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SOME CURVATURE PROPERTIES OF KENMOTSU MANIFOLDS WITH SCHOUTEN-VAN KAMPEN CONNECTION

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Abstract

The aim of the present paper is to study the concircular curvature tensor, projective curvature tensor, Weyl conformal curvature tensor of Kenmotsu manifolds admitting Schouten-van Kampen connection and an example is given to verify our results.

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1 Introduction

The Schouten-van Kampen connection has been introduced for studying nonholomorphic manifolds. It preserves by parallelism, a pair of complementary distributions on a differentiable manifold endowed with an affine connection [2] [6] [13]. Then Olszak has studied the Schouten-van Kampen connection to adapt to an almost contact metric structure [11]. He has characterized some classes of almost contact metric manifolds with the Schouten-van Kampen connection and established certain curvature properties with respect to this connection. Recently Gopal Ghosh [5], Nagaraja [8] and Yildiz [16] have studied the Schouten-van Kampen connection in Sasakian manifolds and f-Kenmotsu manifolds respectively. Kenmotsu manifolds introduced by Kenmotsu in 1971[7] have been extensively studied by many authors [9] [10] [12].

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The present paper is organized as follows: After a brief review of Kenmotsu manifolds in section 2, we prove that if the curvature tensor with respect to the Schouten-van Kampen connection ∇^* vanishes, then the Kenmotsu manifold is locally isometric to the hyperbolic space $H^{2n+1}(-1)$. Thereafter, we study concircularly flat, ξ-concircularly flat, pseudo-concircularly flat and ϕ-concircularly semisymmetric Kenmotsu manifolds with respect to Schouten-van Kampen connection and proved $R^*.C^* = R^*.R^*$. Further, we study the projective curvature tensor, Weyl projective curvature tensor and recurrent conditions of Kenmotsu manifold with respect to the Schouten-van Kampen connection. Finally, in the last section we give an example of a 5-dimensional Kenmotsu manifold admitting Schouten-van Kampen connection to verify our results.

2 Preliminaries

A $(2n+1)$-dimensional smooth manifold M is said to be an almost contact metric manifold if it admits an almost contact metric structure (ϕ, ξ, η, g) consisting of a tensor field ϕ of type $(1,1)$, a vector field ξ, a 1-form η and a Riemannian metric g compatible with (ϕ, ξ, η) satisfying

$$\phi^2 X = -X + \eta(X)\xi,\ \phi\xi = 0,\ g(X,\xi) = \eta(X),\ \eta(\xi) = 1, \eta \circ \phi = 0 \tag{1}$$

and

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2}$$

An almost contact metric manifold is said to be a Kenmotsu manifold [3] if

$$(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X, \tag{3}$$

where ∇ denotes the Riemannian connection of g.
In a Kenmotsu manifold the following relations hold [4].

$$\nabla_X \xi = X - \eta(X)\xi, \tag{4}$$

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y), \tag{5}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{6}$$

$$R(\xi, X)Y = \eta(Y)X - g(X,Y)\xi, \tag{7}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{8}$$

$$S(X,\xi) = -2n\eta(X), \tag{9}$$

$$Q\xi = -2n\xi, \tag{10}$$

$$S(\phi X, \phi Y) = S(X,Y) + 2n\eta(X)\eta(Y), \tag{11}$$

for any vector fields X, Y, Z on M, where R denotes the curvature tensor of type $(1,3)$ on M.

3 Some curvature properties of Kenmotsu manifolds with Schouten-van Kampen connection

Throughout this paper we associate $*$ with the quantities with respect to the Schouten-van Kampen connection. The Schouten-van Kampen connection ∇^* associated to the Levi-Civita connection ∇ is given by [11]

$$\nabla^*_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi, \tag{12}$$

for any vector fields X, Y on M.
Using (4) and (5), the above equation yields,

$$\nabla^*_X Y = \nabla_X Y + g(X,Y)\xi - \eta(Y)X. \tag{13}$$

By taking $Y = \xi$ in (13) and using (4) we obtain

$$\nabla^*_X \xi = 0. \tag{14}$$

We now calculate the Riemann curvature tensor R^* using (13) as follows:

$$R^*(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(X,Z)Y. \tag{15}$$

Using (6) and taking $Z = \xi$ in (15) we get

$$R^*(X,Y)\xi = 0. \tag{16}$$

On contracting (15), we obtain the Ricci tensor S^* of a Kenmotsu manifold with respect to the Schouten-van Kampen connection ∇^* as

$$S^*(Y,Z) = S(Y,Z) + 2ng(Y,Z). \tag{17}$$

This gives

$$Q^*Y = QY + 2nY. \tag{18}$$

Contracting with respect to Y and Z in (17), we get

$$r^* = r + 2n(2n+1), \tag{19}$$

where r^* and r are the scalar curvatures with respect to the Schouten-van Kampen connection ∇^* and the Levi-Civita connection ∇ respectively.

Definition 1. *A Kenmotsu manifold with respect to the Levi-Civita connection is of constant curvature if its curvature tensor R is of the form*

$$g(R(X,Y)Z,U) = k\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\},$$

where k is a constant.

If $R^* = 0$, then the equation (15) becomes

$$R(X, Y, Z, U) = -\{g(Y, Z)g(X, U) - g(X, Z)g(Y, U)\}. \tag{20}$$

From which it follows that the Kenmotsu manifold with respect to the Levi-Civita connection is of constant curvature -1.
This leads to the following :

Theorem 1. *If the curvature tensor of a Kenmotsu manifold with respect to Schouten-van Kampen connection ∇^* vanishes, then the Kenmotsu manifold is locally isometric to the hyperbolic space $H^{2n+1}(-1)$.*

Definition 2. *[1] For each plane p in the tangent space $T_x(M)$, the sectional curvature $K(p)$ is defined by $K(p) = \frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y)-g(X,Y)^2}$, where $\{X, Y\}$ is the orthonormal basis for p. Clearly $K(p)$ is independent of the choice of the orthonormal basis $\{X, Y\}$.*

Taking $Z = X,\ U = Y$ in (20), we get

$$R(X, Y, X, Y) = \{g(X, X)g(Y, Y) - g(X, Y)g(X, Y)\}. \tag{21}$$

Then, from the above equation we conclude that

$$K(p) = \frac{R(X, Y, X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2} = -1. \tag{22}$$

Thus, we can state the following theorem :

Theorem 2. *If in a Kenmotsu manifold, the curvature tensor of a Schouten-van Kampen connection ∇^* vanishes, then the sectional curvature of the plane determined by two vectors $X, Y \in \xi^\perp$ is -1.*

Now, an interesting invariant of a concircular transformation is the concircular curvature tensor. The concircular curvature tensor [14] C^* with respect to the Schouten-van Kampen connection ∇^* is defined by

$$C^*(X, Y)Z = R^*(X, Y)Z - \frac{r^*}{2n(2n+1)}\{g(Y, Z)X - g(X, Z)Y\}, \tag{23}$$

for all vector fields X, Y, Z on M.

By interchanging X and Y in (23), we have

$$C^*(Y, X)Z = R^*(Y, X)Z - \frac{r^*}{2n(2n+1)}\{g(X, Z)Y - g(Y, Z)X\}. \tag{24}$$

By adding (23) and (24) and using the fact that $R(X, Y)Z + R(Y, X)Z = 0$, we get

$$C^*(X, Y)Z + C^*(Y, X)Z = 0. \tag{25}$$

From (15), (23) and first Bianchi identity $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ with respect to ∇, we obtain

$$C^*(X,Y)Z + C^*(Y,Z)X + C^*(Z,X)Y = 0. \tag{26}$$

Hence, (25) and (26), show that concircular curvature tensor with respect to the Schouten-van Kampen connection in a Kenmotsu manifold is skew-symmetric and cyclic.

Next, we assume that the manifold M with respect to the Schouten-van Kampen connection is concircularly flat, that is, $C^*(X,Y)Z = 0$. Then from (23), it follows that

$$R^*(X,Y)Z = \frac{r^*}{2n(2n+1)}\{g(Y,Z)X - g(X,Z)Y\}. \tag{27}$$

Taking the inner product of the above equation with ξ, we have

$$g(R^*(X,Y)Z,\xi) = \frac{r^*}{2n(2n+1)}\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}. \tag{28}$$

Using (1), (8), (15) and (19) in (28), we get

$$\frac{r+2n(2n+1)}{2n(2n+1)}\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} = 0. \tag{29}$$

This implies that either the scalar curvature of M is $r = -2n(2n+1)$ or

$$g(Y,Z)\eta(X) - g(X,Z)\eta(Y) = 0. \tag{30}$$

Taking $Y = \xi$ in (30) and using (1), yields

$$\eta(X)\eta(Z) - g(X,Z) = 0. \tag{31}$$

Replacing X by QX in (31) and using (10), we get

$$S(X,Z) = -2n\eta(X)\eta(Z). \tag{32}$$

Hence we can state the following theorem:

Theorem 3. *For a concircularly flat Kenmotsu manifold with respect to the Schouten-van Kampen connection, either the scalar curvature is $-2n(2n+1)$ or the manifold is a special type of η-Einstein manifold.*

Definition 3. *A Kenmotsu manifold with respect to the Schouten-van Kampen connection ∇^* is said to be ξ- concircularly flat if $C^*(X,Y)\xi = 0$.*

Now, we assume that the manifold M with respect to the Schouten-van Kampen connection is ξ-concircularly flat, that is, $C^*(X,Y)\xi = 0$. Then from (23), it follows that

$$R^*(X,Y)\xi = \frac{r^*}{2n(2n+1)}\{\eta(Y)X - \eta(X)Y\}. \tag{33}$$

In view of (16) and (19), we have

$$\frac{r+2n(2n+1)}{2n(2n+1)}\{\eta(Y)X-\eta(X)Y\}=0. \tag{34}$$

Taking $Y=\xi$ in (34) and using (1), we get

$$\frac{r+2n(2n+1)}{2n(2n+1)}\{X-\eta(X)\xi\}=0. \tag{35}$$

Taking the inner product of the above equation with U, we have

$$\frac{r+2n(2n+1)}{2n(2n+1)}\{g(X,U)-\eta(X)\eta(U)\}=0. \tag{36}$$

This implies that either the scalar curvature of M is $r=-2n(2n+1)$ or

$$g(X,U)-\eta(X)\eta(U)=0. \tag{37}$$

Now, replacing X by QX in (37) and using (10), we get

$$S(X,U)=-2n\eta(X)\eta(U). \tag{38}$$

Hence we can state the following theorem:

Theorem 4. *For a ξ-concircularly flat Kenmotsu manifold with respect to the Schouten-van Kampen connection, either the scalar curvature is $-2n(2n+1)$ or the manifold is a special type of η-Einstein manifold.*

Definition 4. *A Kenmotsu manifold is said to be pseudo-concircularly flat with respect to the Schouten-van Kampen connection ∇^* if it satisfies*

$$g(C^*(\phi X,Y)Z,\phi W)=0, \tag{39}$$

for any vector fields X,Y,Z on M.

In view of (23) and (39), we have

$$g(R^*(\phi X,Y)Z-\frac{r^*}{2n(2n+1)}\{g(Y,Z)\phi X-g(\phi X,Z)Y\},\phi W)=0. \tag{40}$$

Making use of (15) and (19) in (40), we get

$$g(R(\phi X,Y)Z,\phi W)-\frac{r}{2n(2n+1)}\{g(Y,Z)g(\phi X,\phi W)-g(\phi X,Z)Y,g(Y,\phi W)\}=0. \tag{41}$$

Let $\{e_1,e_2,e_3,.......e_{2n+1}\}$ be a local orthonormal basis of vector fields in M. Then by putting $Y=Z=e_i$ in (41) and summing up with respect to $i, 1\leq i\leq 2n+1$, we obtain

$$S(\phi X,\phi W)=\frac{r}{2n+1}g(\phi X,\phi W). \tag{42}$$

On using (1) and (11) in (42), we get

$$S(X,W) = \frac{r}{2n+1}g(X,W) - \{2n + \frac{r}{2n+1}\eta(X)\eta(W)\}. \tag{43}$$

Again taking $X = W = e_i$ in (43) and summing up with respect to $i, 1 \leq i \leq 2n+1$, we obtain

$$r = -2n(2n+1). \tag{44}$$

By virtue of (43) and (44), we get

$$S(X,W) = -2ng(X,W). \tag{45}$$

Thus, M is an Einstein manifold.
Hence, we state the following:

Theorem 5. *Let the Kenmotsu manifold M with Schouten-van Kampen connection be pseudo-concircularly flat if and only if $S(Y,Z) = -2ng(Y,Z)$.*

Definition 5. *[17] A Kenmotsu manifold is said to be ϕ-concircularly semisymmetric with respect to Schouten-van Kampen connection ∇^* if $C^*(X,Y).\phi = 0$ holds on M.*

Now, we consider ϕ-concircularly semisymmetric Kenmotsu manifold with respect to Schouten-van Kampen connection. Then

$$(C^*(X,Y).\phi)Z = C^*(X,Y)\phi Z - \phi C^*(X,Y)Z = 0, \tag{46}$$

for all X, Y, Z.
Taking $Z = \xi$ in (46), we get

$$\phi(C^*(X,Y)\xi) = 0. \tag{47}$$

Using (23) and (6) in (47), we get

$$\frac{r + 2n(2n+1)}{2n(2n+1)}\{\eta(X)\phi Y - \eta(Y)\phi X\} = 0. \tag{48}$$

Replace Y by ξ and X by ϕX in (48) and using (1), we get

$$\frac{r + 2n(2n+1)}{2n(2n+1)}\{X - \eta(X)\xi\} = 0. \tag{49}$$

Taking the inner product of the above equation with U, we have

$$\frac{r + 2n(2n+1)}{2n(2n+1)}\{g(X,U) - \eta(X)\eta(U)\} = 0. \tag{50}$$

This implies that either the scalar curvature of M is $r = -2n(2n+1)$ or

$$g(X,U) - \eta(X)\eta(U) = 0. \tag{51}$$

Now, replacing X by QX in (51) and using (10), we get

$$S(X,U) = -2n\eta(X)\eta(U). \tag{52}$$

Hence we can state the following:

Theorem 6. *For a ϕ-concircularly semisymmetric Kenmotsu manifold with respect to the Schouten-van Kampen connection, either the scalar curvature is $-2n(2n+1)$ or the manifold is a special type of η-Einstein manifold.*

Further, we have

$$\begin{aligned}(R^*(X,Y).C^*)(U,V,W) = R^*(X,Y)C^*(U,V)W - C^*(R^*(X,Y)U,V)W \\ - C^*(U,R^*(X,Y)V)W - C^*(U,V)R^*(X,Y)W.\end{aligned} \tag{53}$$

With the use of (23), (53) becomes

$$\begin{aligned}&(R^*(X,Y).C^*)(U,V,W) = R^*(X,Y)R^*(U,V)W - R^*(R^*(X,Y)U,V)W \\ &- R^*(U,R^*(X,Y)V)W - R^*(U,V)R^*(X,Y)W \\ &+ \frac{r^*}{2n(2n+1)}\{g(R^*(X,Y)V,W)U + g(V,R^*(X,Y)W)U - g(R^*(X,Y)U,W)V \\ &- g(U,R^*(X,Y)W)V\}.\end{aligned} \tag{54}$$

By the symmetric properties of the curvature tensor R^* [12, **?**], we get

$$\begin{aligned}(R^*(X,Y).C^*)(U,V,W) = R^*(X,Y)R^*(U,V)W - R^*(R^*(X,Y)U,V)W \\ - R^*(U,R^*(X,Y)V)W - R^*(U,V)R^*(X,Y)W.\end{aligned} \tag{55}$$

Finally, we get

$$(R^*(X,Y).C^*)(U,V,W) = (R^*(X,Y).R^*)(U,V,W). \tag{56}$$

Thus we state the following:

Theorem 7. *Let M be a Kenmotsu manifold with Schouten-van Kampen connection. Then $R^*.C^* = R^*.R^*$.*

Now, the projective curvature tensor [15] P^* with respect to the Schouten-van Kampen connection ∇^* is defined by

$$P^*(X,Y)Z = R^*(X,Y)Z - \frac{1}{2n}\{S^*(Y,Z)X - S^*(X,Z)Y\}. \tag{57}$$

By interchanging X and Y in (57), we have

$$P^*(Y,X)Z = R^*(Y,X)Z - \frac{1}{2n}\{S^*(X,Z)Y - S^*(Y,Z)X\}. \tag{58}$$

By adding (57) and (58) and using the fact that $R(X,Y)Z + R(Y,X)Z = 0$, we get

$$P^*(X,Y)Z + P^*(Y,X)Z = 0. \tag{59}$$

From (15), (57) and the first Bianchi identity $R(X,Y)Z+R(Y,Z)X+R(Z,X)Y = 0$ with respect to ∇, we obtain

$$P^*(X,Y)Z + P^*(Y,Z)X + P^*(Z,X)Y = 0. \tag{60}$$

Hence, (59) and (60), show that the projective curvature tensor with respect to the Schouten-van Kampen connection in a Kenmotsu manifold is skew-symmetric and cyclic.
Now, by taking $Z = \xi$ in (57), using (16) and (17), we get

$$P^*(X, Y)\xi = 0. \tag{61}$$

Thus, we can state the following:

Theorem 8. *Let M be a Kenmotsu manifold with the Schouten-van Kampen connection being ξ-projectively flat.*

Definition 6. *A Kenmotsu manifold is said to be ϕ-projectively semisymmetric with respect to the Schouten-van Kampen connection ∇^* if*

$$P^*(X, Y).\phi = 0, \tag{62}$$

for any vector fields X, Y on M.

Now, (62) turns into

$$(P^*(X, Y).\phi)Z = P^*(X, Y)\phi Z - \phi P^*(X, Y)Z = 0. \tag{63}$$

Making use of (57), (15) and (17) in (63), we get

$$\begin{aligned} &R(X, Y)\phi Z - \phi R(X, Y)Z - \frac{1}{2n}\{S(Y, \phi Z)X - S(X, \phi Z)Y \\ &+ S(X, Z)\phi Y - S(Y, Z)\phi X\} = 0. \end{aligned} \tag{64}$$

Taking Y by ξ in (64), using (6) and (9), we get

$$S(X, \phi Z)\xi = -2ng(X, \phi Z)\xi. \tag{65}$$

Taking an inner product with ξ and Replacing X by ϕX, using (2) and (11) in (65), we get

$$S(Y, Z) = -2ng(Y, Z). \tag{66}$$

and

$$r = -2n(2n + 1). \tag{67}$$

Again by substituting (67) in (57), we obtain

$$P^*(X, Y)Z = R(X, Y)Z + \{g(Y, Z)X - g(X, Z)Y\}. \tag{68}$$

Thus we can state the following :

Theorem 9. *Let M be a Kenmotsu manifold with the Schouten-van Kampen connection being ϕ-projectively semisymmetric if and only if $S(Y, Z) = -2ng(Y, Z)$. Further, if $P^* = 0$ then M is isomorphic to the hyperbolic space $H^{2n+1}(-1)$.*

In a Riemannian manifold, the Weyl conformal curvature tensor K^* with respect to the Schouten-van Kampen connection ∇^* is defined as

$$K^*(X,Y)Z = R^*(X,Y)Z - \frac{1}{2n-1}\{S^*(Y,Z)X - S^*(X,Z)Y + g(Y,Z)Q^*X - g(X,Z)Q^*Y\} + \frac{r^*}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}. \tag{69}$$

By making use of (15), (17), (18), (19) in (69) yields

$$K^*(X,Y)Z = K(X,Y)Z. \tag{70}$$

for all X, Y, Z. Thus we state the following:

Theorem 10. *The Weyl conformal curvature tensor of Kenmotsu manifold with respect to the LeviCivita connection and the Schouten-van Kampen connection are equivalent.*

Definition 7. *A Kenmotsu manifold with respect to the Schouten-van Kampen connection ∇^* is called recurrent, if its curvature tensor R^* satisfies the condition*

$$(\nabla^*_W R^*)(X,Y)Z = A(W)R^*(X,Y)Z, \tag{71}$$

where R^* is the curvature tensor with respect to the connection ∇^*.

Using (71), we can write

$$\nabla^*_W R^*(X,Y)Z - R^*(\nabla^*_W X,Y)Z - R^*(X,\nabla^*_W Y)Z - R^*(X,Y)\nabla^*_W Z = A(W)R^*(X,Y)Z. \tag{72}$$

Making use of (13), (15) and (17) in (72), we get

$$g(W,R(X,Y)Z)\xi - g(W,X)R(\xi,Y)Z - g(W,Y)R(X,\xi)Z - g(W,Z)R(X,Y)\xi - \eta(R(X,Y)Z)W + \eta(X)R(W,Y)Z + \eta(Y)R(X,W)Z + \eta(Z)R(X,Y)W - \eta(W)\{\phi R(X,Y)Z - R(\phi X,Y)Z - R(X,\phi Y)Z - R(X,Y)\phi Z\} = A(W)\{g(Y,Z)X - g(X,Z)Y\}. \tag{73}$$

Replacing Z by ξ and using (1), (6), (7) and (8), we get

$$A(W)\{\eta(Y)X - \eta(X)Y\} = g(W,Y)X - g(W,X)Y + R(X,Y)W. \tag{74}$$

Taking an inner product with U in (74), we have

$$A(W)\{\eta(Y)g(X,U) - \eta(X)g(Y,U)\} = g(W,Y)g(X,U) - g(W,X)g(Y,U) + R(X,Y,W,U). \tag{75}$$

Let $\{e_1, e_2, e_3, ...e_{2n+1}\}$ be a local orthonormal basis of vector fields in M. Then by putting $X = U = e_i$ in (75) and summing up with respect to $i, 1 \leq i \leq 2n+1$, we obtain

$$S(Y, W) = -2n\{g(Y, W) + \eta(Y)A(W)\}. \tag{76}$$

Suppose the associated 1-form A is equal to the associated 1-form η, then from (76), we get

$$S(Y, W) = -2n\{g(Y, W) + \eta(Y)\eta(W)\}. \tag{77}$$

Thus we state the following :

Theorem 11. *If a Kenmotsu manifold whose curvature tensor of manifold is covariant constant with respect to the Schouten-van Kampen connection, the manifold is recurrent and the associated* 1*-form* A *is equal to the associated* 1*-form* η*, then the manifold is an* η*-Einstein manifold.*

4 Example of a 5-dimensional Kenmotsu manifold with respect to the Schouten-van Kampen connection

We consider the five-dimensional manifolod $M = \{(x, y, z, u, v) \in R^5\}$, where (x, y, z, u, v) are the standard coordinates in R^5. The vector fields

$$E_1 = e^{-v}\frac{\partial}{\partial x}, \quad E_2 = e^{-v}\frac{\partial}{\partial y}, \quad E_3 = e^{-v}\frac{\partial}{\partial z}, \quad E_4 = e^{-v}\frac{\partial}{\partial u}, \quad E_5 = e^{-v}\frac{\partial}{\partial v}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, E_3)$ for any $Z \in \chi(M)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi E_1 = E_3, \phi E_2 = E_4, \phi E_3 = -E_1, \phi E_4 = -E_2, \phi E_5 = 0$. Then using the linearity of ϕ and g we have

$$\eta(E_5) = 1, \quad \phi^2(Z) = -Z + \eta(Z)E_5, \quad g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U),$$

for any $Z, U \in \chi(M)$. Thus for $E_5 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[E_1, E_2] = [E_1, E_3] = [E_1, E_4] = [E_2, E_3] = 0, \quad [E_1, E_5] = E_1,$$
$$[E_4, E_5] = E_4, \quad [E_2, E_4] = [E_3, E_4] = 0, \quad [E_2, E_5] = E_2, \quad [E_3, E_5] = E_3.$$

The Riemannian connection ∇ of metric g is given by the Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

By Koszul's formula, we get

$$\begin{aligned}
&\nabla_{E_1}E_1 = -E_5, \quad \nabla_{E_1}E_2 = 0, \quad \nabla_{E_1}E_3 = 0, \quad \nabla_{E_1}E_4 = 0, \quad \nabla_{E_1}E_5 = E_1,\\
&\nabla_{E_2}E_1 = 0, \quad \nabla_{E_2}E_2 = -E_5, \quad \nabla_{E_2}E_3 = 0, \quad \nabla_{E_2}E_4 = 0, \quad \nabla_{E_2}E_5 = E_2,\\
&\nabla_{E_3}E_1 = 0, \quad \nabla_{E_3}E_2 = 0, \quad \nabla_{E_3}E_3 = -E_5, \quad \nabla_{E_3}E_4 = 0, \quad \nabla_{E_3}E_5 = E_3,\\
&\nabla_{E_4}E_1 = 0, \quad \nabla_{E_4}E_2 = 0, \quad \nabla_{E_4}E_3 = 0, \quad \nabla_{E_4}E_4 = -E_5, \quad \nabla_{E_4}E_5 = E_4,\\
&\nabla_{E_5}E_1 = 0, \quad \nabla_{E_5}E_2 = 0, \quad \nabla_{E_5}E_3 = 0, \quad \nabla_{E_5}E_4 = 0, \quad \nabla_{E_5}E_5 = 0.
\end{aligned}$$

Further, we obtain the following:

$$\nabla^*_{E_i}E_j = 0, \quad i, \quad j = 1,2,3,4,5.$$

and hence

$$(\nabla^*_{E_i}\phi)E_j = 0, \quad i, \quad j = 1,2,3,4,5.$$

From the above expressions it follows that the manifold satisfies (2), (3) and (4) for $\xi = E_5$. Hence the manifold is a Kenmotsu manifold. With the help of the above results we can verify the following results.

$$\begin{aligned}
&R(E_1,E_2)E_2 = R(E_1,E_3)E_3 = R(E_1,E_4)E_4 = R(E_1,E_5)E_5 = -E_1,\\
&R(E_1,E_2)E_1 = E_2, \quad R(E_1,E_3)E_1 = R(E_5,E_3)E_5 = R(E_2,E_3)E_5 = E_3,\\
&R(E_2,E_3)E_3 = R(E_2,E_4)E_4 = R(E_2,E_5)E_5 = -E_2, \quad R(E_3,E_4)E_4 = -E_3,\\
&R(E_2,E_5)E_2 = R(E_1,E_5)E_1 = R(E_4,E_5)E_4 = R(E_3,E_5)E_3 = E_5,\\
&R(E_1,E_4)E_1 = R(E_2,E_4)E_2 = R(E_3,E_4)E_3 = R(E_5,E_4)E_5 = E_4
\end{aligned}$$

and

$$R^*(E_i,E_j)E_k = 0, \quad i, \quad j, \quad k = 1,2,3,4,5.$$

From the above expressions of the curvature tensor of the Kenmotsu manifold it can be easily seen that the manifold has a constant sectional curvature -1. Making use of the above results we obtain the Ricci tensors as follows:

$$\begin{aligned}
S(E_1,E_1) = {}& g(R(E_1,E_2)E_2,E_1) + g(R(E_1,E_3)E_3,E_1) + g(R(E_1,E_4)E_4,E_1)\\
&+ g(R(E_1,E_5)E_5,E_1) = -4.
\end{aligned}$$

Similarly, we have

$$S(E_2,E_2) = S(E_3,E_3) = S(E_3,E_3) = S(E_4,E_4) = S(E_5,E_5) = -4 \tag{78}$$

and

$$S^*(E_1,E_1) = S^*(E_2,E_2) = S^*(E_3,E_3) = S^*(E_4,E_4) = S^*(E_5,E_5) = 0.$$

and

$$r = \sum_{i=1}^{5} S(e_i,e_i) = -20 \quad \text{and} \quad r^* = \sum_{i=1}^{5} S^*(e_i,e_i) = 0.$$

Therefore, from (78) it can be easily verified that the manifold is an Einstein manifold with respect to the Levi-Civita connection.

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