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ON k-MODIFIED HARMONIC POLYNOMIALS AND THEIR DIMENSIONS[1]

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Abstract

In this article we find out the dimensions of the various spaces of homogeneous k-modified harmonic polynomials for every degree of homogeneity and for all exceptional values of k, the general case in k having been studied before.

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1 Introduction

A function u on a domain in $\mathbb{R}^d$ $(d \geq 2)$ is called *k-modified harmonic* if and only if it satisfies the so-called Weinstein equation,

$$x_d \cdot \Delta u(x_1, \ldots, x_d) + k \cdot \frac{\partial u(x_1, \ldots, x_d)}{\partial x_d} = 0\,, \tag{1}$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_d^2}$ and $k \in \mathbb{R}$. For historical information and the justification of the term the reader is referred to [3]. If u is a polynomial and satisfies (1), it is called *k-modified harmonic polynomial*.

We denote by $\mathcal{H}_n^{(k)}(\mathbb{R}^d)$ the real vector space of all k-modified harmonic polynomials on $\mathbb{R}^d$ that are homogeneous of degree n, whereas $\mathcal{P}_n(\mathbb{R}^d)$ will stand for the real vector space of all homogeneous polynomials of degree n on $\mathbb{R}^d$ $(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$. Since the monomials

$$x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \ldots \cdot x_d^{\alpha_d}$$

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for $\alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{N}_0$, $\alpha_1 + \alpha_2 + \ldots + \alpha_d = n$ form a basis of $\mathcal{P}_n(\mathbb{R}^d)$,

$$\dim \mathcal{P}_n(\mathbb{R}^d) = \binom{d+n-1}{n}.$$

The purpose of this note is to establish the dimension of $\mathcal{H}_n^{(k)}(\mathbb{R}^d)$, in particular for certain exceptional values of k, thus completing the dimensional studies in [3].

We consider the (well-defined) linear map

$$W_n^{(k)} : \mathcal{P}_n(\mathbb{R}^d) \longrightarrow \mathcal{P}_{n-1}(\mathbb{R}^d), \quad W_n^{(k)}(u)(x) := x_d \Delta u(x) + k \cdot \frac{\partial u(x)}{\partial x_d}$$

(for $n = 0$ we define $\mathcal{P}_{-1}(\mathbb{R}^d) := \{0\}$). Obviously, the kernel of $W_n^{(k)}$ is the space $\mathcal{H}_n^{(k)}(\mathbb{R}^d)$. Therefore, if $W_n^{(k)}$ is surjective, the dimension formula for linear maps gives

$$\dim \mathcal{H}_n^{(k)}(\mathbb{R}^d) = \dim \mathcal{P}_n(\mathbb{R}^d) - \dim \mathcal{P}_{n-1}(\mathbb{R}^d)$$
$$= \binom{d+n-1}{n} - \binom{d+n-2}{n-1} = \binom{d+n-2}{n}.$$

This is the general case, as we have proven in [3]:

Theorem 1. *The operator $W_n^{(k)}$ is surjective if and only if $-k \notin 2\mathbb{N}_0 \cap [0, n-1]$. Then,*

$$\dim \mathcal{H}_n^{(k)}(\mathbb{R}^d) = \binom{d+n-2}{n}.$$

2 Study of the exceptional cases

We now focus our attention on the exceptional cases and set $k = -2m$, where $m \in \mathbb{N}_0$, $m \leq \frac{n-1}{2}$.

For $u \in \mathcal{P}_n(\mathbb{R}^d)$, expressed in ascending powers of x_d as

$$u(x_1, \ldots, x_d) = \sum_{j=0}^{n} x_d^j p_j(x_1, \ldots, x_{d-1})$$

with $p_j \in \mathcal{P}_{n-j}(\mathbb{R}^{d-1})$, we have:

$$W_n^{(-2m)}(u) = x_d \Delta u - 2m \cdot \frac{\partial u}{\partial x_d} = x_d \tilde{\Delta} u + x_d \cdot \frac{\partial^2 u}{\partial x_d^2} - 2m \cdot \frac{\partial u}{\partial x_d},$$

where $\tilde{\Delta}$ denotes the Laplacian in the first $d-1$ coordinates. So, we have

$$W_n^{(-2m)}(u) = \sum_{j=0}^{n} x_d^{j+1} \tilde{\Delta} p_j + \sum_{j=0}^{n} j(j-1-2m) x_d^{j-1} p_j$$

$$= \sum_{l=1}^{n-1} x_d^l \tilde{\Delta} p_{l-1} + \sum_{l=0}^{n-1} (l+1)(l-2m) x_d^l p_{l+1} = \sum_{l=0}^{n-1} \left[\tilde{\Delta} p_{l-1} + (l+1)(l-2m) p_{l+1} \right] x_d^l, \tag{2}$$

where we define p_{-1} as the zero polynomial.

We are interested in the image of $W_n^{(-2m)}$ and therefore examine the solvability in u of an equation of the form

$$W_n^{(-2m)}(u) = \sum_{l=0}^{n-1} x_d^l q_l(x_1, \ldots, x_{d-1}), \tag{3}$$

given $q_l \in \mathcal{P}_{n-1-l}(\mathbb{R}^{d-1})$ for $0 \leq l \leq n-1$. At this point we observe that the polynomials q_l for $l > 2m$ do not affect the solvability of (3), because p_{l+1} in (2) can be taken adequately, since its coefficient $(l+1)(l-2m)$ does not vanish. Thus, for the solvability issue, (3) may be "reduced" to

$$W_n^{(-2m)}(u) = \sum_{l=0}^{2m} x_d^l q_l(x_1, \ldots, x_{d-1}). \tag{4}$$

Considering (2), the solving of (4) amounts to finding $p_j \in \mathcal{P}_{n-j}(\mathbb{R}^{d-1})$, $0 \leq j \leq 2m+1$, such that

$$\tilde{\Delta} p_{l-1} + (l+1)(l-2m) p_{l+1} = q_l \tag{5}$$

for $0 \leq l \leq 2m$. There is no problem for the odd values of l: one may set, for instance, $p_0 := 0$, and define inductively

$$p_{2j+2} := \frac{q_{2j+1} - \tilde{\Delta} p_{2j}}{(2j+2)(2j+1-2m)} \quad \text{for } 0 \leq j \leq m-1\,.$$

For the even values of l, however, (5) is not solvable in general. One has to take $p_1 = -\frac{q_0}{2m}$ (for $l = 0$, and here there is a problem if $m = 0$), and the rest of the p_{2j+1} are determined inductively, which amounts to

$$p_{2j+1} = \sum_{l=0}^{j} \frac{\tilde{\Delta}^{j-l} q_{2l} \cdot (-1)^{j-l} \cdot (2l-1)!!}{(2j+1)!!(2l-2m)(2l+2-2m)\ldots(2j-2m)}$$

for $0 \leq j \leq m-1$ ($(-1)!!$ is defined as 1), as can be easily verified. Therefore, (5) imposes the condition ($l = 2m$) $\tilde{\Delta} p_{2m-1} = q_{2m}$, that is,

$$\sum_{l=0}^{m-1} \frac{(-1)^{m-1-l}(2l-1)!!\tilde{\Delta}^{m-l} q_{2l}}{(2m-1)!!(2l-2m)(2l+2-2m)\ldots(-2)} = q_{2m} \iff$$

$$\sum_{l=0}^{m-1} \frac{-(2l-1)!!\tilde{\Delta}^{m-l} q_{2l}}{(2m-1)!!(2m-2l)!!} = q_{2m} \iff \sum_{l=0}^{m} \frac{(2l-1)!!}{(2m-2l)!!} \tilde{\Delta}^{m-l} q_{2l} = 0\,.$$

This leads us to consider the operator $T : \mathcal{P}_{n-1}(\mathbb{R}^d) \longrightarrow \mathcal{P}_{n-1-2m}(\mathbb{R}^{d-1})$,

$$v := \sum_{l=0}^{n-1} x_d^l q_l(x_1, \ldots, x_{d-1}) \longmapsto T(v) := \sum_{l=0}^{m} \frac{(2l-1)!!}{(2m-2l)!!} \tilde{\Delta}^{m-l} q_{2l} \,.$$

Obviously, the image $\mathrm{Im} W_n^{(-2m)}$ of $W_n^{(-2m)}$ is equal to the kernel $\ker T$ of T. Moreover, T is surjective; to see this, given an equation $T(v) = w$, it suffices to set $q_l = 0$ for $l \neq 0$, by which $T(v) = w$ reduces to $\frac{1}{(2m)!!}\tilde{\Delta}^m q_0 = w$, and then to observe that $\tilde{\Delta}^m$ is surjective, since the Laplacian on the space of polynomials is surjective (see [1]). Therefore,

$$\begin{aligned} \dim \ker T &= \dim \mathcal{P}_{n-1}(\mathbb{R}^d) - \dim \mathcal{P}_{n-1-2m}(\mathbb{R}^{d-1}) \\ &= \binom{d+n-2}{n-1} - \binom{d+n-2m-3}{n-1-2m}, \end{aligned}$$

and it follows that

$$\begin{aligned} \dim \mathcal{H}_n^{(-2m)}(\mathbb{R}^d) &= \dim \ker W_n^{(-2m)} = \dim \mathcal{P}_n(\mathbb{R}^d) - \dim \mathrm{Im} W_n^{(-2m)} \\ = \dim \mathcal{P}_n(\mathbb{R}^d) - \dim \ker T &= \binom{d+n-1}{n} - \binom{d+n-2}{n-1} + \binom{d+n-2m-3}{n-1-2m} \\ &= \binom{d+n-2}{n} + \binom{d+n-3-2m}{d-2}. \end{aligned}$$

This result shows that in comparison to the general case, the dimension of the space of k-modified harmonic polynomials that are homogeneous of degree n increases by $\binom{d+n-3-2m}{d-2}$ in the exceptional case $k = -2m$.

Remark 1. If $d = 2$, the surplus of dimensions is always equal to 1.

Remark 2. For $d \geq 3$, the surplus of dimensions decreases as m increases. Recalling that the k-modified harmonic polynomials are in fact the harmonic polynomials with respect to the Laplace-Beltrami operator for the line-element

$$\mathrm{d}l^2 = x_d^{\frac{2k}{d-2}} (\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \ldots \mathrm{d}x_d^2) \,,$$

we see that the Euclidean setting ($k = 0$) provides the richest space of harmonic polynomials.

Remark 3. Recently, Heinz Leutwiler published further results on k-modified harmonic polynomials in the case in which k is a negative even integer and the dimension $d = 3$ (see [2]).

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