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# SOME PROPERTIES OF PSEUDO-SLANT SUBMANIFOLDS OF A SASAKIAN MANIFOLD

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**Abstract**

In this paper, we study pseudo-slant submanifolds of a Sasakian manifold. We find some results and investigate the integrability conditions of distributions which are involved in the definition of pseudo-slant submanifolds. Finally, we obtain the necessary and sufficient condition for a pseudo-slant submanifold to be pseudo-slant product.

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## 1 Introduction

D. Blair studied contact manifolds in Riemannian geometry in [4]. The slant submanifolds of an almost contact metric manifold were defined and studied by A. Lotta [15]. After that, such submanifolds were studied in [5] and by J.L. Cabrerizo et al, of Sasakian manifolds [6] and [7]. The differential geometry of slant submanifolds has shown an increasing development since B.Y. Chen defined slant submanifolds in complex manifolds as a natural generalization of both invariant and anti-invariant submanifolds [8],[9]. After that, many research articles have been published on the existence of these submanifolds in various known spaces. Semi-slant submanifolds of Keahler manifold N. Papaghuice [16], as a generalization of slant submanifolds. Bi-slant submanifolds were introduced in an almost Hermitian manifold. Recently Carriazo defined and studied bi-slant submanifolds in an almost Hermitian manifold and gave the notion of pseudo-slant submanifold in an almost Hermitian manifold. V.A. Khan and M.A Khan [12] defined

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and studied the contact version of pseudo-slant submanifold in a Sasakian manifold. Also, U.C. De and Avijit Sarkar in [10] studied pseudo-slant submanifolds of trans-Sasakian manifolds. Many results on totally umbilical hemi-slant submanifolds of Cosymplectic manifolds were obtained by M.A. Khan in [14]. Recently M. Atceken [2], has studied slant and pseudo-slant submanifolds in $(LCS)_n$-manifolds and CR-submanifolds of Kenmotsu manifolds in [1] along with the geometry of pseudo-slant submanifolds of a Kenmotsu manifold in [3] as well as in [11] for nearly Cosymplectic manifold. Warped product pseudo-slant submanifolds of a nearly Cosymplectic manifold were studied in [18] by S. Uddin and A.A. Mustafa. S. Uddin and M.A. Khan have found a classification on totally umbilical proper slant and hemi-slant submanifolds of nearly trans-Sasakian manifolds in [13] and for nearly Kenmotsu manifold in [17]. Motivated by the above study, we obtain some interesting results on pseudo-slant submanifolds of a Sasakian manifold.

The paper is organized in following manner.

In this paper, we find some properties of pseudo-slant submanifolds of a Sasakian manifold. In section two, we give some basic definitions and formulas for a Sasakian manifold and their submanifolds. In section three, we recall the some basic results and definitions of a pseudo-slant submanifold of almost contact metric manifolds. We obtain the integrability conditions of distributions on the pseudo-slant submanifolds of a Sasakian manifold and then analogous results for these submanifolds in the setting of Sasakian manifolds. At last, we obtain the necessary and sufficient condition for a pseudo-slant submanifold to be a pseudo-slant product.

## 2 Preliminaries

In this section, we recall some basic definitions and formulas from the theory of Sasakian manifold and their submanifolds. We give some notations used throughout this paper.

Let $\bar{M}$ be an odd dimensional $C^\infty$-differentiable manifold with the almost contact metric structure $(J, \xi, \eta, g)$, where $J$ is a tensor field of type $(1,1)$, $\xi$ is a vector field, $\eta$ is 1-form and $g$ is a Riemannian metric on $\bar{M}$, satisfying

$$J^2X = -X + \eta(X)\xi, \tag{2.1}$$

$$J\xi = 0,\ \eta \circ \phi = 0,\ \eta(\xi) = 1,\ g(X, \xi) = \eta(X), \tag{2.2}$$

and

$$g(JX, JY) = g(X, Y) - \eta(X)\eta(Y),\ g(JX, Y) = -g(X, JY), \tag{2.3}$$

for any vector fields $X, Y \in \Gamma(T\bar{M})$ An almost contact structure $(J, \xi, \eta, g)$ is said to be normal if the almost complex structure $\phi$ on the product manifold $\bar{M} \times R$ given by

$$\phi(X, f\frac{d}{dt}) = (JX - f\xi, \eta(X)\frac{d}{dt}),$$

where $f$ is the $C^\infty$-function on $\bar{M} \times R$. The condition for normality in terms of $J, \xi$ and $\eta$ is $[J, J] + 2d\eta \otimes \xi = 0$ on $\bar{M}$, where $[J, J](X, Y) = J^2[X, Y] + [JX, JY] - J[JX, Y] - J[X, JY]$ is the Nijenhuis tensor of $J$. Finally, the fundamental 2-form $\Phi$ is defined by $\Phi(X, Y) = g(X, \phi Y)$. A normal almost contact metric structure is called Sasakian structure, which satisfies

$$(\nabla_X J) = g(X, Y)\xi - \eta(Y)X \tag{2.4}$$

and

$$(\nabla_X \xi) = -JX \tag{2.5}$$

for any vector fields $X, Y \in \Gamma(T\bar{M})$. Then almost contact metric structure $(\bar{M}, J, \xi, \eta, g)$ is called Sasakian manifold.
Now, let $M$ be a submanifold of a contact metric manifold $\bar{M}$ with induced metric $g$. Also let $\nabla$ and $\nabla^\perp$ be the induced connections on the tangent bundle $TM$ and the normal bundle $T^\perp M$ of $M$, respectively. Then the Gauss and Wiengarten formulas are, respectively given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.6}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^\perp V, \tag{2.7}$$

where $h$ and $A_V$ are the second fundamental form and the shape operator corresponding to the normal vector field $V$, respectively, for the immersion of $M$ into $\bar{M}$. The second fundamental form and shape operator are related by formula

$$g(h(X, Y), V) = g(A_V X, Y) \tag{2.8}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$.
$M$ is said to be totally geodesic submanifold if $h(X, Y) = 0$ for each $X, Y \in \Gamma(TM)$.

**Example 1.** *We consider $R^{2n+1}$ with Cartesian coordinates $(x_i, y_i, z_i)(i = 1, ...., n)$ and its usual contact form*

$$\eta = \frac{1}{2}(dz - \sum y_i dx_i).$$

*The characteristic vector field $\xi$ is given by $2\frac{\partial}{\partial z}$ and its Riemannian metric $g$ and its tensor field $J$ are given by*

$$g = \eta \otimes \eta + \frac{1}{4}(\sum((dx_i)^2 + (dy_i)^2), \; J = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_j & 0 \end{pmatrix}, i = 1, ...., n$$

*This gives a contact structure on $R^{2n+1}$. The vector fields $E_i = 2\frac{\partial}{\partial y_i}, E_{n+i} = 2(\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial z})$, $\xi$ form a $J-$ basis for the contact metric structure. On the other hand, it can be shown that $R^{2n+1}(J, \xi, \eta, g)$ is a Sasakian manifold.*

## 3 Pseudo-slant submanifolds of a Sasakian manifold

In this section, we will get the integrability conditions of the distributions of pseudo-slant submanifolds of a Sasakian manifold. At last, we obtain necessary and sufficient conditions for a pseudo-slant submanifold to be pseudo-slant product. In contact geometry A. Lotta introduced slant submanifold as follows [15],

**Definition 1.** *A submanifold $M$ of an almost contact metric manifold $\bar{M}$ is said to be a slant submanifold if for any $p \in M$ and $X \in T_pM - \{\xi\}$, the angle between $JX$ and $T_pM$ is constant. the constant angle $\theta X \in [0, \frac{\pi}{2}]$ is called slant angle of $M$ in $\bar{M}$.*

*(1) If $\theta = 0$ the submanifold is invariant submanifold.*

*(2) If $\theta = \frac{\pi}{2}$ then it is anti-invariant submanifold.*

*(3) If $\theta \neq 0, \frac{\pi}{2}$ then it is proper slant submanifold.*

*The tangent bundle $TM$ of $M$ is decomposed as $TM = D \oplus < \xi >$, where the orthogonal complementary distribution $D$ of $< \xi >$ is known as the slant distribution on $M$*

**Definition 2.** *Let $M$ be a submanifold of an almost contact metric manifold $\bar{M}$. $M$ is said to be pseudo-slant of $\bar{M}$ if there exist two orthogonal distributions $D_\theta$ and $D^\perp$ on $M$ such that*

*(1) TM has the orthogonal direct decomposition $TM = D^\perp \oplus D_\theta \oplus < \xi >$.*

*(2) The distribution $D^\perp$ is an anti-invariant submanifold.*

*(3) The distribution $D_\theta$ is a slant, that is the slant angle between of $D_\theta$ and $JD_\theta$ is constant.*

*Let $m = dim(D^\perp)$ and $n = dim(D_\theta)$. We distinguish the following five cases.*

*(1) If $n = 0$ or $\theta = \frac{\pi}{2}$, then $M$ is an anti-invariant submanifold.*

*(2) If $m = 0$ and $\theta = 0$, then $M$ is invariant submanifold.*

*(3) If $m = 0$ and $\theta \neq 0, \frac{\pi}{2}$, then $M$ is a proper slant submanifold.*

*(4) If $m, n \neq 0$ and $\theta = 0$, then $M$ is semi-invariant submanifold.*

*(5) If $m, n \neq 0$ and $\theta \neq 0, \frac{\pi}{2}$, then $M$ is pseudo-slant submanifold [12].*

Now, we give the following results in the setting of almost contact manifolds given by Cabrerizo et.al.

**Theorem 1.** *Let $M$ be a slant submanifold of an almost contact metric manifold $\bar{M}$ such that $\xi \in \Gamma(TM)$. Then $M$ is slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that*

Now, let $M$ be a submanifold of an almost contact metric manifold $\bar{M}$. Then for any $X \in \Gamma(TM)$, we can write

$$JX = \phi X + \omega X, \tag{3.1}$$

where $\phi X$ and $\lambda X$ are the tangential and normal component of $JX$ respectively. Similarly, for $V \in \Gamma(T^\perp M)$, we have

$$JV = BV + CV \tag{3.2}$$

where $BV$ and $CV$ are the tangential and normal component of $JV$. Then, using (2.1), (3.1) and (3.2), we have

$$\phi^2 = -I + \eta \otimes \xi - B\omega,\ \omega\phi + C\omega = 0, \tag{3.3}$$

and

$$\phi B + BC = 0,\ \omega B + C^2 = -I. \tag{3.4}$$

Furthermore, for any $X, Y \in \Gamma(TM)$, we have $g(\phi X, Y) = -g(X, \phi Y)$ and $U, V \in \Gamma(T^{\perp}M)$, we get $g(U, CV) = -g(U, CV)$. These show that $\phi$ and $C$ are skew symmetric tensor fields. Moreover, for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, we get

$$g(\omega X, V) = -g(X, BV) \tag{3.5}$$

which gives the relation between $\omega$ and $B$.
Furthermore, the covariant derivatives of the tensor field $\phi$, $\omega$, $B$ and $C$ are, respectively defined by

$$(\nabla_X \phi)Y = \nabla_X \phi Y - \phi \nabla_X Y, \tag{3.6}$$

$$(\nabla_X \omega)Y = \nabla^{\perp}_X \omega Y - \phi \omega_X Y, \tag{3.7}$$

$$(\nabla_X B)Y = \nabla_X BY - B\nabla^{\perp}_X Y, \tag{3.8}$$

$$(\nabla_X C)Y = \nabla^{\perp}_X CY - C\nabla^{\perp}_X Y. \tag{3.9}$$

A submanifold $M$ is said to be invariant if $\omega$ is identically zero, that is $JX \in \Gamma(TM)$ for all $X \in \Gamma(TM)$. On the other hand, $M$ is said to be anti-invariant if $\phi$ is identically zero, that is $JX \in \Gamma(T^{\perp}M)$ for all $X \in \Gamma(TM)$. Now, we get easily

$$(\nabla_X \phi)Y = A_{\omega Y}X + Bh(X, Y), \tag{3.10}$$

and

$$(\nabla_X \omega)Y = Ch(X, Y) - h(X, \phi Y), \tag{3.11}$$

similarly, for any $V \in \Gamma(T^2M)$ and $X \in \Gamma(TM)$, we obtain

$$(\nabla_X B)Y = A_{CY}X + \phi A_V X, \tag{3.12}$$

and

$$(\nabla_X C)Y = -h(BV, X) - \omega A_V X. \tag{3.13}$$

since $M$ is tangent to $\xi$, then using (2.5), (2.6),(2.8)and (3.1)

$$\nabla_\xi \xi = 0,\ h(\xi, \xi) = 0,\ A_V \xi = 0 \tag{3.14}$$

for all $V \in \Gamma(T^{\perp}M)$ and $\xi \in \Gamma(TM)$.
Now, we have the following result of an almost contact manifold given by Cabrerizo et.al.

**Theorem 2.** *Let $M$ be a slant submanifold of an almost contact manifold of $\bar{M}$ such that $\xi \in \Gamma(TM)$. Then, $M$ is slant submanifold if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$\phi^2 = -\lambda(I - \eta \otimes \xi) \tag{3.15}$$

*furthermore, in this case, if $\theta$ is slant angle of $M$, then $\lambda = cos^2\theta$ [6].*

**Corollary 1.** *Let $M$ be a slant submanifold of an almost contact manifold of $\bar{M}$ with slant angle $\theta$. then for any $X, Y \in \Gamma(TM)$, we have*

$$g(\phi X, \phi Y) = cos^2\theta(g(X,Y) - \eta(X)\eta(Y)), \tag{3.16}$$

*and*

$$g(\omega X, \omega Y) = sin^2\theta(g(X,Y) - \eta(X)\eta(Y)). \tag{3.17}$$

By using (3.10) and (3.14), we get

$$\eta((\nabla_X T)Y) = g(X,Y) - \eta(X)\eta(Y) \tag{3.18}$$

for $X, Y \in \Gamma(D_\theta)$.
If we denote the projection on $D^\perp$ and $D_\theta$ by $P$ and $Q$ respectively then for any vector field $X \in \Gamma(TM)$, we can write

$$X = PX + QX + \eta(X)\xi \tag{3.19}$$

Now operating $J$ on both sides of equation (3.19), we get

$$JX = JPX + JQX$$

and

$$\phi X + \omega X = \omega PX + \phi QX + \omega QX.$$

we can easily see that

$$\phi X = \phi QX, \omega X = \omega PX + \omega QX,$$

and

$$JPX = \omega PX, \phi PX = 0, JQX = \phi QX + \omega QX, \phi QX \in \Gamma(D_\theta).$$

If we denote the orthogonal complementary of $J(TM)$ in $T^\perp M$ by $\mu$, then the normal bundle $T^\perp M$ can be decomposed as follows

$$T^\perp M = \omega(D^\perp) \oplus \omega(D_\theta) \oplus \mu. \tag{3.20}$$

We can easily see that bundle $\mu$ is an invariant subbundle with respect to $J$. Since $D^\perp$ and $D_\theta$ are orthogonal distributions on $M$, $g(Z,X) = 0$ for each $Z \in (D^\perp)$ and $X \in \Gamma(D_\theta)$. Thus, by equation (2.3) and (3.1), we can write

$$g(\omega Z, \omega X) = g(JZ, JX) = g(Z,X) = 0,$$

that is distributions $\omega(D^\perp)$ and $\omega(D_\theta)$ are also mutually perpendicular. In fact decomposition (3.20) is an orthogonal direct decomposition.

**Theorem 3.** *Let $M$ be a submanifold of an almost contact metric manifold $\bar{M}$. Then $D_\theta$ is slant distribution if and only if there is a constant $\lambda \in [0,1]$ such that*

$$(\phi Q)^2 X = -\lambda X. \tag{3.21}$$

*for any $X \in \Gamma(D_\theta)$. In this case, the slant angle $\theta$ satisfies $\lambda = cos^2\theta$.*

Moreover, for any $Z, W \in \Gamma(D^\perp)$ and $U \in \Gamma(TM)$, also by using (2.4), (2.7) and (2.8), we get

$$\begin{aligned} g(A_{\omega Z}W - A_{\omega W Z}, U) &= g(h(W,U), \omega Z) - g(h(Z,U), \omega W) \\ &= g(\bar{\nabla}_U W, JZ) - g(\bar{\nabla}_U Z, JW) \\ &= -g(J\bar{\nabla}_U W, Z) + g(J\bar{\nabla}_U Z, W) \\ &= g(\bar{\nabla}_U JZ - (\bar{\nabla}_U J)Z, W) \\ &+ g((\bar{\nabla}_U J)W - \bar{\nabla}_U JW, Z) \\ &= g(\bar{\nabla}_U JZ, W) - g(\bar{\nabla}_U JW, Z) \\ &= -g(A_{\omega Z}U, W) + g(A_{\omega W}U, Z) \\ &= g(A_{\omega W}Z - A_{\omega Z}W, U). \end{aligned}$$

It follows that

$$A_{\omega W}Z = A_{\omega Z}W. \tag{3.22}$$

**Theorem 4.** *Let $M$ be a pseudo-slant submanifold of a Sasakian manifold $\bar{M}$, then*

$$\nabla^\perp_W \omega Z - \nabla^\perp_Z \omega W \in \omega(D^\perp)$$

*for any $Z, W \in \Gamma(D^\perp)$.*

Proof. For any $Z, W \in \Gamma(D^\perp)$ and $V \in \mu$ and using (2.4), (3.22), we obtain

$$\begin{aligned} g(\nabla^\perp_W \omega Z - \nabla^\perp_Z \omega W, V) &= g(\bar{\nabla}_W JZ + A_{JZ}W - \bar{\nabla}_Z JW + A_{JW}Z, V) \\ &= g(\bar{\nabla}_W JZ - \bar{\nabla}_Z JW, V) \\ &= g((\bar{\nabla}_W J)Z + J\bar{\nabla}_W Z, V) \\ &- g((\bar{\nabla}_Z J)W + J\bar{\nabla}_Z W, V) \\ &= g(J\bar{\nabla}_W Z, V) - g(J\bar{\nabla}_Z W, V) \\ &= g(\bar{\nabla}_W Z, JV) - g(\bar{\nabla}_Z W, JV) \\ &= g(\nabla_W Z, V) - g(\nabla_Z W, V) \\ &+ g(h(Z,W), JV) - g(h(W,Z), JV) \\ &= 0 \end{aligned}$$

Thus the proof is complete.

**Theorem 5.** *Let $M$ be a pseudo-slant submanifold of a Sasakian manifold $\bar{M}$. Then the anti-invariant distribution $D^\perp$ is completely integrable and its maximal integral submanifold is an anti-invariant submanifold of $\bar{M}$.*

Proof. For any $Z, W \in \Gamma(D^{\perp})$ and $X \in \Gamma(D_\theta)$, by using (2.4), (2.6), (2.7) and (2.8), we get

$$\begin{aligned}
g([Z,W],X) &= g(\bar{\nabla}_Z W - \bar{\nabla}_Z W, X) \\
&= g(\bar{\nabla}_W X, Z) - g(\bar{\nabla}_Z X, W) \\
&= g(J\bar{\nabla}_W X, JZ) - g(J\bar{\nabla}_Z X, JW) \\
&= g(\bar{\nabla}_W JX, JZ) - g(\bar{\nabla}_Z JX, JW) \\
&- g((\bar{\nabla}_W J)X, JZ) + g((\bar{\nabla}_Z J)X, JW) \\
&= g(\bar{\nabla}_W \phi X + \bar{\nabla}_W \omega X, \omega Z) \\
&- g(\bar{\nabla}_Z \phi X + \bar{\nabla}_Z \omega X, \omega W) \\
&= g(h(\phi X, W), \omega Z) - g(h(\phi X, Z), \omega W) \\
&+ g(\nabla^{\perp}_W \omega X, \omega Z) - g(\nabla^{\perp}_Z \omega X, \omega W) \\
&= g(A_{\omega Z} W - A_{\omega W} Z, \phi X) + g(\nabla^{\perp}_W \omega X, \omega Z) \\
&- g(\nabla^{\perp}_Z \omega X, \omega W),
\end{aligned}$$

by using (3.7), (3.11) and (3.22), we obtain

$$\begin{aligned}
g([Z,W],X) &= g(\nabla^{\perp}_W \omega X, \omega Z) - g(\nabla^{\perp}_Z \omega X, \omega W) \\
&= g((\nabla_W \omega)X + \omega \nabla_W X, \omega Z) \\
&- g((\nabla_Z \omega)X + \omega \nabla_Z X, \omega W) \\
&= g(Ch(W,X) - h(W,\phi X), \omega Z) - g(Ch(Z,X) - h(Z,\phi X), \omega W) \\
&+ g(\omega \nabla_W X, \omega Z) - g(\omega \nabla_Z X, \omega W) \\
&= -g(h(W,\phi X), \omega Z) + g(h(Z,\phi X), \omega W) \\
&+ g(\omega \nabla_W X, \omega Z) - g(\omega \nabla_Z X, \omega W)
\end{aligned}$$

by using (3.17), we have

$$\begin{aligned}
g([Z,W],X) &= sin^2\theta g(\nabla_W X, Z) - sin^\theta g(\nabla_Z X, W) \\
&= sin^2\theta g(\nabla_Z W, X) - sin^\theta g(\nabla_W Z, X) \\
&= sin^2\theta g([Z,W],X),
\end{aligned}$$

hence

$$cos^2\theta g([Z,W],X) = 0.$$

Thus $[Z,W] \in \Gamma(D^{\perp})$, that is, anti-invariant distribution $D^{\perp}$ is always integrable and its integral submanifold is an anti-invariant submanifold of $\bar{M}$.
Thus the proof is complete.
Now, by using (2.4), we get

$$(\bar{\nabla}_X J)Y = \bar{\nabla}_X JY - J\bar{\nabla}_X Y = g(X,Y)\xi - \eta(Y)X.$$

Hence by using (2.6), (2.7), (3.1) and (3.2), we have

$$-A_{\omega Y}X + \nabla^{\perp}_X \omega Y - \phi \nabla_X Y - Bh(X,Y) - Ch(X,Y) = g(X,Y)\xi - \eta(Y)X,$$

for any $X, Y \in \Gamma(D^{\perp})$. From the tangent component of this last equation, we have

$$A_{\omega Y}X + \phi\nabla_X Y + Bh(X,Y) + g(X,Y)\xi = 0. \tag{3.23}$$

By interchanging roles of $X$ and $Y$ in (3.23), we have

$$A_{\omega X}Y + \phi\nabla_Y X + Bh(Y,X) + g(Y,X)\xi = 0, \tag{3.24}$$

which is equivalent to

$$T[X,Y] = A_{\omega X}Y - A_{\omega Y}X.$$

From (3.22), we can easily to see that the anti-invariant distribution $D^{\perp}$ is always integrable.
Since the ambient manifold $\bar{M}$ is Sasakian, for any $Z, W \in \Gamma(D^{\perp})$

$$(\bar{\nabla}_Z J)W = g(Z,W)\xi - \eta(W)Z,$$

which implies that

$$\bar{\nabla}_Z JW - J\bar{\nabla}_Z W = \bar{\nabla}_Z \omega W - J(\nabla_Z W + h(W,Z)) - g(Z,W)\xi.$$

So we have

$$-A_{\omega W}Z + \nabla^{\perp}_Z \omega W - \phi\nabla_Z W - \omega\nabla_Z W - Bh(W,Z) - Ch(W,Z) - g(Z,W)\xi = 0.$$

From the tangential components of the last equation, we have

$$A_{\omega W}Z + \phi\nabla_Z W + Ch(W,Z) + g(Z,W)\xi.$$

from the above equation, we obtain

$$T[W,Z] = A_{\omega W}Z + \phi\nabla_Z W + Ch(W,Z)$$

The anti-invariant distribution $D^{\perp}$ is integrable, $J[Z,W] = \omega[Z,W]$ because tangential component of $J[Z,W]$ is zero. So we have

$$A_{\omega W}Z + \phi\nabla_Z W + Ch(W,Z) = 0. \tag{3.25}$$

Similarly, we get

$$A_{\omega Z}W + \phi\nabla_W Z + Ch(Z,W) = 0. \tag{3.26}$$

Here, by using (3.22), (3.25) and (3.26), we have

$$(\nabla_Z \phi)W = (\nabla_W \phi)Z.$$

**Lemma 1.** *Let $M$ be a pseudo-slant submanifold of a Sasakian manifold $\bar{M}$, Then we get*

$$(\nabla_Z \phi)W = (\nabla_W \phi)Z, \tag{3.27}$$

*for any $Z, W \in \Gamma(D^{\perp})$.*

**Theorem 6.** *Let $M$ be a pseudo-slant submanifold of a Sasakian manifold $\bar{M}$. Then the slant distribution $D_\theta$ is integrable if and only if*

$$P_1\{\nabla_X\phi Y - \phi\nabla_X Y - A_{\omega Y}X - Bh(X,Y) + g(X,Y)\xi - \eta(Y)X\} = 0, \quad (3.28)$$

*for any $X, Y \in \Gamma(D_\theta)$.*

Proof. For any $X, Y \in \Gamma(D_\theta)$, by using (2.4),and considering the tangential component, we have

$$T[X,Y] = \nabla_X\phi Y - \phi\nabla_Y X - A_{\omega Y}X - Bh(X,Y) + g(X,Y)\xi - \eta(Y)X. \quad (3.29)$$

Applying $P_1$ to (3.29), we have (3.28)

**Theorem 7.** *Let $M$ be a pseudo-slant submanifold of a Sasakian manifold $\bar{M}$. Then the slant distribution $D_\theta$ is integrable if and only if*

$$\nabla^\perp_Z\omega W - \nabla^\perp_W\omega Z + h(Z,\phi W) - h(W,\phi Z) \in \mu \oplus \omega(D_\theta),$$

*for any $Z, W \in \Gamma(D_\theta)$.*

Proof. For any $Z, W \in \Gamma(D_\theta)$ and $X \in \Gamma(D^\perp)$, by using (2.3), we obtain

$$\begin{aligned} g([Z,W],X) &= g(\bar{\nabla}_Z W, X) - g(\bar{\nabla}_W Z, X) \\ &= g(J\bar{\nabla}_Z W, JX) + \eta(\bar{\nabla}_Z W)\eta(X) \\ &\quad - g(J\bar{\nabla}_W Z, JX) - \eta(\bar{\nabla}_W Z)\eta(X). \end{aligned}$$

Thus, we have

$$\begin{aligned} g([Z,W],X) &= g(\bar{\nabla}_Z JW, \omega X) - g((\bar{\nabla}_Z J)W, \omega X) \\ &\quad - g(\bar{\nabla}_W JZ, \omega X) + g((\bar{\nabla}_W J)Z, \omega X). \end{aligned}$$

Taking into account (2.4) and (3.1), we get

$$g([Z,W],X) = g(\bar{\nabla}_Z(\phi W + \omega W), \omega X) - g(\bar{\nabla}_W(\phi Z + \omega Z), \omega X).$$

Then from the Gauss and Weingarten formulas the above equation takes the form, we obtain

$$\begin{aligned} g([Z,W],X) &= g(h(Z,\phi W), \omega X) + g(\nabla^\perp_Z\omega W, \omega X) \\ &\quad - g(h(W,\phi Z), \omega X) + g(\nabla^\perp_W\omega Z\omega X). \end{aligned}$$

Since, we have $\omega X \in (D^\perp) \subseteq (T^\perp M)$, we conclude

$$\nabla^\perp_Z\omega W - \nabla^\perp_W\omega Z + h(Z,\phi W) - h(W,\phi Z) \in \mu \oplus \omega(D_\theta).$$

**Theorem 8.** *Let $M$ be a pseudo-slant submanifold of a Sasakian manifold $\bar{M}$. Then the slant distribution $D_\theta$ is integrable if and only if*

$$\phi A_{\omega U}X + A_{\omega U}\phi X = 0,$$

*for any $U \in (D^\perp)$ and $X \in \Gamma(D_\theta)$.*

Proof. For any $U \in (D^{\perp})$ and $X, Y \in \Gamma(D_\theta)$, by direct calculation, we get

$$\begin{aligned} g([X,Y],U) &= g(\bar{\nabla}_X Y - \bar{\nabla}_Y X, U) \\ &= g(J\bar{\nabla}_X Y, JU) - g(J\bar{\nabla}_Y X, JU) \\ &= g(J\bar{\nabla}_X Y, \omega U) - g(J\bar{\nabla}_Y X, \omega U) \\ &= g(\bar{\nabla}_X JY, \omega U) - g(\bar{\nabla}_Y JX, \omega U) \\ &- g((\bar{\nabla}_X J)Y, \omega U) + g((\bar{\nabla}_Y J)X, \omega U) \end{aligned}$$

Hence, by using (2.4) and (3.1), we get

$$\begin{aligned} g([X,Y],U) &= g(\bar{\nabla}_Y \omega U, JX) - g(\bar{\nabla}_X \omega U, JY) \\ &= g(\bar{\nabla}_Y \omega U, \phi X) + g(\bar{\nabla}_Y \omega U, \omega X) \\ &- g(\bar{\nabla}_X \omega U, \phi Y) - g(\bar{\nabla}_X \omega U, \omega Y) \end{aligned}$$

On the other hand, using (2.4), (2.6) and (2.7), we obtain

$$\begin{aligned} (\bar{\nabla}_X J)U &= \bar{\nabla}_X JU - J\bar{\nabla}_X U \\ g(X,U)\xi - \eta(U)X &= \bar{\nabla}_X \omega U - \phi\nabla_X U - \omega\nabla_X U - Bh(X,U) - Ch(X,U) \\ 0 &= \bar{\nabla}_X \omega U - \phi\nabla_X U - \omega\nabla_X U - Bh(X,U) - Ch(X,U), \end{aligned}$$

that is,

$$-A_{\omega U}X + \nabla^{\perp}_X \omega U = \phi\nabla_X U + \omega\nabla_X U + Bh(X,U) + Ch(X,U)$$

From the tangential components, we have

$$\begin{aligned} -A_{\omega U}X &= \phi\nabla_X U + Bh(X,U) \\ (\nabla_X \omega)U &= Ch(X,U). \end{aligned} \tag{3.30}$$

Also, by using (3.7) and (3.30), we obtain

$$\begin{aligned} g([X,Y],U) &= g(A_{\omega U}X, \phi Y) - g(A_{\omega U}Y, \phi X) + g(\nabla^{\perp}_Y \omega U, \omega X) - g(\nabla^{\perp}_X \omega U, \omega Y) \\ &= -g(\phi A_{\omega U}X, Y) - g(A_{\omega U}\phi X, Y) + g((\nabla_Y \omega)U + \omega\nabla_Y U, \omega X) \\ &- g((\nabla_X \omega)U + \omega\nabla_X U, \omega Y) \\ &= -g(\phi A_{\omega U}X, Y) - g(A_{\omega U}\phi X, Y) + g(Ch(Y,U), \omega X) + g(C\nabla_Y U, \omega X) \\ &- g(Ch(X,U), \omega Y) - g(\omega\nabla_X U, \omega Y) \\ &= -g(\phi A_{\omega U}X, Y) - g(A_{\omega U}\phi X, Y) + g(\omega\nabla_Y U, \omega X) - g(\omega\nabla_X U, \omega Y) \\ &= -g(\phi A_{\omega U}X, Y) - g(A_{\omega U}\phi X, Y) + sin^2\theta\{g(\nabla_Y U, X) - g(\nabla_X U, Y)\} \\ &= -g(\phi A_{\omega U}X, Y) - g(A_{\omega U}\phi X, Y) + sin^2\theta\{g(\nabla_X Y, U) - g(\nabla_Y X, U)\} \\ &= -g(\phi A_{\omega U}X, Y) - g(A_{\omega U}\phi X, Y) + sin^2\theta\{g([X,Y],U)\}. \end{aligned}$$

So we have

$$cos^2\theta\{g([X,Y],U)\} = -g(\phi A_{\omega U}X, Y) - g(A_{\omega U}\phi X, Y)$$

Which completes our assertion.
For a pseudo-slant submanifold $M$ of $\bar{M}$, the slant and anti-invariant distributions are totally geodesic in $M$, then $M$ is called pseudo-slant product.
The following theorem characterizes the pseudo-slant product in Sasakian manifold.

**Theorem 9.** *Let $M$ be a pseudo-slant submanifold of a Sasakian manifold $\bar{M}$. Then $M$ is a pseudo-slant product if and only if the second fundamental form $h$ satisfies*

$$Bh(X,Z) = 0, \tag{3.31}$$

*for all $X \in \Gamma(D_\theta)$ and $Z \in \Gamma(TM)$.*

Proof. For all $X, Y \in \Gamma(D_\theta)$ and $U, V \in \Gamma(D^\perp)$, we get

$$\begin{aligned} g(\nabla_X Y, U) &= -g(\nabla_X U, Y) = -g(\bar{\nabla}_X U, Y) \\ &= -g(J\bar{\nabla}_X U, JY) - \eta(\bar{\nabla}_X U)\eta(Y) \\ &= -g((\bar{\nabla}_X J)U - \bar{\nabla}_X JU, JY) \\ &- g(\nabla_X U + h(X,U), \xi)\eta(Y) \\ &= -g(\bar{\nabla}_X JU, JY) - g(\nabla_X U, \xi)\eta(Y) \\ &= -g(\bar{\nabla}_X JU, JY) + g(\nabla_X \xi, U)\eta(Y) \\ &= -g(\bar{\nabla}_X JU, \phi Y) - g(\bar{\nabla}_X JU, \omega Y). \end{aligned}$$

Now, put $JU = \omega U$ and using (3.14), we obtain

$$g(\nabla_X Y, U) = -g(\bar{\nabla}_X \omega U, \phi Y) - g(\bar{\nabla}_X \omega U, \omega Y).$$

Using (2.6) and (2.7), we get

$$\begin{aligned} g(\nabla_X Y, U) &= g(A_{\omega U} X - \nabla^\perp_X \omega U, \phi Y) + g(A_{\omega U} X - \nabla^\perp_X \omega U, \omega Y) \\ &= (A_{\omega U} X, \phi Y) - g((\nabla_X \omega)U, \omega Y) - g(\omega \nabla_X U, \omega Y) \\ &= (A_{\omega U} X, \phi Y) - g(\omega \nabla_X U, \omega Y) - g(Ch(X,U), \omega Y), \end{aligned}$$

Hence using (3.14) and (3.17), we have

$$\begin{aligned} g(\nabla_X Y, U) &= g(A_{\omega U} X, \phi Y) - g(\omega \nabla_X U, \omega Y) \\ &= g(A_{\omega U} X, \phi Y) - sin^2\theta\{g(\nabla_X U, Y) - \eta(\nabla_X U)\eta(Y)\} \\ &= g(h(X,\phi Y), \omega U) - sin^2\theta g(\nabla_X U, Y) + sin^2\theta g(\nabla_X U, \xi)\eta(Y)\} \\ &= g(h(X,\phi Y), \omega U) + sin^2\theta g(\nabla_X Y, U) - sin^2\theta g(\nabla_X \xi, U)\eta(Y)\} \\ &= g(h(X,\phi Y), \omega U) + sin^2\theta g(\nabla_X Y, U) \end{aligned}$$

that is

$$cos^2\theta g(\nabla_X Y, U) = g(h(X,\phi Y), \omega U) = -g(Bh(X,\phi Y), U). \tag{3.32}$$

In the same way, we can obtain

$$\begin{aligned} g(\nabla_V U, X) &= g(\bar{\nabla}_V U, X) = -g(\bar{\nabla}_V X, U) \\ &= -g(J\bar{\nabla}_V X, JU) - \eta(\bar{\nabla}_V X)\eta(U) \\ &= g((\bar{\nabla}_V J)X, JU) - g(\bar{\nabla}_V JX, JU) \end{aligned}$$

For $U, V \in \Gamma(D^{\perp})$, since the tangent component of $JU$ and $\phi U$ are zero, we get

$$\begin{aligned} g(\nabla_V U, X) &= g((\bar{\nabla}_V J)X, \omega U) - g(\bar{\nabla}_V JX, \omega U) \\ &= g(\bar{\nabla}_V JX, \omega U) = -g(\bar{\nabla}_V \phi X, \omega U) - g(\bar{\nabla}_V \omega X, \omega U) \\ &= -g(\nabla_V \phi X + h(\phi X, V), \omega U) + g(A_{\omega X} V - \nabla^{\perp}_V \omega X, \omega U) \\ &= -g(h(\phi X, V), \omega U) - g(\nabla^{\perp}_V \omega X, \omega U) \\ &= -g(h(\phi X, V), \omega U) - g((\nabla_V \omega)X + \omega \nabla_V X, \omega U), \end{aligned}$$

hence using (3.14) we have

$$\begin{aligned} g(\nabla_V U, X) &= -g(h(V, \phi X), \omega U) - g(\omega \nabla_V X, \omega U) \\ &\quad + g(h(V, \phi X), \omega U) - g(Ch(V, X), \omega U) \\ &= -g(\omega \nabla_V X, \omega U) - g(Ch(V, X), \omega U) \\ &= g(Ch(V, X), \omega U) + sin^2\theta g(\nabla_V U, X), \end{aligned}$$

that is

$$cos^2\theta g(\nabla_V U, X) = -g(Ch(V, X), \omega U) = g(Bh(V, X), U). \tag{3.33}$$

From equations (3.32) and (3.33). Thus $D_\theta$ and $D^{\perp}$ are totally geodesic in $M$ if and only if (3.31) is satisfied.

**Theorem 10.** *Let $M$ be a pseudo-slant submanifold of a Sasakian manifold $\bar{M}$. If $\omega$ is parallel on $D_\theta$, then either $M$ is a $D_\theta$-geodesic submanifold or $h(X, Y)$ is an eigenvector of $C^2$ with eigenvalues $-cos^2\theta$, for any $X, Y \in \Gamma(D_\theta)$.*

Proof. For any $X, Y \in \Gamma(D_\theta)$, from (3.11), we have

$$Ch(X, Y) - h(X, \phi Y) = 0 \tag{3.34}$$

On the other hand, since $D_\theta$ is a slant distribution, we have

$$\begin{aligned} 0 &= Ch(X, Y - \eta(Y)\xi) - h(X, \phi(Y - \eta(Y)\xi)) \\ &= Ch(X, Y - \eta(Y)\xi) - h(X, \phi Y), \end{aligned}$$

that is

$$Ch(X, Y - \eta(Y)\xi) = h(X, \phi Y). \tag{3.35}$$

Now, applying $C$ to (3.35), we obtain

$$C^2 h(X, Y - \eta(Y)\xi) = Ch(X, \phi Y).$$

On the other hand, by interchanging of $Y$ and $\phi Y$ in (3.34), we get

$$h(X, \phi^2 Y) = Ch(X, \phi Y).$$

Hence, using (3.15), we obtain

$$C^2 h(X, Y - \eta(Y)\xi) = Ch(X, \phi Y) = h(X, \phi^2 Y) = -cos^2\theta h(X, Y - \eta(Y)\xi).$$

This implies that either $h$ vanishes on $D_\theta$ or $h$ is eigenvector of $C^2$ with eigenvalues $-cos^2\theta$.

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