\tau_1|_Q, \tau_2|_Q)$-$bI|_Q$-compact.*

Proof. Let a subset Q be (τ_1, τ_2)-bI-compact relative to X and $\{(P_\mu \cap Q) : \mu \in \Lambda\}$ be a cover of Q by $(\tau_1|_Q, \tau_2|_Q)$-$bI|_Q$-open sets in Q.

Then, $\{P_\mu : \mu \in \Lambda\}$ is a cover of Q by (τ_1, τ_2)-bI-open sets of X. Since, Q is (τ_1, τ_2)-bI-compact relative to X, thus there exists a finite subset Λ_0 of Λ such that $M = Q \setminus \cup\{P_\mu : \mu \in \Lambda_0\} \in I$. Then, $Q \cap M \in I|_Q$. This implies $Q \setminus \cup\{(P_\mu \cap Q) : \mu \in \Lambda_0\} \in I|_Q$. Hence, Q is $(\tau_1|_Q, \tau_2|_Q)$-$bI|_Q$-compact.

Conversely, let $\{P_\mu : \mu \in \Lambda\}$ be a cover of Q by (τ_1, τ_2)-bI-open sets of X. Thus, $\{(P_\mu \cap Q) : \mu \in \Lambda\}$ is a cover of Q by $(\tau_1|_Q, \tau_2|_Q)$-$bI|_Q$-open sets of Q. Hence, there exists a finite subset Λ_0 of Λ such that $Q \setminus \cup\{(P_\mu \cap Q) : \mu \in \Lambda_0\} \in I|_Q \subset I$. Since $Q \setminus \cup\{P_\mu : \mu \in \Lambda_0\} \subseteq Q \setminus \cup\{(P_\mu \cap Q) : \mu \in \Lambda_0\} \in I$, Q is (τ_1, τ_2)-bI-compact relative to X.

□

Definition 3.5. *An ideal bitopological space (X, τ_1, τ_2, I) is said to be (τ_1, τ_2)-bI-countably compact, if for each countable cover $\{P_\mu : \mu \in \Lambda\}$ of X by (τ_1, τ_2)-bI-open sets of X, there exists a finite subset Λ_0 of Λ such that $X \setminus \cup\{P_\mu : \mu \in \Lambda_0\} \in I$.*

The following results are stated without proof.

Theorem 3.9. *For a space (X, τ_1, τ_2, I), the following are equivalent:*

(i) (X, τ_1, τ_2, I) is (τ_1, τ_2)-bI-countably compact,

(ii) if the countable collection $\{P_\mu : \mu \in \Lambda\}$ of (τ_1, τ_2)-bI-closed sets in X is such that $\cap\{P_\mu : \mu \in \Lambda\} = \emptyset$, then there exists a finite subset Λ_0 of Λ such that $\cap\{P_\mu : \mu \in \Lambda_0\} \in I$.

Theorem 3.10. *If $f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2, f(I))$ is a (τ_1, τ_2)-bI-irresolute surjection and X is (τ_1, τ_2)-bI-countably compact, then $(Y, \sigma_1, \sigma_2, f(I))$ is (σ_1, σ_2)-$bf(I)$-countably compact.*

Theorem 3.11. *Let $f : (X, \tau_1, \tau_2, f^{-1}(I)) \to (Y, \sigma_1, \sigma_2, I)$ be a (τ_1, τ_2)-strongly-bI-open bijection. If Y is (σ_1, σ_2)-bI-countably compact, then $(X, \tau_1, \tau_2, f^{-1}(I))$ is (τ_1, τ_2)-$bf^{-1}(I)$-countably compact.*

Conflict of interest: Authors declare that there is no conflict of interest.

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