Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 6(68), No. 1 - 2026, 201-230
https://doi.org/10.31926/but.mif.2026.6.68.1.14

SOME IMPORTANT CHARACTERIZATIONS OF PSEUDO $\mathcal{W}_8$-CURVATURE TENSOR WITHIN THE FRAMEWORK OF $(LCS)_n$-MANIFOLDS

Pranjal SHARMA[1] and Gyanvendra Pratap SINGH[*,2]

Abstract

This study investigates the pseudo $\mathcal{W}_8$-curvature tensor $\widehat{\mathcal{W}}_8$ on $(LCS)_n$-manifolds. The study examines pseudo $\mathcal{W}_8$-flat and ζ-pseudo $\mathcal{W}_8$-flat $(LCS)_n$-manifolds, leading to notable findings. An $(LCS)_n$-manifold satisfying the φ-pseudo $\mathcal{W}_8$- semisymmetric condition is found to be a generalized η-Einstein manifold. Curvature conditions $\widehat{\mathcal{W}}_8 \cdot Q = 0$, $\widehat{\mathcal{W}}_8 \cdot R = 0$, and $R(\zeta, \mathcal{H}_1) \cdot \widehat{\mathcal{W}}_8 = 0$ lead to the manifold being Einstein or η-Einstein. The pseudo $\mathcal{W}_8$-Ricci pseudosymmetric condition and η-Ricci solitons under $\widehat{\mathcal{W}}_8(\zeta, \mathcal{H}_1) \cdot S = 0$ and $S(\zeta, \mathcal{H}_1) \cdot \widehat{\mathcal{W}}_8 = 0$ yield notable results. Conservative, irrotational, cyclic parallel, and η-parallel Ricci tensors are also analyzed.

2020 Mathematics Subject Classification: 53B30, 53C15, 53C25, 53D15.
Key words: $\mathcal{W}_8$-curvature tensor, $(LCS)_n$-manifolds, φ-semisymmetric, Ricci pseudosymmetric, η-Einstein manifolds, η-Ricci solitons.

1 Introduction

In 2003, Shaikh [30] introduced Lorentzian concircular structure manifolds, or $(LCS)_n$-manifolds, providing an example that extended the concept of Lorentzian para-Sasakian manifolds previously defined by Matsumoto [20] as well as by Mihai and Rosca [21]. Later, Shaikh and Baishya [33, 34] explored the relevance of $(LCS)_n$-manifolds to general relativity and cosmology. Various researchers have since examined these manifolds, including Atceken and collaborators [1, 2, 15], Hui [14], Hui and Chakraborty [16], Narain and Yadav [22], Prakasha [26], and Shaikh along with co-authors [31, 32, 35, 36, 37, 39, 40], among others.

[1]Department of Mathematics and Statistics, Deen Dayal Upadhyaya Gorakhpur University, Gorakhpur-273009, Uttar Pradesh, e-mail: pranjal.sharma.gkp@gmail.com
[2*] *Corresponding author*, Department of Mathematics and Statistics, Deen Dayal Upadhyaya Gorakhpur University, Gorakhpur-273009, Uttar Pradesh, e-mail: gpsingh.singh700@gmail.com

The Pseudo $\mathcal{W}_8$-curvature tensor has been defined by Prasad, Yadav and Pandey [29], as

$$\begin{aligned}\widehat{\mathcal{W}}_8(\mathcal{H}_1,\mathcal{H}_2)\mathcal{H}_3 &= aR(\mathcal{H}_1,\mathcal{H}_2)\mathcal{H}_3 + b\left[S(\mathcal{H}_1,\mathcal{H}_2)\mathcal{H}_3 - S(\mathcal{H}_2,\mathcal{H}_3)\mathcal{H}_1\right] \\ &\quad - \frac{r}{n}\left[\frac{a}{(n-1)} - b\right]\left[g(\mathcal{H}_1,\mathcal{H}_2)\mathcal{H}_3 - g(\mathcal{H}_2,\mathcal{H}_3)\mathcal{H}_1\right], \qquad (1)\end{aligned}$$

where a and b are constants with the condition that $a, b \neq 0$, $R(\mathcal{H}_1,\mathcal{H}_2)\mathcal{H}_3$ denotes the *curvature tensor* and $S(\mathcal{H}_1,\mathcal{H}_2)$ denote the *Ricci tensor* of the manifold. If $a = 1$ and $b = \frac{1}{n-1}$, then (1) becomes

$$\widehat{\mathcal{W}}_8(\mathcal{H}_1,\mathcal{H}_2)\mathcal{H}_3 = R(\mathcal{H}_1,\mathcal{H}_2)\mathcal{H}_3 + \frac{1}{(n-1)}\left[S(\mathcal{H}_1,\mathcal{H}_2)\mathcal{H}_3 - S(\mathcal{H}_2,\mathcal{H}_3)\mathcal{H}_1\right], \qquad (2)$$

This represents the $\mathcal{W}_8$-curvature tensor as described by Tripathi and Gupta [41]. Hence, we observe that the $\mathcal{W}_8$-curvature tensor is a special case of the tensor $\widehat{\mathcal{W}}_8$. For this reason, $\widehat{\mathcal{W}}_8$ is referred to as the pseudo $\mathcal{W}_8$-curvature tensor.

In 1982, Hamilton [12] introduced Ricci flow to identify a canonical metric on smooth manifolds. Since its introduction, Ricci flow has become a powerful tool for studying Riemannian manifolds, particularly those with positive curvature. Perelman [24, 25] applied Ricci flow and surgical techniques to prove the Poincaré conjecture. The Ricci flow is an evolution equation governing metrics on a Riemannian manifold and is given by

$$\frac{\partial g}{\partial t} = -2S. \qquad (3)$$

A Ricci soliton is the limiting form of a soliton within the Ricci flow. A solution to the Ricci flow is classified as a Ricci soliton [12, 13] if it evolves under the action of a one-parameter group of diffeomorphisms and scaling transformations. The equation governing a Ricci soliton is given by

$$\pounds_{\mathcal{H}_1} g + 2S + 2\vartheta g = 0, \qquad (4)$$

where S represents the Ricci tensor, g is the Riemannian metric, $\pounds_{\mathcal{H}_1}$ denotes the Lie derivative with respect to the vector field $\mathcal{H}_1$, and ϑ is a scalar.

In this equation, when $\vartheta < 0$, it describes a shrinking Ricci soliton; when $\vartheta > 0$, it defines an expanding Ricci soliton; and when $\vartheta = 0$, it is referred to as a steady Ricci soliton. Numerous researchers have studied Ricci solitons, including those mentioned in references [8, 9, 13, 17], along with several others.

The concept of η-Ricci solitons was introduced by Cho and Kimura [6] as a generalization of Ricci solitons for Hopf hypersurfaces in complex space forms. An η-Ricci soliton is defined by a tuple $(g, \zeta, \vartheta, \Psi)$, where ζ is a vector field on $\mathcal{M}$, ϑ and Ψ are real scalars and g is the Riemannian metric that satisfies a specific equation

$$\pounds_{\zeta} g + 2S + 2\vartheta g + 2\Psi \eta \otimes \eta = 0, \qquad (5)$$

where $\pounds_\zeta$ denotes the Lie derivative in the direction of the vector field ζ and S represents the Ricci tensor. In this context, we acknowledge the contributions of Blaga ([4, 3, 5]), Prakasha et al. [28], De and De [7], Kar, D. et al. [18], Eyasmin et al. [11], and many others who have investigated η-Ricci solitons.

In general, if $\Psi = 0$, the η-Ricci soliton reduces to a Ricci soliton. Conversely, if $\Psi \neq 0$, the η-Ricci soliton is termed a proper η-Ricci soliton.

The arrangement of this paper is structured as follows: Section 2 introduces preliminary concepts related to $(LCS)_n$-manifolds, providing the foundational framework for the study. Section 3 addresses pseudo $\mathcal{W}_8$-flat $(LCS)_n$-manifolds, presenting some interesting results. Section 4 examines ζ-pseudo $\mathcal{W}_8$-flat $(LCS)_n$-manifolds, identifying these as particular types of η-Einstein manifolds. Section 5 explores the φ-pseudo $\mathcal{W}_8$-semisymmetric condition in $(LCS)_n$-manifolds, demonstrating that such manifolds are generalized η-Einstein manifolds. In Section 6, the study examines $(LCS)_n$-manifolds satisfying the condition $\widehat{\mathcal{W}}_8 \cdot Q = 0$, showing that they are Einstein manifolds. Section 7 extends this analysis to $(LCS)_n$-manifolds under the condition $\widehat{\mathcal{W}}_8 \cdot R = 0$, classifying them as η-Einstein manifolds. Section 8 considers $(LCS)_n$-manifolds under the condition $R(\zeta, \mathcal{H}_1) \cdot \widehat{\mathcal{W}}_8 = 0$, concluding that these are also η-Einstein manifolds. Section 9 investigates $(LCS)_n$-manifolds that satisfy the pseudo $\mathcal{W}_8$-Ricci pseudosymmetric condition, establishing that they are η-Einstein manifolds as well. Section 10 examines η-Ricci solitons on $(LCS)_n$-manifolds under the condition $\widehat{\mathcal{W}}_8(\zeta, \mathcal{H}_1) \cdot S = 0$, while Section 11 considers $(LCS)_n$-manifolds with η-Ricci solitons satisfying $S(\zeta, \mathcal{H}_1) \cdot \widehat{\mathcal{W}}_8 = 0$, revealing additional notable results. Section 12 discusses conservative pseudo $\mathcal{W}_8$-curvature tensors on $(LCS)_n$-manifolds, presenting interesting findings. Section 13 investigates irrotational pseudo $\mathcal{W}_8$-curvature tensors on $(LCS)_n$-manifolds, showing that these manifolds are Einstein. Finally, Section 14 examines cyclic parallel Ricci tensors, while Section 15 discusses η-parallel Ricci tensors on $(LCS)_n$-manifolds, uncovering further intriguing results.

2 Preliminaries

An n-dimensional Lorentzian manifold $\mathcal{M}$ is a smooth, connected, paracompact Hausdorff manifold equipped with a Lorentzian metric g. This means that $\mathcal{M}$ possesses a smooth symmetric tensor field g of type (0, 2), such that at each point $p \in \mathcal{M}$, the tensor $g_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ defines a non-degenerate inner product with signature $(-, +, \cdots, +)$. Here, $T_p\mathcal{M}$ represents the tangent space to $\mathcal{M}$ at the point p. A non-zero vector $v \in T_p\mathcal{M}$ is classified as timelike, null, or spacelike depending on whether it satisfies $g_p(v, v) < 0$, $g_p(v, v) = 0$, or $g_p(v, v) > 0$, respectively [23].

Definition 2.1 *In a Lorentzian manifold* $(\mathcal{M}, g)$*, a vector field* P*, defined by the condition* $g(\mathcal{H}_1, P) = A(\mathcal{H}_1)$*, where* $\mathcal{H}_1$ *is any element of* $\Gamma(TM)$*, the space* *of all smooth sections of tangent vector fields on M. The vector field P is called a concircular vector field if the following condition is satisfied*([30, 33])

$$(\nabla_{\mathcal{H}_1} A)(\mathcal{H}_2) = \alpha\{g(\mathcal{H}_1, \mathcal{H}_2) + \omega(\mathcal{H}_1)A(\mathcal{H}_2)\}. \tag{6}$$

Here, α is a non-zero scalar function, ω is a closed 1-form, and ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Let $\mathcal{M}$ be an n-dimensional Lorentzian manifold that admits a unit timelike concircular vector field ζ, referred to as the characteristic vector field of the manifold. In this case, the following condition holds

$$g(\zeta, \zeta) = -1. \tag{7}$$

Since ζ is a unit concircular vector field, there exists a non-zero 1-form η such that

$$g(\mathcal{H}_1, \zeta) = \eta(\mathcal{H}_1), \tag{8}$$

for any vector field $\mathcal{H}_1$. The following equation is satisfied

$$(\nabla_{\mathcal{H}_1} \eta)(\mathcal{H}_2) = \alpha\{g(\mathcal{H}_1, \mathcal{H}_2) + \eta(\mathcal{H}_1)\eta(\mathcal{H}_2)\}, \quad \alpha \neq 0, \tag{9}$$

which leads to the expression

$$\nabla_{\mathcal{H}_1} \zeta = \alpha[\mathcal{H}_1 + \eta(\mathcal{H}_1)\zeta], \tag{10}$$

for all vector fields $\mathcal{H}_1$ and $\mathcal{H}_2$. Additionally, α satisfies the condition

$$\nabla_{\mathcal{H}_1} \alpha = (\mathcal{H}_1 \alpha) = d\alpha(\mathcal{H}_1) = \rho\eta(\mathcal{H}_1), \tag{11}$$

where ρ is a scalar function defined by $\rho = -(\zeta\alpha)$. By defining

$$\varphi\mathcal{H}_1 = \frac{1}{\alpha}\nabla_{\mathcal{H}_1} \zeta, \tag{12}$$

we can derive from (10) and (12) that

$$\varphi\mathcal{H}_1 = \mathcal{H}_1 + \eta(\mathcal{H}_1)\zeta. \tag{13}$$

This shows that φ is a symmetric $(1,1)$ tensor known as the structure tensor of the manifold. Consequently, the Lorentzian manifold $\mathcal{M}$, along with the unit timelike concircular vector field ζ, its associated 1-form η, and the $(1,1)$ tensor field φ, is referred to as a Lorentzian concircular structure manifold (abbreviated as $(LCS)_n$-manifold) [31]. Notably, if we set $\alpha = 1$, we can obtain the Lorentzian para-Sasakian structure as defined by Matsumoto [20]. For an $(LCS)_n$-manifold with $n > 2$, the following relations hold [31, 33, 34, 35]

$$\varphi^2\mathcal{H}_1 = \mathcal{H}_1 + \eta(\mathcal{H}_1)\zeta, \tag{14}$$

$$g(\varphi\mathcal{H}_1, \varphi\mathcal{H}_2) = g(\mathcal{H}_1, \mathcal{H}_2) + \eta(\mathcal{H}_1)\eta(\mathcal{H}_2), \tag{15}$$

$$\eta(\zeta) = -1, \quad \varphi\zeta = 0, \quad \eta(\varphi\mathcal{H}_1) = 0, \tag{16}$$

$$R(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 = (\alpha^2 - \rho)[g(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1 - g(\mathcal{H}_1, \mathcal{H}_3)\mathcal{H}_2], \tag{17}$$

$$\eta(R(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3) = (\alpha^2 - \rho)[g(\mathcal{H}_2, \mathcal{H}_3)\eta(\mathcal{H}_1) - g(\mathcal{H}_1, \mathcal{H}_3)\eta(\mathcal{H}_2)], \tag{18}$$

$$R(\mathcal{H}_1, \mathcal{H}_2)\zeta = (\alpha^2 - \rho)[\eta(\mathcal{H}_2)\mathcal{H}_1 - \eta(\mathcal{H}_1)\mathcal{H}_2], \tag{19}$$

$$S(\mathcal{H}_1, \zeta) = (n-1)(\alpha^2 - \rho)\eta(\mathcal{H}_1), \tag{20}$$

$$Q\mathcal{H}_1 = (n-1)(\alpha^2 - \rho)\mathcal{H}_1, \tag{21}$$

$$(\nabla_{\mathcal{H}_1}\varphi)\mathcal{H}_2 = \alpha[g(\mathcal{H}_1, \mathcal{H}_2)\zeta + 2\eta(\mathcal{H}_1)\eta(\mathcal{H}_2)\zeta + \eta(\mathcal{H}_2)\mathcal{H}_1], \tag{22}$$

$$(\mathcal{H}_1\rho) = d\rho(\mathcal{H}_1) = \beta\eta(\mathcal{H}_1), \tag{23}$$

$$S(\varphi\mathcal{H}_1, \varphi\mathcal{H}_2) = S(\mathcal{H}_1, \mathcal{H}_2) + (n-1)(\alpha^2 - \rho)\eta(\mathcal{H}_1)\eta(\mathcal{H}_2), \tag{24}$$

for any vector fields $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ on $\mathcal{M}$ and $\beta = -(\zeta\rho)$ denote a scalar function, where R, S and Q denote the curvature tensor, Ricci tensor and Ricci operator respectively such that $g(Q\mathcal{H}_1, \mathcal{H}_2) = S(\mathcal{H}_1, \mathcal{H}_2)$ on $\mathcal{M}$.

Let $\{e_1, e_2, e_3, \ldots, e_n = \zeta\}$ be an orthonormal basis for the tangent space at any point on the manifold $\mathcal{M}$. The Ricci tensor S and the scalar curvature r of the manifold are given by the following expression

$$S(\mathcal{H}_1, \mathcal{H}_2) = \sum_{i=1}^{n} \varepsilon_i g(R(e_i, \mathcal{H}_1)\mathcal{H}_2, e_i), \tag{25}$$

where ε_i are the signs corresponding to the metric signature.
On $(LCS)_n$-manifolds, the scalar curvature r is given by

$$r = \sum_{i=1}^{n} \varepsilon_i S(e_i, e_i), \tag{26}$$

where ε_i are the signs corresponding to the metric signature. Additionally, we have

$$g(\mathcal{H}_1, \mathcal{H}_2) = \sum_{i=1}^{n} \varepsilon_i g(\mathcal{H}_1, e_i)g(\mathcal{H}_2, e_i), \tag{27}$$

where $\mathcal{H}_1, \mathcal{H}_2 \in \chi(\mathcal{M})$ and $\varepsilon_i = g(e_i, e_i) = \pm 1$.

Definition 2.2 *An $(LCS)_n$-manifold $\mathcal{M}$ is referred to as a generalized η-Einstein manifold if its Ricci tensor S takes the form*

$$S(\mathcal{H}_1, \mathcal{H}_2) = \nu_1 g(\mathcal{H}_1, \mathcal{H}_2) + \nu_2\eta(\mathcal{H}_1)\eta(\mathcal{H}_2) + \nu_3\Phi(\mathcal{H}_1, \mathcal{H}_2), \tag{28}$$

where $\Phi(\mathcal{H}_1, \mathcal{H}_2) = g(\varphi\mathcal{H}_1, \mathcal{H}_2)$ and ν_1, ν_2, and ν_3 are smooth functions on $\mathcal{M}$. If $\nu_3 = 0$, then $\mathcal{M}$ is said to be an η-Einstein manifold.

If $\nu_2 = 0, \nu_3 = 0$, then $\mathcal{M}$ is said to be an Einstein manifold.

In view of (28), we have

$$Q\mathcal{H}_1 = \nu_1\mathcal{H}_1 + \nu_2\eta(\mathcal{H}_1)\zeta + \nu_3\varphi\mathcal{H}_1. \tag{29}$$

Let us consider $(LCS)_n$-manifold. Then putting $\mathcal{H}_1 = \mathcal{H}_2 = e_i$ in (28), $i = 1, 2, 3, ..., n$ and taking summation for $1 \leq i \leq n$, we have

$$r = \nu_1 n - \nu_2 + \nu_3\varrho, \tag{30}$$

where $\varrho = trace(\varphi)$.
Now putting $\mathcal{H}_1 = \mathcal{H}_2 = \zeta$ in (28) and using (8), (16) and (20), we have

$$(\alpha^2 - \rho)(n-1) = \nu_1 - \nu_2. \tag{31}$$

Now, assuming $\nu_3 = 0$ and by virtue of (30) and (31), we obtain

$$\nu_1 = \frac{r}{(n-1)} - (\alpha^2 - \rho) \;\; and \;\; \nu_2 = \frac{r}{(n-1)} - (\alpha^2 - \rho)n. \tag{32}$$

where r denotes the scalar curvature of $\mathcal{M}$.
Thus from (28), we obtain

$$\begin{aligned} S(\mathcal{H}_1, \mathcal{H}_2) = &\left[\frac{r}{(n-1)} - (\alpha^2 - \rho)\right] g(\mathcal{H}_1, \mathcal{H}_2) \\ &+ \left[\frac{r}{(n-1)} - (\alpha^2 - \rho)n\right] \eta(\mathcal{H}_1)\eta(\mathcal{H}_2). \end{aligned} \tag{33}$$

Let $(g, \zeta, \vartheta, \Psi)$ be an η-Ricci soliton on an $(LCS)_n$-manifold $\mathcal{M}$. Taking (5) in consideration and writing $\pounds_\zeta g$ in terms of Levi-Civita connection ∇, we obtain

$$2S(\mathcal{H}_1, \mathcal{H}_2) = -g(\nabla_{\mathcal{H}_1}\zeta, \mathcal{H}_2) - g(\mathcal{H}_1, \nabla_{\mathcal{H}_2}\zeta) - 2\vartheta g(\mathcal{H}_1, \mathcal{H}_2) - 2\Psi\eta(\mathcal{H}_1)\eta(\mathcal{H}_2), \tag{34}$$

for any vector fields $\mathcal{H}_1$ and $\mathcal{H}_2$ on $\mathcal{M}$.

Using (10) in (34), we obtain

$$S(\mathcal{H}_1, \mathcal{H}_2) = -(\alpha + \vartheta)g(\mathcal{H}_1, \mathcal{H}_2) - (\alpha + \Psi)\eta(\mathcal{H}_1)\eta(\mathcal{H}_2). \tag{35}$$

Substituting $\mathcal{H}_2$=ζ in (35), we have

$$S(\mathcal{H}_1, \zeta) = (\Psi - \vartheta)\eta(\mathcal{H}_1). \tag{36}$$

Further substituting $\mathcal{H}_1 = \zeta$ in (36), we obtain

$$S(\zeta, \zeta) = (\vartheta - \Psi). \tag{37}$$

Moreover, comparing (20) and (36), we have

$$\Psi - \vartheta = (n-1)(\alpha^2 - \rho). \tag{38}$$

3 $(LCS)_n$-Manifolds satisfying Pseudo $\mathcal{W}_8$-flatness Condition

In this section, we explore $(LCS)_n$-manifolds satisfying pseudo $\mathcal{W}_8$-flatness condition.

Definition 3.1 *An $(LCS)_n$-manifold is said to satisfy pseudo $\mathcal{W}_8$-flatness condition if it fulfills*

$$\widehat{\mathcal{W}}_8(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 = 0, \tag{39}$$

for any vector fields $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ on $\mathcal{M}$.

By virtue of (1), we have

$$\begin{aligned} &aR(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 + b\left[S(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 - S(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1\right] \\ &-\frac{r}{n}\left[\frac{a}{(n-1)} - b\right]\left[g(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 - g(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1\right] = 0. \end{aligned} \tag{40}$$

Taking inner product of (40) with $\mathcal{F}_3$, we have

$$\begin{aligned} &aR(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{F}_3) + b\left[S(\mathcal{H}_1, \mathcal{H}_2)g(\mathcal{H}_3, \mathcal{F}_3) - S(\mathcal{H}_2, \mathcal{H}_3)g(\mathcal{H}_1, \mathcal{F}_3)\right] \\ &-\frac{r}{n}\left[\frac{a}{(n-1)} - b\right]\left[g(\mathcal{H}_1, \mathcal{H}_2)g(\mathcal{H}_3, \mathcal{F}_3) - g(\mathcal{H}_2, \mathcal{H}_3)g(\mathcal{H}_1, \mathcal{F}_3)\right] = 0. \end{aligned} \tag{41}$$

Let $\{e_1, e_2, e_3, \ldots, e_n = \zeta\}$ be an orthonormal frame field at any point of the manifold. Then substituting $\mathcal{H}_1 = \mathcal{F}_3 = e_i$ and taking summation over $1 \leq i \leq n$, we obtain

$$[a - b(n-1)]\left[S(\mathcal{H}_2, \mathcal{H}_3) + \left(\frac{r}{n}\right) g(\mathcal{H}_2, \mathcal{H}_3)\right] = 0. \tag{42}$$

Therefore, from (42), we can deduce the following cases:

Case I: If $a - b(n-1) = 0$, then

$$n = \left(1 + \frac{a}{b}\right). \tag{43}$$

Case II: If $a - b(n-1) \neq 0$, then

$$S(\mathcal{H}_2, \mathcal{H}_3) = -\left(\frac{r}{n}\right) g(\mathcal{H}_2, \mathcal{H}_3). \tag{44}$$

Thus, based on the discussion above, we can present the following theorem:

Theorem 3.1 *Let $\mathcal{M}$ be an n-dimensional $(LCS)_n$-manifold satisfying the pseudo $\mathcal{W}_8$-flatness condition, then either $n = \left(1 + \frac{a}{b}\right)$ or the manifold is an Einstein manifold, admitting the Ricci tensor of the form (44).*

4 $(LCS)_n$-Manifolds satisfying ζ-Pseudo $\mathcal{W}_8$-flatness Condition

In this section, we explore $(LCS)_n$-manifolds satisfying ζ-pseudo $\mathcal{W}_8$-flatness condition.

Definition 4.1 *An $(LCS)_n$-Manifold is said to satisfy ζ-pseudo $\mathcal{W}_8$-flatness condition if it fulfills*

$$\widehat{\mathcal{W}}_8(\mathcal{H}_1, \mathcal{H}_2)\zeta = 0, \tag{45}$$

for any vector fields $\mathcal{H}_1$ and $\mathcal{H}_2$ on $\mathcal{M}$.

Substituting $\mathcal{H}_3$=ζ in (1) and by virtue of (45), we obtain

$$\begin{aligned} aR(\mathcal{H}_1, \mathcal{H}_2)\zeta + b\left[S(\mathcal{H}_1, \mathcal{H}_2)\zeta - S(\mathcal{H}_2, \zeta)\mathcal{H}_1\right] \\ -\frac{r}{n}\left[\frac{a}{(n-1)} - b\right]\left[g(\mathcal{H}_1, \mathcal{H}_2)\zeta - g(\mathcal{H}_2, \zeta)\mathcal{H}_1\right] = 0. \end{aligned} \tag{46}$$

Using (8), (19) and (20) in (46), we have

$$\begin{aligned} \left[\{a - b(n-1)\}(\alpha^2 - \rho) + \frac{r}{n}\left\{\frac{a}{(n-1)} - b\right\}\right]\eta(\mathcal{H}_2)\mathcal{H}_1 \\ -a(\alpha^2 - \rho)\eta(\mathcal{H}_1)\mathcal{H}_2 + bS(\mathcal{H}_1, \mathcal{H}_2)\zeta - \frac{r}{n}\left[\frac{a}{(n-1)} - b\right]g(\mathcal{H}_1, \mathcal{H}_2)\zeta = 0. \end{aligned} \tag{47}$$

By taking the inner product of (47) with ζ and simplifying further, we obtain

$$\begin{aligned} S(\mathcal{H}_1, \mathcal{H}_2) = {} & \frac{r}{nb}\left[\frac{a}{(n-1)} - b\right]g(\mathcal{H}_1, \mathcal{H}_2) \\ & + \left[\frac{r}{nb}\left\{\frac{a}{(n-1)} - b\right\} - (n-1)(\alpha^2 - \rho)\right]\eta(\mathcal{H}_1)\eta(\mathcal{H}_2). \end{aligned} \tag{48}$$

Let $\{e_1, e_2, e_3, \ldots, e_n = \zeta\}$ be an orthonormal frame field at any point of the manifold. Then substituting $\mathcal{H}_1 = \mathcal{H}_2 = e_i$ and taking summation over $1 \leq i \leq n$, we obtain

$$r = \frac{nb(n-1)(\alpha^2 - \rho)}{2nb - (a+b)}, \qquad 2n \neq \left(\frac{a}{b} + 1\right). \tag{49}$$

Thus, based on the discussion above, we can present the following theorem:

Theorem 4.1 *Let $\mathcal{M}$ be an n-dimensional $(LCS)_n$-manifold satisfying ζ-pseudo $\mathcal{W}_8$-flatness condition, then we have*
(i) The manifold is an η-Einstein manifold, admitting the Ricci tensor of the form (48),
(ii) The scalar curvature r is of the form (49).

5 $(LCS)_n$-Manifolds satisfying φ-Pseudo $\mathcal{W}_8$- semisymmetric Condition

In this section, we explore $(LCS)_n$-Manifolds satisfying φ-Pseudo $\mathcal{W}_8$- semisymmetric condition.

Definition 5.1 *An $(LCS)_n$-Manifold is said to satisfy φ-pseudo $\mathcal{W}_8$- semisymmetric condition if it fulfills*

$$\widehat{\mathcal{W}}_8(\mathcal{H}_1, \mathcal{H}_2) \cdot \varphi = 0, \tag{50}$$

for any vector fields $\mathcal{H}_1$ and $\mathcal{H}_2$ on $\mathcal{M}$.
From (50), we have

$$\left(\widehat{\mathcal{W}}_8(\mathcal{H}_1, \mathcal{H}_2) \cdot \varphi\right) \mathcal{H}_3 = 0. \tag{51}$$

for any vector field $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ on $\mathcal{M}$. From (51), it implies that

$$\widehat{\mathcal{W}}_8(\mathcal{H}_1, \mathcal{H}_2)\varphi\mathcal{H}_3 - \varphi\widehat{\mathcal{W}}_8(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 = 0. \tag{52}$$

By virtue of (1), we have

$$\begin{aligned}\widehat{\mathcal{W}}_8(\mathcal{H}_1, \mathcal{H}_2)\varphi\mathcal{H}_3 = {} & aR(\mathcal{H}_1, \mathcal{H}_2)\varphi\mathcal{H}_3 + b\left[S(\mathcal{H}_1, \mathcal{H}_2)\varphi\mathcal{H}_3 - S(\mathcal{H}_2, \varphi\mathcal{H}_3)\mathcal{H}_1\right] \\ & - \frac{r}{n}\left[\frac{a}{(n-1)} - b\right]\left[g(\mathcal{H}_1, \mathcal{H}_2)\varphi\mathcal{H}_3 - g(\mathcal{H}_2, \varphi\mathcal{H}_3)\mathcal{H}_1\right],\end{aligned} \tag{53}$$

and

$$\begin{aligned}\varphi\widehat{\mathcal{W}}_8(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 = {} & a\varphi R(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 + b\left[S(\mathcal{H}_1, \mathcal{H}_2)\varphi\mathcal{H}_3 - S(\mathcal{H}_2, \mathcal{H}_3)\varphi\mathcal{H}_1\right] \\ & - \frac{r}{n}\left[\frac{a}{(n-1)} - b\right]\left[g(\mathcal{H}_1, \mathcal{H}_2)\varphi\mathcal{H}_3 - g(\mathcal{H}_2, \mathcal{H}_3)\varphi\mathcal{H}_1\right].\end{aligned} \tag{54}$$

Using (53) and (54) in (52), we obtain

$$\begin{aligned}& aR(\mathcal{H}_1, \mathcal{H}_2)\varphi\mathcal{H}_3 - a\varphi R(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 + b\left[S(\mathcal{H}_2, \mathcal{H}_3)\varphi\mathcal{H}_1 - S(\mathcal{H}_2, \varphi\mathcal{H}_3)\mathcal{H}_1\right] \\ & \quad + \frac{r}{n}\left[\frac{a}{(n-1)} - b\right]\left[g(\mathcal{H}_2, \varphi\mathcal{H}_3)\mathcal{H}_1 - g(\mathcal{H}_2, \mathcal{H}_3)\varphi\mathcal{H}_1\right] = 0.\end{aligned} \tag{55}$$

Substituting $\mathcal{H}_1 = \zeta$ in (55) and using (16) and (17), we obtain

$$\begin{aligned}& \left[a(\alpha^2 - \rho) + \frac{r}{n}\left\{\frac{a}{(n-1)} - b\right\}\right] g(\mathcal{H}_2, \varphi\mathcal{H}_3)\zeta \\ & - a(\alpha^2 - \rho)\left[g(\mathcal{H}_2, \mathcal{H}_3)\zeta - \eta(\mathcal{H}_3)\mathcal{H}_2\right] - bS(\mathcal{H}_2, \varphi\mathcal{H}_3)\zeta = 0.\end{aligned} \tag{56}$$

Replace $\mathcal{H}_3$ by $\varphi\mathcal{H}_3$ in (56) and on further simplification, we have

$$\begin{aligned}bS(\mathcal{H}_2, \mathcal{H}_3)\zeta = {} & \left[a(\alpha^2 - \rho) + \frac{r}{n}\left\{\frac{a}{(n-1)} - b\right\}\right] g(\mathcal{H}_2, \mathcal{H}_3)\zeta \\ & + \left[a - b(n-1)\right]\left[\alpha^2 - \rho + \frac{r}{n(n-1)}\right]\eta(\mathcal{H}_2)\eta(\mathcal{H}_3)\zeta \\ & - a(\alpha^2 - \rho)g(\mathcal{H}_2, \varphi\mathcal{H}_3)\zeta.\end{aligned} \tag{57}$$

Taking the inner product of (57) with ζ and on further simplification, we obtain

$$\begin{aligned} S(\mathcal{H}_2,\mathcal{H}_3) = & \frac{1}{b}\left[a(\alpha^2-\rho)+\frac{r}{n}\left\{\frac{a}{(n-1)}-b\right\}\right]g(\mathcal{H}_2,\mathcal{H}_3) \\ & +\frac{1}{b}\left[a-b(n-1)\right]\left[\alpha^2-\rho+\frac{r}{n(n-1)}\right]\eta(\mathcal{H}_2)\eta(\mathcal{H}_3) \\ & -\frac{a}{b}(\alpha^2-\rho)\Phi(\mathcal{H}_2,\mathcal{H}_3). \end{aligned} \tag{58}$$

Let $\{e_1, e_2, e_3, \ldots, e_n = \zeta\}$ be an orthonormal frame field at any point of the manifold. Then substituting $\mathcal{H}_2 = \mathcal{H}_3 = e_i$ and taking summation over $1 \le i \le n$, we obtain

$$r = \frac{n(\alpha^2-\rho)\left[a(n-\varrho-1)+b(n-1)\right]}{2nb-(a+b)}, \quad 2n \neq \left(\frac{a}{b}+1\right). \tag{59}$$

where $\varrho = trace(\varphi)$.
Thus, based on the discussion above, we can present the following theorem:

Theorem 5.1 *Let $\mathcal{M}$ be an n-dimensional $(LCS)_n$-manifold satisfying φ-pseudo $\mathcal{W}_8$-semisymmetric condition, then we have*
(i) The manifold is a generalized η-Einstein manifold, admitting the Ricci tensor of the form (58),
(ii) The scalar curvature r is of the form (59).

6 $(LCS)_n$-Manifolds satisfying $\widehat{\mathcal{W}}_8 \cdot Q = 0$ Condition

Consider an $(LCS)_n$-Manifold satisfying the condition

$$\widehat{\mathcal{W}}_8 \cdot Q = 0. \tag{60}$$

Then from the relation (60), it follows that

$$\widehat{\mathcal{W}}_8(\mathcal{H}_1,\mathcal{H}_2)Q\mathcal{H}_3 - Q\widehat{\mathcal{W}}_8(\mathcal{H}_1,\mathcal{H}_2)\mathcal{H}_3 = 0, \tag{61}$$

for any vector fields $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ on $\mathcal{M}$.
Substituting $\mathcal{H}_2$=ζ in (61) and by virtue of (1), we obtain

$$\begin{aligned} & aR(\mathcal{H}_1,\zeta)Q\mathcal{H}_3 - aQR(\mathcal{H}_1,\zeta)\mathcal{H}_3 + b\left[S(\zeta,\mathcal{H}_3)Q\mathcal{H}_1 - S(\zeta,Q\mathcal{H}_3)\mathcal{H}_1\right] \\ & -\frac{r}{n}\left[\frac{a}{(n-1)}-b\right]\left[\eta(\mathcal{H}_3)Q\mathcal{H}_1 - \eta(Q\mathcal{H}_3)\mathcal{H}_1\right] = 0. \end{aligned} \tag{62}$$

Using (17), (20) and (21) in (62), we have

$$S(\mathcal{H}_1,\mathcal{H}_3) = (n-1)(\alpha^2-\rho)g(\mathcal{H}_1,\mathcal{H}_3). \tag{63}$$

Let $\{e_1, e_2, e_3, \ldots, e_n = \zeta\}$ be an orthonormal frame field at any point of the manifold. Then substituting $\mathcal{H}_1 = \mathcal{H}_3 = e_i$ and taking summation over $1 \leq i \leq n$, we obtain

$$r = n(n-1)(\alpha^2 - \rho). \tag{64}$$

Thus, based on the discussion above, we can present the following theorem:

Theorem 6.1 *Let $\mathcal{M}$ be an n-dimensional $(LCS)_n$-manifold satisfying $\widehat{\mathcal{W}}_8 \cdot Q = 0$ condition, then we have*
(i) The manifold is an Einstein manifold, admitting the Ricci tensor of the form (63),
(ii) The scalar curvature r is of the form (64).

7 $(LCS)_n$-Manifolds satisfying $\widehat{\mathcal{W}}_8 \cdot R = 0$ Condition

Consider an $(LCS)_n$-Manifold satisfying the condition

$$\widehat{\mathcal{W}}_8 \cdot R = 0. \tag{65}$$

Then from the relation (65), it follows that

$$\widehat{\mathcal{W}}_8(\zeta, \mathcal{F}_1)R(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 - R(\widehat{\mathcal{W}}_8(\zeta, \mathcal{F}_1)\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 - R(\mathcal{H}_1, \widehat{\mathcal{W}}_8(\zeta, \mathcal{F}_1)\mathcal{H}_2)\mathcal{H}_3 \\ -R(\mathcal{H}_1, \mathcal{H}_2)\widehat{\mathcal{W}}_8(\zeta, \mathcal{F}_1)\mathcal{H}_3 = 0, \tag{66}$$

for any vector fields $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ and $\mathcal{F}_1$ on $\mathcal{M}$.
Substituting $\mathcal{H}_3$=ζ in (66) and using (19), we obtain

$$(\alpha^2 - \rho)\left[\eta(\widehat{\mathcal{W}}_8(\zeta, \mathcal{F}_1)\mathcal{H}_1)\mathcal{H}_2 - \eta(\widehat{\mathcal{W}}_8(\zeta, \mathcal{F}_1)\mathcal{H}_2)\mathcal{H}_1\right] - R(\mathcal{H}_1, \mathcal{H}_2)\widehat{\mathcal{W}}_8(\zeta, \mathcal{F}_1)\zeta = 0. \tag{67}$$

By virtue of (1), we have

$$\eta(\widehat{\mathcal{W}}_8(\zeta, \mathcal{F}_1)\mathcal{H}_1)\mathcal{H}_2 = -\left[a(\alpha^2 - \rho) + \frac{r}{n}\left\{\frac{a}{(n-1)} - b\right\}\right] g(\mathcal{F}_1, \mathcal{H}_1)\mathcal{H}_2 \\ + \left[(\alpha^2 - \rho)\{b(n-1) - a\} - \frac{r}{n}\left\{\frac{a}{(n-1)} - b\right\}\right] \eta(\mathcal{H}_1)\eta(\mathcal{F}_1)\mathcal{H}_2 + bS(\mathcal{F}_1, \mathcal{H}_1)\mathcal{H}_2, \tag{68}$$

$$\eta(\widehat{\mathcal{W}}_8(\zeta, \mathcal{F}_1)\mathcal{H}_2)\mathcal{H}_1 = -\left[a(\alpha^2 - \rho) + \frac{r}{n}\left\{\frac{a}{(n-1)} - b\right\}\right] g(\mathcal{F}_1, \mathcal{H}_2)\mathcal{H}_1 \\ + \left[(\alpha^2 - \rho)\{b(n-1) - a\} - \frac{r}{n}\left\{\frac{a}{(n-1)} - b\right\}\right] \eta(\mathcal{H}_2)\eta(\mathcal{F}_1)\mathcal{H}_1 + bS(\mathcal{F}_1, \mathcal{H}_2)\mathcal{H}_1, \tag{69}$$

and

$$\widehat{\mathcal{W}}_8(\zeta,\mathcal{F}_1)\zeta = a(\alpha^2-\rho)\left[\mathcal{F}_1+\eta(\mathcal{F}_1)\zeta\right]. \tag{70}$$

Using (68), (69) and (70) in (67), we obtain

$$\begin{aligned}(\alpha^2-\rho)\left[\frac{r}{n}\left\{\frac{a}{(n-1)}-b\right\}\{g(\mathcal{F}_1,\mathcal{H}_2)\mathcal{H}_1-g(\mathcal{F}_1,\mathcal{H}_1)\mathcal{H}_2\}\right.\\ +\left\{(\alpha^2-\rho)\{b(n-1)-a\}-\frac{r}{n}\left\{\frac{a}{(n-1)}-b\right\}\right\}\{\eta(\mathcal{H}_1)\eta(\mathcal{F}_1)\mathcal{H}_2-\eta(\mathcal{H}_2)\eta(\mathcal{F}_1)\mathcal{H}_1\}\\ \left.+b\left[S(\mathcal{F}_1,\mathcal{H}_1)\mathcal{H}_2-S(\mathcal{F}_1,\mathcal{H}_2)\mathcal{H}_1\right]-a\eta(\mathcal{F}_1)R(\mathcal{H}_1,\mathcal{H}_2)\zeta\right]=0.\end{aligned} \tag{71}$$

Since $(\alpha^2-\rho)\neq 0$. Then, substituting $\mathcal{H}_2=\zeta$ in (71), we have

$$\begin{aligned}-\frac{r}{n}\left[\frac{a}{(n-1)}-b\right]g(\mathcal{F}_1,\mathcal{H}_1)\zeta+\left[b(\alpha^2-\rho)(n-1)-\frac{r}{n}\left\{\frac{a}{(n-1)}-b\right\}\right]\eta(\mathcal{F}_1)\eta(\mathcal{H}_1)\zeta\\ +bS(\mathcal{F}_1,\mathcal{H}_1)\zeta=0.\end{aligned} \tag{72}$$

Taking the inner product of (72) with ζ, we obtain

$$\begin{aligned}S(\mathcal{F}_1,\mathcal{H}_1)=&\frac{r}{nb}\left[\frac{a}{(n-1)}-b\right]g(\mathcal{F}_1,\mathcal{H}_1)\\ &-\left[(\alpha^2-\rho)(n-1)-\frac{r}{nb}\left\{\frac{a}{(n-1)}-b\right\}\right]\eta(\mathcal{F}_1)\eta(\mathcal{H}_1).\end{aligned} \tag{73}$$

Let $\{e_1,e_2,e_3,\ldots,e_n=\zeta\}$ be an orthonormal frame field at any point of the manifold. Then substituting $\mathcal{F}_1=\mathcal{H}_1=e_i$ and taking summation over $1\leq i\leq n$, we obtain

$$r=\frac{nb(n-1)(\alpha^2-\rho)}{2nb-(a+b)},\qquad 2n\neq\left(\frac{a}{b}+1\right). \tag{74}$$

Thus, based on the discussion above, we can present the following theorem:

Theorem 7.1 *Let $\mathcal{M}$ be an n-dimensional $(LCS)_n$-manifold satisfying $\widehat{\mathcal{W}}_8\cdot R=0$ condition, then we have*
(i) The manifold is an η-Einstein manifold, admitting the Ricci tensor of the form (73),
(ii) The scalar curvature r is of the form (74).

8 $(LCS)_n$-manifolds satisfying $R(\zeta,\mathcal{H}_1)\cdot\widehat{\mathcal{W}}_8=0$ condition

Consider an $(LCS)_n$-Manifold satisfying the condition

$$R(\zeta,\mathcal{H}_1)\cdot\widehat{\mathcal{W}}_8=0, \tag{75}$$

Then from (75), we infer

$$R(\zeta,\mathcal{H}_1)\widehat{\mathcal{W}}_8(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3-\widehat{\mathcal{W}}_8(R(\zeta,\mathcal{H}_1)\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3-\widehat{\mathcal{W}}_8(\mathcal{F}_1,R(\zeta,\mathcal{H}_1)\mathcal{F}_2)\mathcal{F}_3 \\ -\widehat{\mathcal{W}}_8(\mathcal{F}_1,\mathcal{F}_2)R(\zeta,\mathcal{H}_1)\mathcal{F}_3=0, \quad (76)$$

for any vector fields $\mathcal{H}_1,\mathcal{F}_1,\mathcal{F}_2$ and $\mathcal{F}_3$ on $\mathcal{M}$.
By virtue of (17), we have

$$(\alpha^2-\rho)\Big[g(\mathcal{H}_1,\widehat{\mathcal{W}}_8(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3)\zeta-\eta(\widehat{\mathcal{W}}_8(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3)\mathcal{H}_1-g(\mathcal{H}_1,\mathcal{F}_1)\widehat{\mathcal{W}}_8(\zeta,\mathcal{F}_2)\mathcal{F}_3 \\ +\eta(\mathcal{F}_1)\widehat{\mathcal{W}}_8(\mathcal{H}_1,\mathcal{F}_2)\mathcal{F}_3-g(\mathcal{H}_1,\mathcal{F}_2)\widehat{\mathcal{W}}_8(\mathcal{F}_1,\zeta)\mathcal{F}_3+\eta(\mathcal{F}_2)\widehat{\mathcal{W}}_8(\mathcal{F}_1,\mathcal{H}_1)\mathcal{F}_3 \\ -g(\mathcal{H}_1,\mathcal{F}_3)\widehat{\mathcal{W}}_8(\mathcal{F}_1,\mathcal{F}_2)\zeta+\eta(\mathcal{F}_3)\widehat{\mathcal{W}}_8(\mathcal{F}_1,\mathcal{F}_2)\mathcal{H}_1\Big]=0. \quad (77)$$

Using (1) in (77), we obtain

$$(\alpha^2-\rho)\big[ag(\mathcal{H}_1,R(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3)\zeta-a\eta(R(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3)\mathcal{H}_1 \\ -a(\alpha^2-\rho)g(\mathcal{H}_1,\mathcal{F}_1)g(\mathcal{F}_2,\mathcal{F}_3)\zeta+a(\alpha^2-\rho)g(\mathcal{H}_1,\mathcal{F}_1)\eta(\mathcal{F}_3)\mathcal{F}_2 \\ -b(\alpha^2-\rho)(n-1)g(\mathcal{H}_1,\mathcal{F}_1)\eta(\mathcal{F}_2)\mathcal{F}_3+a\eta(\mathcal{F}_1)R(\mathcal{H}_1,\mathcal{F}_2)\mathcal{F}_3 \\ +b\eta(\mathcal{F}_1)S(\mathcal{H}_1,\mathcal{F}_2)\mathcal{F}_3-(\alpha^2-\rho)(a-b(n-1))g(\mathcal{H}_1,\mathcal{F}_2)\eta(\mathcal{F}_3)\mathcal{F}_1 \\ +a(\alpha^2-\rho)g(\mathcal{H}_1,\mathcal{F}_2)g(\mathcal{F}_1,\mathcal{F}_3)\zeta-b(\alpha^2-\rho)(n-1)g(\mathcal{H}_1,\mathcal{F}_2)\eta(\mathcal{F}_1)\mathcal{F}_3 \\ +a\eta(\mathcal{F}_2)R(\mathcal{F}_1,\mathcal{H}_1)\mathcal{F}_3+bS(\mathcal{F}_1,\mathcal{H}_1)\eta(\mathcal{F}_2)\mathcal{F}_3-bS(\mathcal{H}_1,\mathcal{F}_3)\eta(\mathcal{F}_2)\mathcal{F}_1 \\ -(\alpha^2-\rho)(a-b(n-1))g(\mathcal{H}_1,\mathcal{F}_3)\eta(\mathcal{F}_2)\mathcal{F}_1+a(\alpha^2-\rho)g(\mathcal{H}_1,\mathcal{F}_3)\eta(\mathcal{F}_1)\mathcal{F}_2 \\ +a\eta(\mathcal{F}_3)R(\mathcal{F}_1,\mathcal{F}_2)\mathcal{H}_1-b\eta(\mathcal{F}_3)S(\mathcal{F}_2,\mathcal{H}_1)\mathcal{F}_1\big]=0. \quad (78)$$

Taking the inner product of (78) with ζ, we have

$$(\alpha^2-\rho)\big[-ag(\mathcal{H}_1,R(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3)-a\eta(R(\mathcal{F}_1,\mathcal{F}_2)\mathcal{F}_3)\eta(\mathcal{H}_1) \\ +a(\alpha^2-\rho)g(\mathcal{H}_1,\mathcal{F}_1)g(\mathcal{F}_2,\mathcal{F}_3) \\ +(\alpha^2-\rho)(a-b(n-1))g(\mathcal{H}_1,\mathcal{F}_1)\eta(\mathcal{F}_2)\eta(\mathcal{F}_3) \\ +a\eta(\mathcal{F}_1)\eta(R(\mathcal{H}_1,\mathcal{F}_2)\mathcal{F}_3)-a(\alpha^2-\rho)g(\mathcal{H}_1,\mathcal{F}_2)\eta(\mathcal{F}_3)\eta(\mathcal{F}_1) \\ -a(\alpha^2-\rho)g(\mathcal{H}_1,\mathcal{F}_2)g(\mathcal{F}_1,\mathcal{F}_3)+a\eta(\mathcal{F}_2)\eta(R(\mathcal{F}_1,\mathcal{H}_1)\mathcal{F}_3) \\ +b\eta(\mathcal{F}_2)\eta(\mathcal{F}_3)S(\mathcal{F}_1,\mathcal{H}_1)-b\eta(\mathcal{F}_2)\eta(\mathcal{F}_1)S(\mathcal{H}_1,\mathcal{F}_3) \\ +b(\alpha^2-\rho)(n-1)g(\mathcal{H}_1,\mathcal{F}_3)\eta(\mathcal{F}_2)\eta(\mathcal{F}_1)+a\eta(\mathcal{F}_3)\eta(R(\mathcal{F}_1,\mathcal{F}_2)\mathcal{H}_1)\big]=0. \quad (79)$$

Using (18) in (79), we obtain

$$(\alpha^2-\rho)\big[-aR(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3,\mathcal{H}_1)+a(\alpha^2-\rho)g(\mathcal{H}_1,\mathcal{F}_1)g(\mathcal{F}_2,\mathcal{F}_3) \\ -b(\alpha^2-\rho)(n-1)g(\mathcal{H}_1,\mathcal{F}_1)\eta(\mathcal{F}_2)\eta(\mathcal{F}_3) \\ -a(\alpha^2-\rho)g(\mathcal{H}_1,\mathcal{F}_2)g(\mathcal{F}_1,\mathcal{F}_3)+b\eta(\mathcal{F}_2)\eta(\mathcal{F}_3)S(\mathcal{F}_1,\mathcal{H}_1) \\ -b\eta(\mathcal{F}_2)\eta(\mathcal{F}_1)S(\mathcal{H}_1,\mathcal{F}_3)+b(\alpha^2-\rho)(n-1)g(\mathcal{H}_1,\mathcal{F}_3)\eta(\mathcal{F}_2)\eta(\mathcal{F}_1)\big]=0. \quad (80)$$

Let $\{e_1, e_2, e_3, \ldots, e_n = \zeta\}$ be an orthonormal frame field at any point of the manifold. Then substituting $\mathcal{F}_1 = \mathcal{H}_1 = e_i$ and taking summation over $1 \leq i \leq n$, we obtain

$$(\alpha^2 - \rho)\big[-aS(\mathcal{F}_2, \mathcal{F}_3) + a(\alpha^2 - \rho)(n-1)g(\mathcal{F}_2, \mathcal{F}_3) + b\left\{r - (\alpha^2 - \rho)(n-1)n\right\}\eta(\mathcal{F}_2)\eta(\mathcal{F}_3)\big] = 0. \tag{81}$$

Since $(\alpha^2 - \rho) \neq 0$, we have

$$S(\mathcal{F}_2, \mathcal{F}_3) = (\alpha^2 - \rho)(n-1)g(\mathcal{F}_2, \mathcal{F}_3) + \frac{b}{a}\left[r - (\alpha^2 - \rho)(n-1)n\right]\eta(\mathcal{F}_2)\eta(\mathcal{F}_3). \tag{82}$$

Further substituting $\mathcal{F}_2 = \mathcal{F}_3 = e_i$ and taking summation over $1 \leq i \leq n$, we obtain

$$r = n(n-1)(\alpha^2 - \rho). \tag{83}$$

Thus, based on the discussion above, we can present the following theorem:

Theorem 8.1 *Let $\mathcal{M}$ be an n-dimensional $(LCS)_n$-manifold satisfying $R(\zeta, \mathcal{H}_1) \cdot \widehat{\mathcal{W}}_8 = 0$ condition, then we have*
(i) The manifold is an η-Einstein manifold, admitting the Ricci tensor of the form (82),
(ii) The scalar curvature r is of the form (83).

9 $(LCS)_n$-manifolds satisfying pseudo $\mathcal{W}_8$-Ricci pseudosymmetric condition

In this section, we explore $(LCS)_n$-manifolds satisfying pseudo $\mathcal{W}_8$-Ricci pseudosymmetric condition.

Definition 9.1 *An $(LCS)_n$-manifold is said to satisfy the pseudo $\mathcal{W}_8$-Ricci pseudosymmetric condition if its curvature tensor fulfills the following relation:*

$$(\widehat{\mathcal{W}}_8(\mathcal{H}_1, \mathcal{H}_2) \cdot S)(\mathcal{H}_3, \mathcal{F}_1) = L_S Q(g, S)(\mathcal{H}_3, \mathcal{F}_1; \mathcal{H}_1, \mathcal{H}_2), \tag{84}$$

which holds on $\Omega_S = \left\{x \in \mathcal{M} : S \neq \frac{r}{n} g \text{ at } x\right\}$, where L_S is a function defined on Ω_S.

From (84), it infer that

$$\begin{aligned} &S(\widehat{\mathcal{W}}_8(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3, \mathcal{F}_1) + S(\mathcal{H}_3, \widehat{\mathcal{W}}_8(\mathcal{H}_1, \mathcal{H}_2)\mathcal{F}_1) \\ &= L_S\big[g(\mathcal{H}_2, \mathcal{H}_3)S(\mathcal{H}_1, \mathcal{F}_1) - g(\mathcal{H}_1, \mathcal{H}_3)S(\mathcal{H}_2, \mathcal{F}_1) \\ &\quad + g(\mathcal{H}_2, \mathcal{F}_1)S(\mathcal{H}_1, \mathcal{H}_3) - g(\mathcal{H}_1, \mathcal{F}_1)S(\mathcal{H}_2, \mathcal{H}_3)\big]. \end{aligned} \tag{85}$$

Substituting $\mathcal{H}_3=\zeta$ and using (20), we obtain

$$\begin{aligned} &S(\widehat{\mathcal{W}}_8(\mathcal{H}_1,\mathcal{H}_2)\zeta,\mathcal{F}_1)+(\alpha^2-\rho)(n-1)\eta(\widehat{\mathcal{W}}_8(\mathcal{H}_1,\mathcal{H}_2)\mathcal{F}_1)\\ &\qquad = L_S\big[\eta(\mathcal{H}_2)S(\mathcal{H}_1,\mathcal{F}_1)-\eta(\mathcal{H}_1)S(\mathcal{H}_2,\mathcal{F}_1)\\ &+(\alpha^2-\rho)(n-1)\left\{\eta(\mathcal{H}_1)g(\mathcal{H}_2,\mathcal{F}_1)-\eta(\mathcal{H}_2)g(\mathcal{H}_1,\mathcal{F}_1)\right\}\big]. \end{aligned} \tag{86}$$

By virtue of (1), we have

$$\begin{aligned} S(\widehat{\mathcal{W}}_8(\mathcal{H}_1,\mathcal{H}_2)\zeta,\mathcal{F}_1) = &\{a-b(n-1)\}\left[\alpha^2-\rho+\frac{r}{n(n-1)}\right]\eta(\mathcal{H}_2)S(\mathcal{H}_1,\mathcal{F}_1)\\ &-a(\alpha^2-\rho)\eta(\mathcal{H}_1)S(\mathcal{H}_2,\mathcal{F}_1)\\ &+b(\alpha^2-\rho)(n-1)\eta(\mathcal{F}_1)S(\mathcal{H}_1,\mathcal{H}_2)\\ &-\frac{r}{n}(\alpha^2-\rho)\left\{a-b(n-1)\right\}\eta(\mathcal{F}_1)g(\mathcal{H}_1,\mathcal{H}_2), \end{aligned} \tag{87}$$

$$\begin{aligned} \eta(\widehat{\mathcal{W}}_8(\mathcal{H}_1,\mathcal{H}_2)\mathcal{F}_1) = &\left[a(\alpha^2-\rho)+\frac{r}{n}\left\{\frac{a}{(n-1)}-b\right\}\right]g(\mathcal{H}_2,\mathcal{F}_1)\eta(\mathcal{H}_1)\\ &-a(\alpha^2-\rho)g(\mathcal{H}_1,\mathcal{F}_1)\eta(\mathcal{H}_2)\\ &-\frac{r}{n}\left\{\frac{a}{(n-1)}-b\right\}g(\mathcal{H}_1,\mathcal{H}_2)\eta(\mathcal{F}_1)\\ &+b\left[S(\mathcal{H}_1,\mathcal{H}_2)\eta(\mathcal{F}_1)-S(\mathcal{H}_2,\mathcal{F}_1)\eta(\mathcal{H}_1)\right], \end{aligned} \tag{88}$$

Substituting (87) and (88) in(86), we obtain

$$\begin{aligned} &\{a-b(n-1)\}\left[\alpha^2-\rho+\frac{r}{n(n-1)}\right]\eta(\mathcal{H}_2)S(\mathcal{H}_1,\mathcal{F}_1)-a(\alpha^2-\rho)\eta(\mathcal{H}_1)S(\mathcal{H}_2,\mathcal{F}_1)\\ &\quad+b(\alpha^2-\rho)(n-1)\eta(\mathcal{F}_1)S(\mathcal{H}_1,\mathcal{H}_2)-\frac{r}{n}(\alpha^2-\rho)\left\{a-b(n-1)\right\}\eta(\mathcal{F}_1)g(\mathcal{H}_1,\mathcal{H}_2)\\ &\quad+(\alpha^2-\rho)(n-1)\left[a(\alpha^2-\rho)+\frac{r}{n}\left\{\frac{a}{(n-1)}-b\right\}\right]g(\mathcal{H}_2,\mathcal{F}_1)\eta(\mathcal{H}_1)\\ &\quad-a(\alpha^2-\rho)g(\mathcal{H}_1,\mathcal{F}_1)\eta(\mathcal{H}_2)-\frac{r}{n}\left\{\frac{a}{(n-1)}-b\right\}g(\mathcal{H}_1,\mathcal{H}_2)\eta(\mathcal{F}_1)\\ &\quad+b\left[S(\mathcal{H}_1,\mathcal{H}_2)\eta(\mathcal{F}_1)-S(\mathcal{H}_2,\mathcal{F}_1)\eta(\mathcal{H}_1)\right]\\ &\quad=L_S\left[\eta(\mathcal{H}_2)S(\mathcal{H}_1,\mathcal{F}_1)-\eta(\mathcal{H}_1)S(\mathcal{H}_2,\mathcal{F}_1)\right.\\ &\quad\left.+(\alpha^2-\rho)(n-1)\left\{g(\mathcal{H}_2,\mathcal{F}_1)\eta(\mathcal{H}_1)-g(\mathcal{H}_1,\mathcal{F}_1)\eta(\mathcal{H}_2)\right\}\right]. \end{aligned} \tag{89}$$

Substituting $\mathcal{H}_2=\zeta$ and on further simplification, we have

$$\begin{aligned} S(\mathcal{H}_1,\mathcal{F}_1) = &\left[\frac{(n-1)(\alpha^2-\rho)\left\{L_S-a(\alpha^2-\rho)\right\}}{L_S-\{a-b(n-1)\}\left\{\alpha^2-\rho+\frac{r}{n(n-1)}\right\}}\right]g(\mathcal{H}_1,\mathcal{F}_1)\\ &+\left[\frac{-(\alpha^2-\rho)\left\{b(\alpha^2-\rho)(n-1)^2-\frac{r}{n}\left\{a-b(n-1)\right\}\right\}}{L_S-\{a-b(n-1)\}\left\{\alpha^2-\rho+\frac{r}{n(n-1)}\right\}}\right]\eta(\mathcal{H}_1)\eta(\mathcal{F}_1), \end{aligned} \tag{90}$$

provided $L_S \neq \{a-b(n-1)\}\left\{\alpha^2-\rho+\frac{r}{n(n-1)}\right\}$.

Let $\{e_1, e_2, e_3, \ldots, e_n=\zeta\}$ be an orthonormal frame field at any point of the manifold. Then substituting $\mathcal{H}_1=\mathcal{F}_1=e_i$ and taking summation over $1 \leq i \leq n$, we obtain

$$\begin{aligned}(n-1)\left[nL_S-(\alpha^2-\rho)(n-1)\{a-b(n-1)\}\right]r-\{a-b(n-1)\}\,r^2\\ =n(n-1)^2(\alpha^2-\rho)\left[nL_S-(\alpha^2-\rho)\{(a-b)n+b\}\right].\end{aligned} \tag{91}$$

Thus, based on the discussion above, we can present the following theorem:

Theorem 9.1 *Let $\mathcal{M}$ be an n-dimensional $(LCS)_n$-manifold satisfying pseudo $\mathcal{W}_8$-Ricci pseudosymmetric condition, then we have*
(i) The manifold is an η-Einstein manifold, admitting the Ricci tensor of the form (90),
(ii) The scalar curvature r satisfies (91).

10 η-Ricci solitons on $(LCS)_n$-manifold satisfying $\widehat{\mathcal{W}}_8(\zeta,\mathcal{H}_1)\cdot S=0$ condition

In this section, we examine η-Ricci solitons on an $(LCS)_n$-manifold satisfying $\widehat{\mathcal{W}}_8(\zeta,\mathcal{H}_1)\cdot S=0$ condition. The condition that must be satisfied by S is [3]

$$S(\widehat{\mathcal{W}}_8(\zeta,\mathcal{H}_1)\mathcal{H}_2,\mathcal{H}_3)+S(\mathcal{H}_2,\widehat{\mathcal{W}}_8(\zeta,\mathcal{H}_1)\mathcal{H}_3)=0, \tag{92}$$

for any vector fields $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ on $\mathcal{M}$.

By virtue of (1), (35) and (36), we have

$$\begin{aligned}S(\widehat{\mathcal{W}}_8(\zeta,\mathcal{H}_1)\mathcal{H}_2,\mathcal{H}_3)&=(\Psi-\vartheta)\left[a(\alpha^2-\rho)+\frac{r}{n}\left\{\frac{a}{(n-1)}-b\right\}\right.\\ &\left.+b(\alpha+\vartheta)\right]\eta(\mathcal{H}_3)g(\mathcal{H}_1,\mathcal{H}_2)+a(\alpha^2-\rho)(\alpha+\vartheta)\eta(\mathcal{H}_2)g(\mathcal{H}_1,\mathcal{H}_3)\\ &-(\alpha+\vartheta)\left[b(\Psi-\vartheta)-\frac{r}{n}\left\{\frac{a}{(n-1)}-b\right\}\right]\eta(\mathcal{H}_1)g(\mathcal{H}_2,\mathcal{H}_3)\\ &+(\alpha+\Psi)\left[a(\alpha^2-\rho)+\frac{r}{n}\left\{\frac{a}{(n-1)}-b\right\}\right]\eta(\mathcal{H}_1)\eta(\mathcal{H}_2)\eta(\mathcal{H}_3),\end{aligned} \tag{93}$$

and

$$\begin{aligned}S(\mathcal{H}_2,\widehat{\mathcal{W}}_8(\zeta,\mathcal{H}_1)\mathcal{H}_3)&=(\Psi-\vartheta)\left[a(\alpha^2-\rho)+\frac{r}{n}\left\{\frac{a}{(n-1)}-b\right\}\right.\\ &\left.+b(\alpha+\vartheta)\right]\eta(\mathcal{H}_2)g(\mathcal{H}_1,\mathcal{H}_3)+a(\alpha^2-\rho)(\alpha+\vartheta)\eta(\mathcal{H}_3)g(\mathcal{H}_1,\mathcal{H}_2)\\ &-(\alpha+\vartheta)\left[b(\Psi-\vartheta)-\frac{r}{n}\left\{\frac{a}{(n-1)}-b\right\}\right]\eta(\mathcal{H}_1)g(\mathcal{H}_2,\mathcal{H}_3)\\ &+(\alpha+\Psi)\left[a(\alpha^2-\rho)+\frac{r}{n}\left\{\frac{a}{(n-1)}-b\right\}\right]\eta(\mathcal{H}_1)\eta(\mathcal{H}_2)\eta(\mathcal{H}_3).\end{aligned} \tag{94}$$

Using (93) and (94) in (92), we obtain

$$\begin{aligned}(\Psi-\vartheta)&\left[a(\alpha^2-\rho)+\frac{r}{n}\left\{\frac{a}{(n-1)}-b\right\}+b(\alpha+\vartheta)\right][\eta(\mathcal{H}_3)g(\mathcal{H}_1,\mathcal{H}_2)\\&+\eta(\mathcal{H}_2)g(\mathcal{H}_1,\mathcal{H}_3)]+a(\alpha^2-\rho)(\alpha+\vartheta)[\eta(\mathcal{H}_2)g(\mathcal{H}_1,\mathcal{H}_3)+\eta(\mathcal{H}_3)g(\mathcal{H}_1,\mathcal{H}_2)]\\&-2(\alpha+\vartheta)\left[b(\Psi-\vartheta)-\frac{r}{n}\left\{\frac{a}{(n-1)}-b\right\}\right]\eta(\mathcal{H}_1)g(\mathcal{H}_2,\mathcal{H}_3)\\&+2(\alpha+\Psi)\left[a(\alpha^2-\rho)+\frac{r}{n}\left\{\frac{a}{(n-1)}-b\right\}\right]\eta(\mathcal{H}_1)\eta(\mathcal{H}_2)\eta(\mathcal{H}_3)=0.\end{aligned}\tag{95}$$

Substituting $\mathcal{H}_1$=ζ in (95) and on further simplification, we have

$$(\alpha+\vartheta)\left[b(\Psi-\vartheta)-\frac{r}{n}\left\{\frac{a}{(n-1)}-b\right\}\right][g(\mathcal{H}_2,\mathcal{H}_3)+\eta(\mathcal{H}_2)\eta(\mathcal{H}_3)]=0.\tag{96}$$

Therefore, from (96), we can deduce the following cases:
Case I: If $(\alpha+\vartheta)=0$. Then, it infer that

$$\vartheta=-\alpha,\quad \Psi=-\alpha+(n-1)(\alpha^2-\rho).\tag{97}$$

Case II: If $\left[b(\Psi-\vartheta)-\frac{r}{n}\left\{\frac{a}{(n-1)}-b\right\}\right]=0$, Then, it infer that

$$(\Psi-\vartheta)=\frac{r}{nb}\left[\frac{a}{(n-1)}-b\right].\tag{98}$$

Case III: If $g(\mathcal{H}_2,\mathcal{H}_3)+\eta(\mathcal{H}_2)\eta(\mathcal{H}_3)=0$. Then, replacing $\mathcal{H}_2$ by $Q\mathcal{H}_2$ and using (21), we obtain

$$S(\mathcal{H}_2,\mathcal{H}_3)=-(n-1)(\alpha^2-\rho)\eta(\mathcal{H}_2)\eta(\mathcal{H}_3).\tag{99}$$

Let $\{e_1,e_2,e_3,\dots,e_n=\zeta\}$ be an orthonormal frame field at any point of the manifold. Then substituting $\mathcal{H}_2=\mathcal{H}_3=e_i$ and taking summation over $1\le i\le n$, we obtain

$$r=(n-1)(\alpha^2-\rho).\tag{100}$$

Thus, based on the discussion above, we can present the following theorem:

Theorem 10.1 *Let $\mathcal{M}$ be an n-dimensional $(LCS)_n$-manifold and let (g,ζ,ϑ,Ψ) be an η-Ricci soliton on $\mathcal{M}$. If $\mathcal{M}$ satisfies $\widehat{\mathcal{W}}_8(\zeta,\mathcal{H}_1)\cdot S=0$, then either of the following statements are true:*
(i) $\vartheta=-\alpha,\quad \Psi=-\alpha+(n-1)(\alpha^2-\rho)$,
(ii) $(\Psi-\vartheta)=\frac{r}{nb}\left[\frac{a}{(n-1)}-b\right]$,
(iii) $\mathcal{M}$ is a special type of an η-Einstein manifold, admitting the Ricci tensor of the form (99) and having scalar curvature $r=(n-1)(\alpha^2-\rho)$.

11 η-Ricci solitons on $(LCS)_n$-manifold satisfying $S(\zeta, \mathcal{H}_1) \cdot \widehat{\mathcal{W}}_8 = 0$ condition

In this section, we examine η-Ricci solitons on an $(LCS)_n$-manifold satisfying $S(\zeta, \mathcal{H}_1) \cdot \widehat{\mathcal{W}}_8 = 0$ condition. The condition that must be satisfied by S is [3]

$$\begin{aligned} &S(\mathcal{H}_1, \widehat{\mathcal{W}}_8(\mathcal{H}_2, \mathcal{H}_3)\mathcal{F}_1)\zeta - S(\zeta, \widehat{\mathcal{W}}_8(\mathcal{H}_2, \mathcal{H}_3)\mathcal{F}_1)\mathcal{H}_1 + S(\mathcal{H}_1, \mathcal{H}_2)\widehat{\mathcal{W}}_8(\zeta, \mathcal{H}_3)\mathcal{F}_1 \\ &\quad - S(\zeta, \mathcal{H}_2)\widehat{\mathcal{W}}_8(\mathcal{H}_1, \mathcal{H}_3)\mathcal{F}_1 + S(\mathcal{H}_1, \mathcal{H}_3)\widehat{\mathcal{W}}_8(\mathcal{H}_2, \zeta)\mathcal{F}_1 - S(\zeta, \mathcal{H}_3)\widehat{\mathcal{W}}_8(\mathcal{H}_2, \mathcal{H}_1)\mathcal{F}_1 \\ &\quad + S(\mathcal{H}_1, \mathcal{F}_1)\widehat{\mathcal{W}}_8(\mathcal{H}_2, \mathcal{H}_3)\zeta - S(\zeta, \mathcal{F}_1)\widehat{\mathcal{W}}_8(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1 = 0, \end{aligned} \tag{101}$$

for any vector fields $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ and $\mathcal{F}_1$ on $\mathcal{M}$.
Taking the inner product of (101) with ζ, we obtain

$$\begin{aligned} &- S(\mathcal{H}_1, \widehat{\mathcal{W}}_8(\mathcal{H}_2, \mathcal{H}_3)\mathcal{F}_1) - S(\zeta, \widehat{\mathcal{W}}_8(\mathcal{H}_2, \mathcal{H}_3)\mathcal{F}_1)\eta(\mathcal{H}_1) \\ &\quad + S(\mathcal{H}_1, \mathcal{H}_2)\eta(\widehat{\mathcal{W}}_8(\zeta, \mathcal{H}_3)\mathcal{F}_1) - S(\zeta, \mathcal{H}_2)\eta(\widehat{\mathcal{W}}_8(\mathcal{H}_1, \mathcal{H}_3)\mathcal{F}_1) \\ &\quad + S(\mathcal{H}_1, \mathcal{H}_3)\eta(\widehat{\mathcal{W}}_8(\mathcal{H}_2, \zeta)\mathcal{F}_1) - S(\zeta, \mathcal{H}_3)\eta(\widehat{\mathcal{W}}_8(\mathcal{H}_2, \mathcal{H}_1)\mathcal{F}_1) \\ &\quad + S(\mathcal{H}_1, \mathcal{F}_1)\eta(\widehat{\mathcal{W}}_8(\mathcal{H}_2, \mathcal{H}_3)\zeta) - S(\zeta, \mathcal{F}_1)\eta(\widehat{\mathcal{W}}_8(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1) = 0. \end{aligned} \tag{102}$$

Substituting $\mathcal{H}_2$=$\mathcal{H}_3$=ζ in (102) and using (1), we obtain

$$\left[2b(\alpha + \vartheta)(\vartheta - \Psi) + (2\vartheta + \alpha - \Psi)\frac{r}{n}\left\{\frac{a}{(n-1)} - b\right\}\right][g(\mathcal{H}_1, \mathcal{F}_1) + \eta(\mathcal{H}_1)\eta(\mathcal{F}_1)] = 0. \tag{103}$$

Therefore from (103), we can deduce the following cases:
Case I: If $\left[2b(\alpha + \vartheta)(\vartheta - \Psi) + (2\vartheta + \alpha - \Psi)\frac{r}{n}\left\{\frac{a}{(n-1)} - b\right\}\right] = 0$. Then, it infer that

$$\vartheta = -\alpha + \frac{r(n-1)(\alpha^2 - \rho)\{a - b(n-1)\}}{ra - b(n-1)\{r + 2n(n-1)(\alpha^2 - \rho)\}}, \tag{104}$$

and

$$\Psi = -\alpha + \frac{2(n-1)(\alpha^2 - \rho)\{r(a-b) - bn(n-1)(\alpha^2 - \rho)\}}{ra - b(n-1)\{r + 2n(n-1)(\alpha^2 - \rho)\}}. \tag{105}$$

Case II: If $g(\mathcal{H}_1, \mathcal{F}_1) + \eta(\mathcal{H}_1)\eta(\mathcal{F}_1) = 0$. Then replacing $\mathcal{H}_1$ by $Q\mathcal{H}_1$, we obtain

$$S(\mathcal{H}_1, \mathcal{F}_1) = -(n-1)(\alpha^2 - \rho)\eta(\mathcal{H}_1)\eta(\mathcal{F}_1). \tag{106}$$

Let $\{e_1, e_2, e_3, \ldots, e_n = \zeta\}$ be an orthonormal frame field at any point of the manifold. Then substituting $\mathcal{H}_1 = \mathcal{F}_1 = e_i$ and taking summation over $1 \leq i \leq n$, we obtain

$$r = (n-1)(\alpha^2 - \rho). \tag{107}$$

Thus, based on the discussion above, we can present the following theorem:

Theorem 11.1 *Let $\mathcal{M}$ be an n-dimensional $(LCS)_n$-manifold and let $(g, \zeta, \vartheta, \Psi)$ be an η-Ricci soliton on $\mathcal{M}$. If $\mathcal{M}$ satisfies $S(\zeta, \mathcal{H}_1) \cdot \widehat{\mathcal{W}}_8 = 0$, then either of the following statements are true:*
(i) ϑ and Ψ are given by (104) and (105) respectively,
(ii) $\mathcal{M}$ is a special type of η-Einstein manifold, admitting the Ricci tensor of the form (106) and having scalar curvature $r = (n-1)(\alpha^2 - \rho)$.

12 Conservative pseudo $\mathcal{W}_8$-curvature tensor on $(LCS)_n$-manifold

In this section, we examine conservative pseudo $\mathcal{W}_8$-curvature tensor on an $(LCS)_n$-manifold.

Definition 12.1 *An n-dimensional $(LCS)_n$-manifold is referred to as pseudo $\mathcal{W}_8$-conservative if it satisfies the condition* [27]

$$div\widehat{\mathcal{W}}_8 = 0, \tag{108}$$

where div denotes the divergence.
Taking covariant derivative of (1) with respect to $\mathcal{F}_1$, we obtain

$$\begin{aligned}(\nabla_{\mathcal{F}_1}\widehat{\mathcal{W}}_8)(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 &= a(\nabla_{\mathcal{F}_1}R)(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 \\ &\quad + b\left[(\nabla_{\mathcal{F}_1}S)(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 - (\nabla_{\mathcal{F}_1}S)(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1\right].\end{aligned} \tag{109}$$

Contracting (109) with respect to $\mathcal{F}_1$, we have

$$\begin{aligned}(div\widehat{\mathcal{W}}_8)(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 &= a(divR)(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 \\ &\quad + b\left[(\nabla_{\mathcal{H}_3}S)(\mathcal{H}_1, \mathcal{H}_2) - (\nabla_{\mathcal{H}_1}S)(\mathcal{H}_2, \mathcal{H}_3)\right].\end{aligned} \tag{110}$$

From ([10]), we infer that

$$divR = (\nabla_{\mathcal{H}_1}S)(\mathcal{H}_2, \mathcal{H}_3) - (\nabla_{\mathcal{H}_2}S)(\mathcal{H}_1, \mathcal{H}_3). \tag{111}$$

By virtue of (108) and (111) in (110), we obtain

$$(a-b)(\nabla_{\mathcal{H}_1}S)(\mathcal{H}_2, \mathcal{H}_3) - a(\nabla_{\mathcal{H}_2}S)(\mathcal{H}_1, \mathcal{H}_3) + b(\nabla_{\mathcal{H}_3}S)(\mathcal{H}_1, \mathcal{H}_2) = 0. \tag{112}$$

Setting $\mathcal{H}_1 = \zeta$ in (112), we get

$$(a-b)(\nabla_{\zeta}S)(\mathcal{H}_2, \mathcal{H}_3) - a(\nabla_{\mathcal{H}_2}S)(\zeta, \mathcal{H}_3) + b(\nabla_{\mathcal{H}_3}S)(\zeta, \mathcal{H}_2) = 0. \tag{113}$$

Further, we know that

$$(\nabla_{\zeta}S)(\mathcal{H}_1, \mathcal{H}_2) = \zeta S(\mathcal{H}_1, \mathcal{H}_2) - S(\nabla_{\zeta}\mathcal{H}_1, \mathcal{H}_2) - S(\mathcal{H}_1, \nabla_{\zeta}\mathcal{H}_2). \tag{114}$$

On further simplifying (114), we obtain

$$(\nabla_\zeta S)(\mathcal{H}_1, \mathcal{H}_2) = (\pounds_\zeta S)(\mathcal{H}_1, \mathcal{H}_2) - S(\nabla_{\mathcal{H}_1}\zeta, \mathcal{H}_2) - S(\mathcal{H}_1, \nabla_{\mathcal{H}_2}\zeta). \tag{115}$$

The Lie derivative of metric g along with vector field $\mathcal{H}_1$ is

$$(\pounds_{\mathcal{H}_1} g)(\mathcal{H}_2, \mathcal{H}_3) = \pounds_{\mathcal{H}_1} g(\mathcal{H}_2, \mathcal{H}_3) - g(\pounds_{\mathcal{H}_1}\mathcal{H}_2, \mathcal{H}_3) - g(\mathcal{H}_2, \pounds_{\mathcal{H}_1}\mathcal{H}_3). \tag{116}$$

Substituting $\mathcal{H}_1 = \zeta$ in (116) and using (10), we obtain

$$(\pounds_\zeta g)(\mathcal{H}_2, \mathcal{H}_3) = 2\alpha\left[g(\mathcal{H}_2, \mathcal{H}_3) + \eta(\mathcal{H}_2)\eta(\mathcal{H}_3)\right]. \tag{117}$$

Notice that $g(Q\mathcal{H}_1, \mathcal{H}_2) = S(\mathcal{H}_1, \mathcal{H}_2)$, then by (117), we have

$$(\pounds_\zeta S)(\mathcal{H}_2, \mathcal{H}_3) = 2\alpha\left[S(\mathcal{H}_2, \mathcal{H}_3) + (n-1)(\alpha^2 - \rho)\eta(\mathcal{H}_2)\eta(\mathcal{H}_3)\right]. \tag{118}$$

Using (10) and (118) in (115), we infer

$$(\nabla_\zeta S)(\mathcal{H}_1, \mathcal{H}_2) = 0, \tag{119}$$

which yields

$$(\nabla_\zeta r) = 0. \tag{120}$$

In view of (113) and making use of (119), we obtain

$$a(\nabla_{\mathcal{H}_2} S)(\zeta, \mathcal{H}_3) - b(\nabla_{\mathcal{H}_3} S)(\zeta, \mathcal{H}_2) = 0. \tag{121}$$

Taking in consideration (20), we have

$$\alpha(a-b)\left[S(\mathcal{H}_2, \mathcal{H}_3) - (n-1)(\alpha^2 - \rho)g(\mathcal{H}_2, \mathcal{H}_3)\right] = 0. \tag{122}$$

Since α is a non-zero scalar, we have

$$(a-b)\left[S(\mathcal{H}_2, \mathcal{H}_3) - (n-1)(\alpha^2 - \rho)g(\mathcal{H}_2, \mathcal{H}_3)\right] = 0. \tag{123}$$

Hence from (123), we infer following cases:

Case I: If (a-b)=0, then

$$a = b. \tag{124}$$

Case II: If (a-b) $\neq 0$, then

$$S(\mathcal{H}_2, \mathcal{H}_3) = (n-1)(\alpha^2 - \rho)g(\mathcal{H}_2, \mathcal{H}_3). \tag{125}$$

Contracting $\mathcal{H}_2$ and $\mathcal{H}_3$, we obtain

$$r = n(n-1)(\alpha^2 - \rho). \tag{126}$$

Thus, based on the discussion above, we can present the following theorem:

Theorem 12.1 *Let $\mathcal{M}$ be an n-dimensional $(LCS)_n$-manifold. If $\mathcal{M}$ is pseudo-$\mathcal{W}_8$ conservative, then either $a = b$, or $\mathcal{M}$ is an Einstein manifold, admitting the Ricci tensor of the form (125) and the scalar curvature is given by (126).*

13 Irrotational pseudo $\mathcal{W}_8$-curvature tensor on $(LCS)_n$-manifold

In this section, we examine irrotational pseudo-$\mathcal{W}_8$ curvature tensor on an $(LCS)_n$-manifold.

Definition 13.1 *The rotation (or curl) of the pseudo $\mathcal{W}_8$-curvature tensor on an $(LCS)_n$-manifold $\mathcal{M}$ is given by* [27]

$$\begin{aligned} Rot\widehat{\mathcal{W}}_8 &= (\nabla_{\mathcal{F}_1}\widehat{\mathcal{W}}_8)(\mathcal{H}_1,\mathcal{H}_2)\mathcal{H}_3 + (\nabla_{\mathcal{H}_1}\widehat{\mathcal{W}}_8)(\mathcal{F}_1,\mathcal{H}_2)\mathcal{H}_3 + (\nabla_{\mathcal{H}_2}\widehat{\mathcal{W}}_8)(\mathcal{H}_1,\mathcal{F}_1)\mathcal{H}_3 \\ &\quad - (\nabla_{\mathcal{H}_3}\widehat{\mathcal{W}}_8)(\mathcal{H}_1,\mathcal{H}_2)\mathcal{F}_1, \end{aligned} \tag{127}$$

for any vector fields $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$, and $\mathcal{F}_1$ on $\mathcal{M}$.

By taking into account Bianchi Second identity for Riemannian connection ∇, (127) takes the form

$$Rot\widehat{\mathcal{W}}_8 = -(\nabla_{\mathcal{H}_3}\widehat{\mathcal{W}}_8)(\mathcal{H}_1,\mathcal{H}_2)\mathcal{F}_1. \tag{128}$$

If the pseudo $\mathcal{W}_8$-curvature tensor is irrotational, then $Curl\widehat{\mathcal{W}}_8 = 0$, and so by (128), we obtain

$$(\nabla_{\mathcal{H}_3}\widehat{\mathcal{W}}_8)(\mathcal{H}_1,\mathcal{H}_2)\mathcal{F}_1 = 0, \tag{129}$$

which implies

$$\begin{aligned} \nabla_{\mathcal{H}_3}\widehat{\mathcal{W}}_8(\mathcal{H}_1,\mathcal{H}_2)\mathcal{F}_1 &= \widehat{\mathcal{W}}_8(\nabla_{\mathcal{H}_3}\mathcal{H}_1,\mathcal{H}_2)\mathcal{F}_1 + \widehat{\mathcal{W}}_8(\mathcal{H}_1,\nabla_{\mathcal{H}_3}\mathcal{H}_2)\mathcal{F}_1 \\ &\quad + \widehat{\mathcal{W}}_8(\mathcal{H}_1,\mathcal{H}_2)\nabla_{\mathcal{H}_3}\mathcal{F}_1. \end{aligned} \tag{130}$$

Substituting $\mathcal{F}_1 = \zeta$ in (130), we have

$$\nabla_{\mathcal{H}_3}\widehat{\mathcal{W}}_8(\mathcal{H}_1,\mathcal{H}_2)\zeta = \widehat{\mathcal{W}}_8(\nabla_{\mathcal{H}_3}\mathcal{H}_1,\mathcal{H}_2)\zeta + \widehat{\mathcal{W}}_8(\mathcal{H}_1,\nabla_{\mathcal{H}_3}\mathcal{H}_2)\zeta + \widehat{\mathcal{W}}_8(\mathcal{H}_1,\mathcal{H}_2)\nabla_{\mathcal{H}_3}\zeta. \tag{131}$$

By virtue of (1) and taking into account (33), (131) infers

$$\begin{aligned} \widehat{\mathcal{W}}_8(\mathcal{H}_1,\mathcal{H}_2)\mathcal{H}_3 &= [a - b(n-1)]\left[(\alpha^2-\rho) + \frac{r}{n(n-1)}\right] g(\mathcal{H}_2,\mathcal{H}_3)\mathcal{H}_1 \\ &\quad - a(\alpha^2-\rho)g(\mathcal{H}_1,\mathcal{H}_3)\mathcal{H}_2 \\ &\quad + \left[b\left\{\frac{r}{(n-1)} - (\alpha^2-\rho)\right\} - \frac{r}{n}\left\{\frac{a}{n-1} - b\right\}\right] g(\mathcal{H}_1,\mathcal{H}_2)\mathcal{H}_3 \\ &\quad + b\left[\frac{r}{(n-1)} - (\alpha^2-\rho)n\right][g(\mathcal{H}_1,\mathcal{H}_3)\eta(\mathcal{H}_2)\zeta \\ &\quad + g(\mathcal{H}_2,\mathcal{H}_3)\eta(\mathcal{H}_1)\zeta + \eta(\mathcal{H}_1)\eta(\mathcal{H}_2)\mathcal{H}_3 + 2\eta(\mathcal{H}_1)\eta(\mathcal{H}_2)\eta(\mathcal{H}_3)\zeta]. \end{aligned} \tag{132}$$

Using (1) and taking inner product with $\mathcal{F}_3$, (132) takes the form

$$\begin{aligned}
&[a-b(n-1)]\left[(\alpha^2-\rho)+\frac{r}{n(n-1)}\right]g(\mathcal{H}_2,\mathcal{H}_3)g(\mathcal{H}_1,\mathcal{F}_3)\\
&-a(\alpha^2-\rho)g(\mathcal{H}_1,\mathcal{H}_3)g(\mathcal{H}_2,\mathcal{F}_3)\\
&+\left[b\left\{\frac{r}{(n-1)}-(\alpha^2-\rho)\right\}-\frac{r}{n}\left\{\frac{a}{n-1}-b\right\}\right]g(\mathcal{H}_1,\mathcal{H}_2)g(\mathcal{H}_3,\mathcal{F}_3)\\
&+b\left[\frac{r}{(n-1)}-(\alpha^2-\rho)n\right][g(\mathcal{H}_1,\mathcal{H}_3)\eta(\mathcal{H}_2)\eta(\mathcal{F}_3)+g(\mathcal{H}_2,\mathcal{H}_3)\eta(\mathcal{H}_1)\eta(\mathcal{F}_3)\\
&+\eta(\mathcal{H}_1)\eta(\mathcal{H}_2)g(\mathcal{H}_3,\mathcal{F}_3)+2\eta(\mathcal{H}_1)\eta(\mathcal{H}_2)\eta(\mathcal{H}_3)\eta(\mathcal{F}_3)]=aR(\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3,\mathcal{F}_3)\\
&+b\left[S(\mathcal{H}_1,\mathcal{H}_2)g(\mathcal{H}_3,\mathcal{F}_3)-S(\mathcal{H}_2,\mathcal{H}_3)g(\mathcal{H}_1,\mathcal{F}_3)\right]\\
&-\frac{r}{n}\left[\frac{a}{n-1}-b\right]\left[g(\mathcal{H}_1,\mathcal{H}_2)g(\mathcal{H}_3,\mathcal{F}_3)-g(\mathcal{H}_2,\mathcal{H}_3)g(\mathcal{H}_1,\mathcal{F}_3)\right]. \qquad (133)
\end{aligned}$$

Contracting $\mathcal{H}_1$ and $\mathcal{F}_3$ in (133) and on further simplification, we obtain

$$S(\mathcal{H}_2,\mathcal{H}_3)=(n-1)(\alpha^2-\rho)g(\mathcal{H}_2,\mathcal{H}_3). \qquad (134)$$

Moreover on contracting $\mathcal{H}_2$ and $\mathcal{H}_3$, we obtain

$$r=n(n-1)(\alpha^2-\rho). \qquad (135)$$

Thus, based on the discussion above, we can present the following theorem:

Theorem 13.1 *Let $\mathcal{M}$ be an n-dimensional $(LCS)_n$-manifold. If pseudo $\mathcal{W}_8$-curvature tensor on $\mathcal{M}$ is irrotational, then $\mathcal{M}$ is an Einstein manifold with the Ricci tensor of the form (134) and the scalar curvature is given by (135).*

14 Cyclic parallel Ricci tensor on $(LCS)_n$-manifold

In this section, we examine cyclic parallel Ricci tensor on an $(LCS)_n$-manifold.

Definition 14.1 *An n-dimensional $(LCS)_n$-manifold is said to have cyclic parallel Ricci tensor if its Ricci tensor S of type (0,2) is non-zero and satisfies the following condition* [13]

$$(\nabla_{\mathcal{F}_1}S)(\mathcal{F}_2,\mathcal{F}_3)+(\nabla_{\mathcal{F}_2}S)(\mathcal{F}_3,\mathcal{F}_1)+(\nabla_{\mathcal{F}_3}S)(\mathcal{F}_1,\mathcal{F}_2)=0, \qquad (136)$$

for any vector fields $\mathcal{F}_1,\mathcal{F}_2$ and $\mathcal{F}_3$ on $\mathcal{M}$.

Taking covariant derivative of (33) along an arbitrary vector field $\mathcal{F}_1$, we obtain

$$\begin{aligned}
(\nabla_{\mathcal{F}_1}S)(\mathcal{F}_2,\mathcal{F}_3)=&\frac{dr(\mathcal{F}_1)}{(n-1)}\left[g(\mathcal{F}_2,\mathcal{F}_3)+\eta(\mathcal{F}_2)\eta(\mathcal{F}_3)\right]\\
&+\left[\frac{r}{(n-1)}-(\alpha^2-\rho)n\right]\left[(\nabla_{\mathcal{F}_1}\eta)(\mathcal{F}_2)\eta(\mathcal{F}_3)+\eta(\mathcal{F}_2)(\nabla_{\mathcal{F}_1}\eta)(\mathcal{F}_3)\right]. \qquad (137)
\end{aligned}$$

Using (9) in (137), we infer

$$(\nabla_{\mathcal{F}_1} S)(\mathcal{F}_2, \mathcal{F}_3) = \frac{dr(\mathcal{F}_1)}{(n-1)} [g(\mathcal{F}_2, \mathcal{F}_3) + \eta(\mathcal{F}_2)\eta(\mathcal{F}_3)] + \alpha \left[\frac{r}{(n-1)} - (\alpha^2 - \rho)n\right] [g(\mathcal{F}_1, \mathcal{F}_2)\eta(\mathcal{F}_3) + g(\mathcal{F}_1, \mathcal{F}_3)\eta(\mathcal{F}_2) + 2\eta(\mathcal{F}_1)\eta(\mathcal{F}_2)\eta(\mathcal{F}_3)]. \tag{138}$$

Along similar lines, we can obtain

$$(\nabla_{\mathcal{F}_2} S)(\mathcal{F}_3, \mathcal{F}_1) = \frac{dr(\mathcal{F}_2)}{(n-1)} [g(\mathcal{F}_3, \mathcal{F}_1) + \eta(\mathcal{F}_3)\eta(\mathcal{F}_1)] + \alpha \left[\frac{r}{(n-1)} - (\alpha^2 - \rho)n\right] [g(\mathcal{F}_2, \mathcal{F}_3)\eta(\mathcal{F}_1) + g(\mathcal{F}_2, \mathcal{F}_1)\eta(\mathcal{F}_3) + 2\eta(\mathcal{F}_1)\eta(\mathcal{F}_2)\eta(\mathcal{F}_3)]. \tag{139}$$

and

$$(\nabla_{\mathcal{F}_3} S)(\mathcal{F}_1, \mathcal{F}_2) = \frac{dr(\mathcal{F}_3)}{(n-1)} [g(\mathcal{F}_1, \mathcal{F}_2) + \eta(\mathcal{F}_1)\eta(\mathcal{F}_2)] + \alpha \left[\frac{r}{(n-1)} - (\alpha^2 - \rho)n\right] [g(\mathcal{F}_3, \mathcal{F}_1)\eta(\mathcal{F}_2) + g(\mathcal{F}_3, \mathcal{F}_2)\eta(\mathcal{F}_1) + 2\eta(\mathcal{F}_1)\eta(\mathcal{F}_2)\eta(\mathcal{F}_3)]. \tag{140}$$

It is known that the Cartan hypersurfaces are manifolds with non-parallel Ricci tensor satisfying (136). It follows from (136) that r is constant.
By virtue of (138), (139) and (140) in (136), we obtain

$$\alpha \left[\frac{r}{(n-1)} - (\alpha^2 - \rho)n\right] [g(\mathcal{F}_1, \mathcal{F}_2)\eta(\mathcal{F}_3) + g(\mathcal{F}_1, \mathcal{F}_3)\eta(\mathcal{F}_2) + g(\mathcal{F}_2, \mathcal{F}_3)\eta(\mathcal{F}_1) + 3\eta(\mathcal{F}_1)\eta(\mathcal{F}_2)\eta(\mathcal{F}_3)] = 0. \tag{141}$$

Since α is a non-zero scalar, we have

$$\left[\frac{r}{(n-1)} - (\alpha^2 - \rho)n\right] [g(\mathcal{F}_1, \mathcal{F}_2)\eta(\mathcal{F}_3) + g(\mathcal{F}_1, \mathcal{F}_3)\eta(\mathcal{F}_2) + g(\mathcal{F}_2, \mathcal{F}_3)\eta(\mathcal{F}_1) + 3\eta(\mathcal{F}_1)\eta(\mathcal{F}_2)\eta(\mathcal{F}_3)] = 0. \tag{142}$$

Replacing $\mathcal{F}_1$ by $\varphi\mathcal{F}_1$ in (142) and on simplification, we obtain

$$\left[\frac{r}{(n-1)} - (\alpha^2 - \rho)n\right] [g(\varphi\mathcal{F}_1, \mathcal{F}_2)\eta(\mathcal{F}_3) + g(\varphi\mathcal{F}_1, \mathcal{F}_3)\eta(\mathcal{F}_2)] = 0. \tag{143}$$

Plugging $\mathcal{F}_3 = \zeta$ in (143), we have

$$\left[\frac{r}{(n-1)} - (\alpha^2 - \rho)n\right] g(\varphi\mathcal{F}_1, \mathcal{F}_2) = 0. \tag{144}$$

Replacing $\mathcal{F}_2$ by $\varphi\mathcal{F}_2$ in (144) and using (15), we obtain

$$\left[\frac{r}{(n-1)} - (\alpha^2-\rho)n\right][g(\mathcal{F}_1,\mathcal{F}_2)+\eta(\mathcal{F}_1)\eta(\mathcal{F}_2)] = 0. \tag{145}$$

Substituting $\mathcal{F}_1$ by $Q\mathcal{F}_1$ in (145) and on simplification, we infer

$$\left[\frac{r}{(n-1)} - (\alpha^2-\rho)n\right]\left[S(\mathcal{F}_1,\mathcal{F}_2)+(n-1)(\alpha^2-\rho)\eta(\mathcal{F}_1)\eta(\mathcal{F}_2)\right] = 0. \tag{146}$$

Hence from (146), we deduce following cases:

Case I: If $\left[\frac{r}{(n-1)} - (\alpha^2-\rho)n\right] = 0$, then

$$r = n(n-1)(\alpha^2-\rho). \tag{147}$$

Case II: If $\left[\frac{r}{(n-1)} - (\alpha^2-\rho)n\right] \neq 0$, then

$$S(\mathcal{F}_1,\mathcal{F}_2) = -(n-1)(\alpha^2-\rho)\eta(\mathcal{F}_1)\eta(\mathcal{F}_2). \tag{148}$$

Contracting $\mathcal{F}_1$ and $\mathcal{F}_2$ in (148), we have

$$r = (n-1)(\alpha^2-\rho). \tag{149}$$

Thus, based on the discussion above, we can present the following theorem:

Theorem 14.1 *Let $\mathcal{M}$ be an n-dimensional $(LCS)_n$-manifold admitting a cyclic parallel Ricci tensor. Then either of the following statements are true:*
(i) The scalar curvature $r = n(n-1)(\alpha^2-\rho)$,
(ii) $\mathcal{M}$ is a special type of an η-Einstein manifold, with a Ricci tensor of the form (148) and the scalar curvature given by $r = (n-1)(\alpha^2-\rho)$.

15 η-parallel Ricci tensor on $(LCS)_n$-manifold

In this section, we examine η-parallel Ricci tensor on an $(LCS)_n$-manifold.

Definition 15.1 *An n-dimensional $(LCS)_n$-manifold $\mathcal{M}$ is said to be η-parallel if its Ricci tensor S satisfies* [19]

$$(\nabla_{\mathcal{F}_3}S)(\varphi\mathcal{F}_1,\varphi\mathcal{F}_2) = 0, \tag{150}$$

for any vector fields $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ on $\mathcal{M}$.
Assume that the Ricci tensor of an $(LCS)_n$-manifold is η-parallel. Then (150) is satisfied.
By virtue of (33), we have

$$S(\varphi\mathcal{F}_1,\varphi\mathcal{F}_2) = \left[\frac{r}{(n-1)} - (\alpha^2-\rho)\right][g(\mathcal{F}_1,\mathcal{F}_2)+\eta(\mathcal{F}_1)\eta(\mathcal{F}_2)]. \tag{151}$$

Taking covariant derivative of (151) along an arbitrary vector field $\mathcal{F}_3$ and using (9) and (22), we have

$$(\nabla_{\mathcal{F}_3}S)(\varphi\mathcal{F}_1,\varphi\mathcal{F}_2) = \frac{dr(\mathcal{F}_3)}{(n-1)}[g(\mathcal{F}_1,\mathcal{F}_2)+\eta(\mathcal{F}_1)\eta(\mathcal{F}_2)]$$
$$+\alpha\left[\frac{r}{(n-1)}-(\alpha^2-\rho)\right][\eta(\mathcal{F}_2)g(\mathcal{F}_3,\mathcal{F}_1)$$
$$+\eta(\mathcal{F}_1)g(\mathcal{F}_2,\mathcal{F}_3)-\eta(\mathcal{F}_1)g(\mathcal{F}_3,\varphi\mathcal{F}_2)-\eta(\mathcal{F}_2)g(\varphi\mathcal{F}_1,\mathcal{F}_3)$$
$$+2\eta(\mathcal{F}_1)\eta(\mathcal{F}_2)\eta(\mathcal{F}_3)]. \quad (152)$$

By virtue of (150), we have

$$\frac{dr(\mathcal{F}_3)}{(n-1)}[g(\mathcal{F}_1,\mathcal{F}_2)+\eta(\mathcal{F}_1)\eta(\mathcal{F}_2)]+\alpha\left[\frac{r}{(n-1)}-(\alpha^2-\rho)\right][\eta(\mathcal{F}_2)g(\mathcal{F}_3,\mathcal{F}_1)$$
$$+\eta(\mathcal{F}_1)g(\mathcal{F}_2,\mathcal{F}_3)-\eta(\mathcal{F}_1)g(\mathcal{F}_3,\varphi\mathcal{F}_2)-\eta(\mathcal{F}_2)g(\varphi\mathcal{F}_1,\mathcal{F}_3)+2\eta(\mathcal{F}_1)\eta(\mathcal{F}_2)\eta(\mathcal{F}_3)]=0. \quad (153)$$

Contracting $\mathcal{F}_1$ and $\mathcal{F}_2$ in (153), we obtain

$$dr(\mathcal{F}_3)=0. \quad (154)$$

Using (154) in (153) and substituting $\mathcal{F}_1=\zeta$, we get

$$\alpha\left[\frac{r}{(n-1)}-(\alpha^2-\rho)\right][-g(\mathcal{F}_2,\mathcal{F}_3)-\eta(\mathcal{F}_2)\eta(\mathcal{F}_3)+g(\varphi\mathcal{F}_3,\mathcal{F}_2)]=0. \quad (155)$$

Contracting $\mathcal{F}_2$ and $\mathcal{F}_3$ in (155), we have

$$\alpha\left[\frac{r}{(n-1)}-(\alpha^2-\rho)\right][\varrho-n+1]=0, \quad (156)$$

where ϱ=$trace(\varphi)$.
Since α is a non-zero scalar, we have

$$\left[\frac{r}{(n-1)}-(\alpha^2-\rho)\right][\varrho-n+1]=0, \quad (157)$$

Hence from (157), we infer following cases:

Case I: If $\left[\frac{r}{(n-1)}-(\alpha^2-\rho)\right]=0$. Then,

$$r=(n-1)(\alpha^2-\rho). \quad (158)$$

Case II: If $\left[\frac{r}{(n-1)}-(\alpha^2-\rho)\right]\neq 0$. Then,

$$\varrho=(n-1). \quad (159)$$

Thus, based on the discussion above, we can present the following theorem:

Theorem 15.1 *Let $\mathcal{M}$ be an n-dimensional $(LCS)_n$-manifold admitting an η-parallel Ricci tensor. Then either of the following statements are true:*
(i) The scalar curvature $r=(n-1)(\alpha^2-\rho)$,
(ii) $\varrho=(n-1)$, where $\varrho=trace(\varphi)$.

16 Conclusions

In summary, this paper provides a detailed analysis of $(LCS)_n$-manifolds by examining a variety of conditions related to curvature tensors and Ricci solitons. Section 2 begins with foundational concepts that establish the framework for the study, setting up a solid basis for investigating the geometry of $(LCS)_n$-manifolds. Section 3 then delves into pseudo $\mathcal{W}_8$-flat $(LCS)_n$-manifolds, uncovering significant relationships and properties that influence later sections.

Sections 4 and 5 explore two specific conditions—ζ-pseudo $\mathcal{W}_8$-flatness and φ-pseudo $\mathcal{W}_8$-semisymmetry, respectively—showing that these manifolds can be characterized as η-Einstein and generalized η-Einstein manifolds. These sections emphasize the intricate connections between curvature properties and Einstein structures in this class of manifolds.

Expanding further, Sections 6, 7, and 8 examine $(LCS)_n$-manifolds under conditions involving the $\widehat{\mathcal{W}}_8$ operator applied to certain tensors, including the Ricci tensor and curvature operator. These conditions continue to classify the manifolds as either Einstein or η-Einstein, reinforcing the prevalence of this classification across a range of geometric conditions.

In Sections 9, 10, and 11, the focus shifts to pseudosymmetric conditions and η-Ricci solitons, providing insights into the broader structure of $(LCS)_n$-manifolds. These sections reveal that $(LCS)_n$-manifolds consistently display η-Einstein properties under diverse curvature and Ricci soliton conditions, underscoring the central role of Einstein structures within this framework.

Section 12 explores conservative pseudo $\mathcal{W}_8$-curvature tensors on $(LCS)_n$-manifolds, presenting intriguing findings. Section 13 investigates irrotational pseudo $\mathcal{W}_8$-curvature tensors, demonstrating that these manifolds are Einstein. Finally, Section 14 addresses cyclic parallel Ricci tensors, while Section 15 discusses η-parallel Ricci tensors, each revealing additional significant results.

Overall, this study offers a comprehensive examination of $(LCS)_n$-manifolds by systematically analyzing the connections between curvature conditions and Einstein properties. The results classify these manifolds as either η-Einstein or generalized η-Einstein and enhance our understanding of their geometric structures across various pseudo-flat and pseudosymmetric conditions. These findings contribute to the broader understanding of Einstein manifolds and their applications in differential geometry, offering valuable avenues for further research into the geometry and topology of $(LCS)_n$-manifolds.

17 Acknowledgement

The authors express their deep gratitude to Prof. Ljubica Velimirovic from the Department of Mathematics, Faculty of Sciences and Mathematics, University of Niš, Serbia for her continuous support and valuable insights, which significantly improved the quality and presentation of this paper.

18 Statements and declarations

Funding: The authors confirm that no financial support, grants, or funding were received during the preparation of this manuscript.
Informed Consent Statement: Informed consent does not apply to this study.
Data Availability Statement: Data from prior studies has been utilized, and all relevant sources are properly cited in the reference list.
Conflicts of Interest: The authors declare that they have no conflicts of interest related to this work.

References

[1] Atceken, M., *On geometry of submanifolds of $(LCS)_n$-manifolds*, Int. J. Math. Math. Sci. **2012** (2012), 1–12.

[2] Atceken, M. and Hui, S. K., *Slant and pseudo-slant submanifolds of $(LCS)_n$-manifolds*, Czechoslovak Math. J. **63** (2013), 177–190.

[3] Blaga, A. M., *η-Ricci solitons on para-Kenmotsu manifolds*, Balkan J. Geom. Appl. **20** (2015), 1–13.

[4] Blaga, A. M., *η-Ricci solitons on Lorentzian para-Sasakian manifolds*, Filomat **30** (2016), 489–496.

[5] Blaga, A. M., Perktas, S. Y., Acet, B. E. and Erdogan, F. E., *η-Ricci solitons in (ϵ)-almost paracontact metric manifolds*, Glas. Mat. **53** (2018), 205–220.

[6] Cho, J. T. and Kimura, M., *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J. **61** (2009), 205–212.

[7] De, K. and De, U. C., *η-Ricci solitons on Kenmotsu 3-manifolds*, An. Univ. Timişoara **56** (2018), 51–63.

[8] Deshmukh, S., *Jacobi-type vector fields on Ricci solitons*, Bull. Math. Soc. Sci. Math. Roumanie **55** (2012), 41–50.

[9] Deshmukh, S., Alodan, H. and Al-Sodais, H., *A note on Ricci solitons*, Balkan J. Geom. Appl. **16** (2011), 48–55.

[10] Eisenhart, L. P., *Riemannian geometry*, Princeton Univ. Press, Princeton, 1926.

[11] Eyasmin, S., Chowdhury, P. R. and Baishya, K. K., *η-Ricci solitons on Kenmotsu manifolds*, Honam Math. J. **40** (2018), 383–392.

[12] Hamilton, R. S., *Three-manifolds with positive Ricci curvature*, J. Differ. Geom. **17** (1982), 255–306.

[13] Hamilton, R. S., *The Ricci flow on surfaces*, Contemp. Math. **71** (1988), 237–261.

[14] Hui, S. K., *On ϕ-pseudo symmetries of $(LCS)_n$-manifolds*, Kyungpook Math. J. **53** (2013), 285–294.

[15] Hui, S. K. and Atceken, M., *Contact warped product semi-slant submanifolds of $(LCS)_n$-manifolds*, Acta Univ. Sapientiae Math. **3** (2011), 212–224.

[16] Hui, S. K. and Chakraborty, D., *Some types of Ricci solitons on $(LCS)_n$-manifolds*, J. Math. Sci. Adv. Appl. **37** (2016), 1–17.

[17] Ivey, T., *Ricci solitons on compact three-manifolds*, Differ. Geom. Appl. **3** (1993), 301–307.

[18] Kar, D., Majhi, P. and De, U. C., *η-Ricci solitons on 3-dimensional N(k)-contact metric manifolds*, Acta Univ. Apulensis **54** (2018), 71–88.

[19] Kon, M., *Invariant submanifolds in Sasakian manifolds*, Math. Ann. **219** (1976), 277–290.

[20] Matsumoto, K., *On Lorentzian almost paracontact manifolds*, Bull. Yamagata Univ. Nat. Sci. **12** (1989), 151–156.

[21] Mihai, I. and Rosca, R., *On Lorentzian para-Sasakian manifolds*, Classical Anal., World Sci. Publ., Singapore, 155–169, 1992.

[22] Narain, D. and Yadav, S., *On weak concircular symmetries of $(LCS)_{2n+1}$-manifolds*, Global J. Sci. Front. Res. **12** (2012), 85–94.

[23] O'Neill, B., *Semi-Riemannian geometry with applications to relativity*, Academic Press, New York, 1983.

[24] Perelman, G., *The entropy formula for the Ricci flow and its geometric applications*, arXiv preprint arXiv:math/0211159 (2002), 1–39.

[25] Perelman, G., *Ricci flow with surgery on three-manifolds*, arXiv preprint arXiv:math/0303109 (2003), 1–22.

[26] Prakasha, D. G., *On Ricci η-recurrent $(LCS)_n$-manifolds*, Acta Univ. Apulensis **24** (2010), 109–118.

[27] Prakasha, D. G. and Chavan, V., *On M-projective curvature tensor of Lorentzian α-Sasakian manifolds*, Int. J. Pure Math. Sci. **18** (2017), 22–31.

[28] Prakasha, D. G. and Hadimani, B. S., *η-Ricci solitons on para-Sasakian manifolds*, J. Geom. **108** (2017), 383–392.

[29] Prasad, B., Yadav, R. P. S. and Pandey, S. N., *Pseudo W_8-curvature tensor on Riemannian manifold*, J. Prog. Sci. **9** (2018), 35–43.

[30] Shaikh, A. A., *On Lorentzian almost paracontact manifolds with a structure of the concircular type*, Kyungpook Math. J. **43** (2003), 305–314.

[31] Shaikh, A. A., *Some results on $(LCS)_n$-manifolds*, J. Korean Math. Soc. **46** (2009), 449–461.

[32] Shaikh, A. A. and Ahmad, H., *Some transformations on $(LCS)_n$-manifolds*, Tsukuba J. Math. **38** (2014), 1–24.

[33] Shaikh, A. A. and Baishya, K. K., *On concircular structure spacetimes*, J. Math. Stat. **1** (2005), 129–132.

[34] Shaikh, A. A. and Baishya, K. K., *On concircular structure spacetimes II*, Am. J. Appl. Sci. **3** (2006), 1790–1794.

[35] Shaikh, A. A., Basu, T. and Eyasmin, S., *On locally ϕ-symmetric $(LCS)_n$-manifolds*, Int. J. Pure Appl. Math. **41** (2007), 1161–1170.

[36] Shaikh, A. A., Basu, T. and Eyasmin, S., *On the existence of ϕ-recurrent $(LCS)_n$-manifolds*, Extracta Math. **23** (2008), 71–83.

[37] Shaikh, A. A. and Binh, T. Q., *On weakly symmetric $(LCS)_n$-manifolds*, J. Adv. Math. Stud. **2** (2009), 75–90.

[38] Shaikh, A. A. and Hui, S. K., *On some classes of generalized quasi-Einstein manifolds*, Commun. Korean Math. Soc. **24** (2009), 415–424.

[39] Shaikh, A. A. and Hui, S. K., *On generalized ϕ-recurrent $(LCS)_n$-manifolds*, AIP Conf. Proc. **1309** (2010), 419–429.

[40] Shaikh, A. A., Matsuyama, Y. and Hui, S. K., *On invariant submanifolds of $(LCS)_n$-manifolds*, J. Egypt. Math. Soc. **24** (2016), 263–269.

[41] Tripathi, M. M. and Gupta, P., *On (N(k); ξ)-semi-Riemannian manifolds: semisymmetries*, Int. J. Pure Appl. Math. **25** (2012), 42-77.