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ON $\lambda-$ PSEUDO BI-STARLIKE FUNCTIONS RELATED TO SOME CONIC DOMAINS

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Abstract

In this paper we introduce a new class $\mathcal{L}_{\Sigma}^{\lambda}(\phi)$ of λ-pseudo bi-starlike functions and determine the bounds for $|a_2|$ and $|a_3|$ where a_2, a_3 are the initial Taylor coefficients of $f \in \mathcal{L}_{\Sigma}^{\lambda}(\phi)$. Furthermore, we estimate the Fekete-Szegö functional for $f \in \mathcal{L}_{\Sigma}^{\lambda}(\phi)$.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$. Further, denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ and normalized by the condition $f(0) = 0 = f'(0) - 1$. One of the important and well-investigated subclasses of $\mathcal{S}$ is the class $\mathcal{S}^*(\alpha)$ of starlike functions of order $\alpha, (0 \leq \alpha < 1)$ defined by the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in \mathbb{U})$$

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and the class $\mathcal{K}(\alpha) \subset \mathcal{S}$ of convex functions of order $\alpha, (0 \le \alpha < 1)$ is defined by the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (z \in \mathbb{U}).$$

An analytic function f is said to be subordinate to an analytic function h, written by $f(z) \prec h(z)$, provided there is an analytic function ω with $\omega(0) = 0$ and such that $|\omega(z)| < 1$ in $\mathbb{U}$ and $f(z) = h(\omega(z))$. Ma and Minda [15] unified the approach to various subclasses of starlike and convex functions which are defined by a condition that either $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ are subordinate to a function ϕ. For this purpose, they considered a class Φ of analytic functions ϕ with positive real part in the unit disk $\mathbb{U}$, $\phi(0) = 1$, $\phi'(0) > 0$, such that ϕ maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions denoted by $\mathcal{S}^*(\phi)$, consists of function $f \in \mathcal{A}$ satisfying the subordination

$$\frac{zf'(z)}{f(z)} \prec \phi(z).$$

Similarly, a function $f \in \mathcal{A}$ is in the class of Ma-Minda convex functions of functions denoted by $\mathcal{K}(\phi)$ if it satisfies

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z).$$

In the sequel, it is assumed that ϕ is in the class Φ.

Example 1.1. *For* $0 < \alpha \le 1$ and $-1 \le B < A \le 1$, *we have that the function*

$$\phi(z) = \left(\frac{1 + Az}{1 + Bz}\right)^{\alpha} = 1 + B_1 z + B_2 z^2 + \cdots, \tag{2}$$

is in the class Φ, *where* $B_1 = \alpha(A - B)$ *and* $B_2 = -\frac{\alpha}{2}[2B(A-B) + (1-\alpha)(A-B)^2]$. *In particular, we have*

$$\left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \quad (0 < \alpha \le 1). \tag{3}$$

Example 1.2. *If we take* $\alpha = 1$ *and* $-1 \le B < A \le 1$, *then* (2) *becomes*

$$\phi(z) = \frac{1 + Az}{1 + Bz} = 1 + (A - B)z + B(A - B)z^2 + \cdots. \tag{4}$$

Further, for some $c \in (0, 1]$, we have $\phi_c \in \Phi$, where

$$\phi_c(z) = \sqrt{1 + cz} = 1 + \frac{c}{2}z - \frac{c^2}{8}z^2 + \ldots. \tag{5}$$

In this case the Ma-Minda class of functions $\mathcal{S}^*(\phi_c)$, consists of functions associated with the right loop of the Cassinian Ovals [5]. In particular if $c = 1$ this class is associated with the right-half of the lemniscate of Bernoulli [20]. If

$$\widetilde{\phi}(z) = z + \sqrt{1 + z^2} = 1 + z + \frac{1}{2}z^2 - \frac{1}{8}z^4 + \ldots. \tag{6}$$

then the class $\mathcal{S}^*(\widetilde{\phi})$ is connected with a right crescent [18]. If

$$\widehat{\phi}(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2, \tag{7}$$

then the class $\mathcal{S}^*(\widehat{\phi})$ is associated with a cardioid [19].

We may consider also the functions $\phi_{k,\alpha}$ related to the conic sections, that were introduced and studied by Kanas et al. [8] – [13], where $(0 \le k < \infty,\ 0 \le \alpha < 1)$ and where

$$\phi_{k,\alpha}(\mathbb{D}) = \{w = u + iv : (u-\alpha)^2 > k^2(u-1)^2 + k^2v^2\},\ \phi_{k,0} = \phi_k. \tag{8}$$

Various classes of functions were defined by a relation to the domain $\phi_{k,\alpha}(\mathbb{D})$. Further, we have

$$\phi_{k,\alpha}(z) = \frac{(1-\alpha)}{1-k^2}\cos\left(A(k)i\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right) - \frac{k^2-\alpha}{1-k^2} \quad (0<k<1), \tag{9}$$

and

$$\phi_{k,\alpha}(z) = \frac{(1-\alpha)}{k^2-1}\sin^2\left(\frac{\pi}{2\mathcal{K}(1,t)}\mathcal{K}\left(\frac{\sqrt{z}}{\sqrt{t}},t\right)\right) + \frac{k^2-\alpha}{k^2-1} \quad (k>1), \tag{10}$$

where $A(k) = \frac{2}{\pi}\arccos k$ and $\mathcal{K}(\omega,t)$ is the Legendre elliptic integral of the first kind

$$\mathcal{K}(\omega,t) = \int_0^\omega \frac{dx}{\sqrt{1-x^2}\sqrt{1-t^2x^2}},$$

with $t \in (0,1)$ chosen such that $k = \cosh\frac{\pi\kappa'(t)}{4\kappa(t)}$. Furthermore,

$$\phi_{1,\alpha}(z) = 1 + \frac{2(1-\alpha)}{\pi^2}\log^2\frac{1+\sqrt{z}}{1-\sqrt{z}} = 1 + \frac{8}{\pi^2}(1-\alpha)z + \frac{16}{3\pi^2}(1-\alpha)z^2 + \dots, \tag{11}$$

and

$$\phi_{0,\alpha}(z) = \frac{1+(1-2\alpha)z}{1-z} = 1 + 2(1-\alpha)z + 2(1-\alpha)z^2 + \cdots. \tag{12}$$

By virtue of the properties of the domains, for $p \prec \phi_{k,\alpha}$, we have

$$\Re\,(p(z)) > \frac{k+\alpha}{k+1}. \tag{13}$$

Note that Kanas and Sugawa [10] proved the positivity of coefficients of the functions $\phi_{k,0}$ implies positivity of coefficients for $0 \le \alpha < 1$ too. Also, we note that the domains $\phi_{k,\alpha}(\mathbb{D})$ are symmetric about real axis and starlike with respect to 1 so $\phi_{k,\alpha} \in \Phi$.

It is well known that every univalent function $f \in \mathcal{S}$ of the form (1), has an inverse $f^{-1}(w)$ defined in $\left(|w| < r_0(f); r_0(f) \ge \frac{1}{4}\right)$ where

$$f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \dots \tag{14}$$

A function $f \in \mathcal{S}$ is said to be bi-univalent in $\mathbb{U}$ if there exists a function $g \in \mathcal{S}$ such that $g(z)$ is a univalent extension of f^{-1} to $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$. The functions $\frac{z}{1-z}$, $-\log(1-z)$, $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ are in the class Σ (see details in [21]). However, the familiar Koebe function is not bi-univalent. Lewin [14] investigated the class of *bi-univalent* functions σ and obtained a bound $|a_2| \leqq 1.51$. Motivated by the work of Lewin [14], Brannan and Clunie [3] conjectured that $|a_2| \leqq \sqrt{2}$. The coefficient estimate problem for $|a_n|$ $(n \in \mathbb{N},\ \ n \geqq 3)$ is still open([21]). Brannan and Taha [4] also worked on certain subclasses of the bi-univalent function class Σ and obtained estimates for their initial coefficients. Various classes of bi-univalent functions were introduced and studied in recent times, the study of *bi-univalent* functions gained momentum mainly due to the work of Srivastava et al.[21]. Motivated by this, many researchers (see [2, 6, 16, 21, 22, 23] also the references cited there in) recently investigated several interesting subclasses of the class Σ and found non-sharp estimates on the first two Taylor-Maclaurin coefficients.

The class $\mathcal{L}_\lambda(\alpha)$ of λ-pseudo-starlike functions of order $\alpha, (0 \le \alpha < 1)$ was introduced and investigated by Babalola [1]. A function f, $f \in \mathcal{A}$ is in the class $\mathcal{L}_\lambda(\alpha)$ if it satisfies

$$\Re\left(\frac{z(f'(z))^\lambda}{f(z)}\right) > \alpha, \qquad (z \in \mathbb{U}).$$

In [1] it was showed that all pseudo-starlike functions are Bazilevič functions of type $(1-1/\lambda)$ and of order $\alpha^{1/\lambda}$ and univalent in the open unit disk $\mathbb{U}$.

Recently Joshi et al. [7] defined the bi-pseudo-starlike functions class and obtained the bounds for the initial coefficients $|a_2|$ and $|a_3|$. In this paper we define a new class $\mathcal{L}^\lambda_\Sigma(\phi)$, λ-bi-pseudo-starlike functions of Σ and determine the bounds for the initial Taylor-Maclaurin coefficients of $|a_2|$ and $|a_3|$ for $f \in \mathcal{L}^\lambda_\Sigma(\phi)$. Further, we consider the Fekete-Szegö problem in this class.

Definition 1. *Assume that $f \in \Sigma$, $\lambda \ge 1$ and $(f'(z))^\lambda$ is analytic in $\mathbb{U}$ with $(f'(0))^\lambda = 1$. Furthermore, assume that $g(z)$ is an extension of f^{-1} to $\mathbb{U}$, and $(g'(z))^\lambda$ is analytic in $\mathbb{U}$ with $(g'(0))^\lambda = 1$. Then $f(z)$ is said to be in the class $\mathcal{L}^\lambda_\Sigma(\phi)$ of λ-bi-pseudo-starlike functions if the following conditions are satisfied:*

$$\frac{z(f'(z))^\lambda}{f(z)} \prec \phi(z) \quad (z \in \mathbb{U}) \tag{15}$$

and

$$\frac{w(g'(w))^\lambda}{g(w)} \prec \phi(w) \quad (w \in \mathbb{U}), \tag{16}$$

where $\phi \in \Phi$ is given by

$$\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots, (B_1 > 0). \tag{17}$$

Remark 1. *For $\lambda = 1$ a function $f \in \Sigma$ is in the class $\mathcal{L}^1_\Sigma(\phi) \equiv \mathcal{S}^*_\Sigma(\phi)$ if the following conditions are satisfied:*

$$\frac{zf'(z)}{f(z)} \prec \phi(z) \quad \text{and} \quad \frac{wg'(w)}{g(w)} \prec \phi(w) \tag{18}$$

where $z, w \in \mathbb{U}$ *and function* g *is described in Definition 1.*

Remark 2. *For* $\lambda = 2$ *a function* $f \in \Sigma$ *is in the class* $\mathcal{L}^2_\Sigma(\phi) \equiv \mathcal{G}_\Sigma(\phi)$ *if the following conditions are satisfied:*

$$f'(z)\frac{zf'(z)}{f(z)} \prec \phi(z) \quad \text{and} \quad g'(w)\frac{wg'(w)}{g(w)} \prec \phi(w) \tag{19}$$

where $z, w \in \mathbb{U}$ *and function* g *is described in Definition 1.*

2 Coefficient estimates for $f \in \mathcal{L}^\lambda_\Sigma(\phi)$.

Using the following lemma we obtain the initial coefficients $|a_2|$ and $|a_3|$ for $f \in \mathcal{L}^\lambda_\Sigma(\phi)$.

Lemma 1. *[17] If* $p \in \mathcal{P}$, *and*

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots, \quad (z \in \mathbb{U}) \tag{20}$$

then $|p_n| \leq 2$ *for* $n \geq 1$, *where* $\mathcal{P}$ *is the family of all functions* p *analytic in* $\mathbb{U}$ *for which*

$$\Re\left(p(z)\right) > 0, \quad (z \in \mathbb{U}). \tag{21}$$

Theorem 1. *Let* $f(z)$ *given by* (1) *be in the class* $\mathcal{L}^\lambda_\Sigma(\phi)$, *then*

$$|a_2| \leq \frac{|B_1|\sqrt{B_1}}{\sqrt{(2\lambda^2-\lambda)B_1^2 - (B_2 - B_1)(2\lambda-1)^2}}, \tag{22}$$

$$|a_3| \leq \frac{|B_1|^2}{|2\lambda-1|^2} + \frac{|B_1|}{|3\lambda-1|}, \tag{23}$$

where $\phi(z)$ *is given by* (17) *and of the form* $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots, (B_1 > 0)$.

Proof. Let g be of the form

$$g(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \ldots$$

Since $f \in \mathcal{L}^\lambda_\Sigma(\phi)$, there exist two analytic functions $u, v : \mathbb{U} \to \mathbb{U}$ with $u(0) = 0 = v(0)$, such that $|u(z)| < 1$, $|v(z)| < 1$ and

$$\frac{z[f'(z)]^\lambda}{f(z)} = \phi(u(z)), \tag{24}$$

$$\frac{w[g'(w)]^\lambda}{g(w)} = \phi(v(w)). \tag{25}$$

Assume that $p(z)$ and $q(z)$ are in $\mathcal{P}$ and they are such that

$$p(z) := \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \cdots$$

and

$$q(z) := \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + \cdots .$$

It follows that,

$$u(z) := \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \cdots\right]$$

and

$$v(z) := \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2 + \cdots\right],$$

so we have

$$\phi(u(z)) = 1 + \frac{1}{2}B_1 p_1 z + \left[\frac{B_1}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2\right] z^2 + \ldots \tag{26}$$

and

$$\phi(v(w)) = 1 + \frac{1}{2}B_1 q_1 w + \left[\frac{B_1}{2}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}B_2 q_1^2\right] w^2 + \ldots . \tag{27}$$

On the other hand, we have

$$\frac{z[f'(z)]^\lambda}{f(z)} = 1 + (2\lambda-1)a_2 z + [(3\lambda-1)a_3 + \left(2\lambda^2 - 4\lambda + 1\right) a_2^2] z^2 + \cdots \tag{28}$$

$$\frac{w[g'(w)]^\lambda}{g(w)} = 1 - (2\lambda-1)a_2 w + [\left(2\lambda^2 + 2\lambda - 1\right) a_2^2 - (3\lambda-1)a_3] w^2 + \cdots . \tag{29}$$

Using (26), (27),(28) and (29) and equating similar coefficients , we get

$$(2\lambda-1)a_2 = \frac{1}{2}B_1 p_1, \tag{30}$$

$$(3\lambda-1)a_3 + \left(2\lambda^2 - 4\lambda + 1\right) a_2^2 = \frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2, \tag{31}$$

$$-(2\lambda-1)a_2 = \frac{1}{2}B_1 q_1, \tag{32}$$

$$\left(2\lambda^2 + 2\lambda - 1\right) a_2^2 - (3\lambda-1)a_3 = \frac{1}{2}B_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}B_2 q_1^2 \tag{33}$$

From (30) and (32), we find that

$$a_2 = \frac{B_1 p_1}{2(2\lambda-1)} = -\frac{B_1 q_1}{2(2\lambda-1)};$$

it follows that

$$p_1 = -q_1 \tag{34}$$

and

$$8(2\lambda-1)^2a_2^2 = B_1^2(p_1^2+q_1^2). \tag{35}$$

Thus,

$$a_2^2 = \frac{B_1^2(p_1^2+q_1^2)}{8(2\lambda-1)^2} \quad (or) \quad p_1^2+q_1^2 = \frac{8(2\lambda-1)^2}{B_1^2}a_2^2 \tag{36}$$

Adding (31) and (33), we have

$$\begin{aligned} &\left(4\lambda^2-2\lambda\right)a_2^2 \\ = \ & \frac{1}{2}B_1(p_1+q_1)+\frac{1}{2}B_1\left[(p_2+q_2)-\frac{1}{2}\left(p_1^2+q_1^2\right)\right]+\frac{1}{4}B_2\left(p_1^2+q_1^2\right) \\ = \ & \frac{1}{2}B_1(p_2+q_2)+\frac{1}{4}(B_2-B_1)\left(p_1^2+q_1^2\right) \end{aligned} \tag{37}$$

Substituting (34) and (36) in (37), we get

$$\begin{aligned} &\left(4\lambda^2-2\lambda\right)a_2^2 = \frac{1}{2}B_1(p_2+q_2)+\frac{1}{4}(B_2-B_1)\frac{8(2\lambda-1)^2}{B_1^2}a_2^2, \\ &\left[\left(4\lambda^2-2\lambda\right)-\frac{2(B_2-B_1)(2\lambda-1)^2}{B_1^2}\right]a_2^2 = \frac{1}{2}B_1(p_2+q_2), \\ &\left[\left(4\lambda^2-2\lambda\right)B_1^2-2(B_2-B_1)(2\lambda-1)^2\right]a_2^2 = B_1^3(p_2+q_2). \end{aligned}$$

Hence

$$a_2^2 = \frac{B_1^3(p_2+q_2)}{2\left[\left(2\lambda^2-\lambda\right)B_1^2-(B_2-B_1)(2\lambda-1)^2\right]}. \tag{38}$$

Applying Lemma 1 in (38), we get the desired inequality (22). From (31) and from (33) and using (36), after simple computation, we obtain

$$a_3 = a_2^2+\frac{B_1(p_2-q_2)}{4(3\lambda-1)}. \tag{39}$$

Again, by (34) we have $p_1^2 = q_1^2$ and applying Lemma 1 then (39) yields the desired inequality. This completes the proof of Theorem 1. □

By taking $\lambda = 1$, we state the following:

Corollary 1. *Let $f(z)$ given by (1) be in the class $\mathcal{L}_\Sigma^1(\phi) \equiv \mathcal{S}_\Sigma^*(\phi)$, then*

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{|B_1^2+B_1-B_2|}} \quad \textit{and} \quad |a_3| \le B_1^2+\frac{B_1}{2}.$$

Remark 3. *For the class of strongly starlike functions, the function ϕ is given by (3) which gives $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$. On the other hand, if we take $\phi(z)$ as in (12), then $B_1 = B_2 = 2(1-\alpha)$ then Corollary 1 yields the bounds of $|a_2|$ and $|a_3|$ given in [7].*

By taking $\lambda = 2$ we state the following new result:

Corollary 2. *Let $f(z)$ given by (1) be in the class $\mathcal{L}_{\Sigma}^{2}(\phi) \equiv \mathcal{G}_{\Sigma}(\phi)$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|6B_1^2 - 9(B_2 - B_1)|}} \quad \text{and} \quad |a_3| \leq \frac{B_1^2}{9} + \frac{B_1}{5}.$$

3 Fekete-Szegö inequalities for the Function Class $\mathcal{L}_{\Sigma}^{\lambda}(\phi)$

Making use of the values of a_2^2 and a_3, and motivated by the recent work of Zaprawa [24] we prove the following Fekete-Szegö result for the function class $\mathcal{L}_{\Sigma}^{\lambda}(\phi)$.

Theorem 2. *Let $f(z) \in \mathcal{L}_{\Sigma}^{\lambda}(\phi)$ and $\mu \in \mathbb{C}$, then*

$$|a_3 - \mu a_2^2| \leq 2B_1 \left| \left(\Theta(\mu) + \frac{1}{4(3\lambda - 1)}\right) + \left(\Theta(\mu) - \frac{1}{4(3\lambda - 1)}\right) \right|. \tag{40}$$

where

$$\Theta(\mu) = \frac{B_1^2(1-\mu)}{2\left[(2\lambda^2 - \lambda) B_1^2 - (B_2 - B_1)(2\lambda - 1)^2\right]}. \tag{41}$$

Proof. From (39) we have

$$a_3 = a_2^2 + \frac{B_1(p_2 - q_2)}{4(3\lambda - 1)}.$$

Using (38), by simple calculation we get

$$a_3 - \mu a_2^2 = B_1 \left[\left(\Theta(\mu) + \frac{1}{4(3\lambda - 1)}\right) p_2 + \left(\Theta(\mu) - \frac{1}{4(3\lambda - 1)}\right) q_2 \right],$$

where

$$\Theta(\mu) = \frac{B_1^2(1-\mu)}{2\left[(2\lambda^2 - \lambda) B_1^2 - (B_2 - B_1)(2\lambda - 1)^2\right]}.$$

Since all B_j are real and $B_1 > 0$, we have

$$|a_3 - \mu a_2^2| \leq 2B_1 \left| \left(\Theta(\mu) + \frac{1}{4(3\lambda - 1)}\right) + \left(\Theta(\mu) - \frac{1}{4(3\lambda - 1)}\right) \right|,$$

which completes the proof. □

Specializing $\lambda = 1$ and $\lambda = 2$ we can state the Fekete-Szegö inequality for the function class $\mathcal{S}_{\Sigma}^{*}(\phi)$ and $\mathcal{G}_{\Sigma}(\phi)$ respectively.

4 Corollaries and its Consequences

Making use (22) , (23) of Theorem 1 and various choices of ϕ given in equations (2) to (7) we state the following new results as Corollaries:

Corollary 3. *Let $f(z)$ given by (1) be in the class $\mathcal{L}_{\Sigma}^{\lambda}\left(\left[\frac{1+Az}{1+Bz}\right]^{\alpha}\right)$, then*

$$|a_2| \leq \frac{\alpha(A-B)}{\sqrt{\frac{1}{2}(2\lambda-1)(A-B)(2\lambda+1-\alpha)}},$$

$$|a_3| \leq \frac{\alpha|A-B|^2}{(2\lambda-1)^2} + \frac{\alpha(A-B)}{(3\lambda-1)}.$$

Remark 4. *By taking $A = 1$* and *$B = -1$, the above Corollary yields the values of $|a_2|$ and $|a_3|$ given in [7].*

Corollary 4. *Let $f(z)$ given by (1) be in the class $\mathcal{L}_{\Sigma}^{\lambda}\left(\frac{1+Az}{1+Bz}\right)$, then*

$$|a_2| \leq \frac{A-B}{\sqrt{(2\lambda-1)(A-B)\lambda}},$$

$$|a_3| \leq \frac{|A-B|^2}{(2\lambda-1)^2} + \frac{A-B}{(3\lambda-1)}.$$

Corollary 5. *Let $f(z)$ given by (1) be in the class $\mathcal{L}_{\Sigma}^{\lambda}(\sqrt{1+cz})$, then*

$$|a_2| \leq \frac{c\sqrt{c}}{\sqrt{2(2\lambda^2-\lambda)c^2 + 3c(2\lambda-1)^2}},$$

$$|a_3| \leq \frac{c^2}{4(2\lambda-1)^2} + \frac{c}{2(3\lambda-1)}.$$

Remark 5. *Let $f(z)$ given by (1) be in the class $\mathcal{L}_{\Sigma}^{\lambda}(\sqrt{1+z})$, then*

$$|a_2| \leq \frac{1}{\sqrt{2(2\lambda^2-\lambda) + 3(2\lambda-1)^2}},$$

$$|a_3| \leq \frac{1}{4(2\lambda-1)^2} + \frac{1}{2(3\lambda-1)}.$$

Corollary 6. *Let $f(z)$ given by (1) be in the class $\mathcal{L}_{\Sigma}^{\lambda}(z+\sqrt{1+z^2})$, then*

$$|a_2| \leq \frac{\sqrt{2}}{\sqrt{2(2\lambda^2-\lambda) + (2\lambda-1)^2}},$$

$$|a_3| \leq \frac{1}{(2\lambda-1)^2} + \frac{1}{(3\lambda-1)}.$$

Corollary 7. *Let $f(z)$ given by (1) be in the class $\mathcal{L}_{\Sigma}^{\lambda}(\psi(z))$, where $\psi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$ then*

$$|a_2| \leq \frac{8}{\sqrt{48(2\lambda^2 - \lambda) + 18(2\lambda - 1)^2}},$$
$$|a_3| \leq \frac{16}{9(2\lambda - 1)^2} + \frac{4}{3(3\lambda - 1)}.$$

Now, similar to the above Corollaries 3 - 7, in view of (9), (11) and (10), we may get some coefficient bounds for functions in the classes associated with the conic domains. In Theorem 1 by replacing ϕ with $\phi_{k,a}$, we may get coefficients estimates (and Fekete-Szegö inequality from Theorem 2) for various subclasses of λ pseudo bi–starlike functions associated with certain conic domains. Further, specializing $\lambda = 1$ and $\lambda = 2$ and suitable choices of ϕ as in above Corollaries 3 - 7, we can state the estimates $|a_2|$ and $|a_3|$ (and Fekete-Szegö inequality from Theorem 2) for the function class $\mathcal{S}_{\Sigma}^{*}(\phi)$ and $\mathcal{G}_{\Sigma}(\phi)$.

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