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CONVERGENCE OF THE STEFFENSEN'S METHOD IN RIEMANNIAN MANIFOLDS

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Abstract

The main purpose of this article is to establish the semilocal convergence analysis of the Steffensen's method in Riemannian manifolds. We establish the semilocal convergence analysis of the Steffensen's method in Riemannian manifolds by using the majorant principle. Finally, some examples are presented to verify the convergence criteria.

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1 Introduction

We consider to find the approximate solution of a nonlinear equation

$$\mathfrak{M}(x) = 0,$$

where $\mathfrak{M}$ is a nonlinear operator defined in an open convex subset Ω of a Banach space B into itself. The most famous second order iterative method to solve a non-linear equation in Banach space is Newton's method. There are many problems in applied sciences and other including engineering, optimization, dynamic economic system, physics, biological problems which is formulated in a equation by using mathematial modelling to find the zeros of a nonlinear equations (see [2, 3, 4, 7, 9]). So many types of iterative methods have been studied in Banach spaces. Now, the work in Riemannian manifolds of numerical iterative methods have been growing interest as there are many types of iterative methods have been studied in manifolds which arises in many contexts. Some problems including eigenvalue problem, minimization problems with orthogonality constraints, optimization problems with equality constraints, invariant subspace computations.

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To solve this problem, we have to find the singular point or zero of a vector field in Riemannian manifolds. Generally, we study the convergence of an iterative methods centered on two types one is semilocal and other is local convergence. The convergence analysis which gives information about a solution and calculates the radius of the convergence, then it is local and when the convergence analysis gives information about an initial point, then it is semilocal. The Steffensen's method which is second order method in Banach space is defined as:

$$\left.\begin{aligned} x_0 &\in \Omega, \\ y_n &= x_n + \mathfrak{M}(x_n), \\ x_{n+1} &= x_n - [x_n, y_n; \mathfrak{M}]^{-1}\mathfrak{M}(x_n),\ \text{for each } n = 0, 1, 2, \ldots, \end{aligned}\right\} \tag{1}$$

where $\mathfrak{M}$ is a nonlinear operator defined in an open convex subset Ω of a Banach space B into itself and $\mathfrak{M}$ is first Fréchet derivative in Ω. The computational efficiency of the Steffensen's method is the same as the Newton's method, when it is applied to find the solution of finite dimensional system of nonlinear equations. Although the use of Steffensen's method is less than Newton's method, its use is interesting since it does not require to evaluate the inverse of first Fréchet derivative in each step. The convergence of this second order method in Banach space has been studied in [5]. In this article, we study the semilocal convergence analysis of the Steffensen's method in Riemannian manifolds.

The article is divided into five sections as follows: Section 1 is the introduction. Section 2, contains all the necessary background on fundamental properties and notation of Riemannian manifolds. In Section 3, we study the semilocal convergence analysis of the Steffensen's method in Riemannian manifolds by using the majorant principle. In Section 4, two numerical examples are given. In Section 5, conclusion of this article is given.

2 Preliminaries

In this section, we introduce some basic definitions and properties of Riemannian manifolds (for more details see [6, 10, 8]).

Throughout the article, we denote $\mathbf{A}$ as a real n- dimensional Riemannian manifold, $\mathfrak{X}(\mathbf{A})$ be the set of all vector fields of class C^∞ on $\mathbf{A}$, the tangent space of $\mathbf{A}$ at u by $T_u\mathbf{A}$ and the tangent bundle by $T\mathbf{A} = \bigcup\limits_{u\in\mathbf{A}} T_u\mathbf{A}$. Suppose $\mathbf{A}$ be equipped with a Riemannian metric $\langle .,. \rangle$ with corresponding norm $\|.\|$. The arc length of piecewise smooth curve $\psi : [0,1] \to \mathbf{A}$ joining u to v is defined by $l(\psi) = \int_0^1 \|\psi'(z)\| dz$ and the Riemannian distance joining u to v is defined by $d(u,v) = \inf_\psi l(\psi)$. The affine connection ∇ on $\mathbf{A}$ is a map

$$\begin{aligned} \nabla : \mathfrak{X}(\mathbf{A}) \times \mathfrak{X}(\mathbf{A}) &\to \mathfrak{X}(\mathbf{A}) \\ (X, \mathbf{Z}) &\mapsto \nabla_X \mathbf{Z} \end{aligned}$$

that satisfies the following properties

$$\begin{aligned}&\text{(i) } \nabla_{fX+g\mathbf{Z}}\mathfrak{V} = f\nabla_X\mathfrak{V} + g\nabla_{\mathbf{Z}}\mathfrak{V}.\\&\text{(ii) } \nabla_X(\mathbf{Z}+\mathfrak{V}) = \nabla_X\mathbf{Z} + \nabla_X\mathfrak{V}.\\&\text{(iii) } \nabla_X(f\mathbf{Z}) = f\nabla_X\mathbf{Z} + X(f)\mathbf{Z},\end{aligned}$$

where X, $\mathbf{Z}$, $\mathfrak{V} \in \mathfrak{X}(\mathbf{A})$ and f, $g \in D(\mathbf{A})$, $D(\mathbf{A})$ is the ring of real-valued functions of class C^k on $\mathbf{A}$. Let $\mathbf{Z}$ be a vector field of class C^1 on $\mathbf{A}$, the covariant derivative of $\mathbf{Z}$ is determined by ∇ which defines at each point $u \in \mathbf{A}$, a linear application as

$$\begin{aligned}D\mathbf{Z}(u) &: T_u\mathbf{A} \to T_u\mathbf{A}\\v &\mapsto D\mathbf{Z}(u)(v) = \nabla_X\mathbf{Z}(u),\end{aligned}$$

where $\mathbf{Z} \in \mathfrak{X}(\mathbf{A})$ of class C^1 on $\mathbf{A}$ and X is a vector field satisfies $X(u) = v$. We define the open and closed geodesic ball with centre u and radius v respectively, as

$$V(u, v) = \{t \in \mathbf{A} : d(u, t) < v\}$$

and

$$V[u, v] = \{t \in \mathbf{A} : d(u, t) \leq v\}.$$

A parametrized curve ψ is said to be a geodesic at $t_0 \in I$, if $\nabla_{\psi'(t)}\psi'(t) = 0$ at the point t_0, where $\psi : I \to \mathbf{A}$. If it is a geodesic for all $t \in I$, then we call ψ is a geodesic. If $[p, q] \subseteq I$ then ψ is a geodesic segment joining $\psi(p)$ to $\psi(q)$. Let By the Hopf-Rinow theorem, if $\mathbf{A}$ is complete metric space, then for any u, $t \in \mathbf{A}$ there exists a geodesic ψ, called the minimizing geodesic joining u to t with

$$l(\psi) = d(u, t).$$

If $v \in T_u\mathbf{A}$, then there exists a unique minimizing geodesic ψ such that $\psi(0) = u$ and $\psi'(0) = v$. The point $\psi(1)$ is called the image of v by the exponential map at u, i.e.

$$\exp_u : T_u\mathbf{A} \to \mathbf{A}$$

such that $\exp_u(v) = \psi(1)$ and for all $p \in [0, 1]$, $\psi(p) = \exp_u(pv)$. Let ψ be a piecewise smooth curve, then for any $x, y \in \mathbb{R}$, the parallel transport along ψ is a mapping from $T_{\psi(x)}\mathbf{A}$ to $T_{\psi(y)}\mathbf{A}$. It is denoted by $\mathbb{M}_{\psi,.,.}$ and given by

$$\begin{aligned}\mathbb{M}_{\psi,x,y} &: T_{\psi(x)}\mathbf{A} \to T_{\psi(y)}\mathbf{A}\\&\qquad v \mapsto V(\psi(y)),\end{aligned}$$

where V is the unique vector field along ψ which satisfies $\nabla_{\psi'(t)}V = 0$ and $V(\psi(x)) = v$.

Definition 1. *Let* $\boldsymbol{Z} \in \mathfrak{X}(\boldsymbol{A})$ *of class* C^k *on* $\boldsymbol{A}$ *and* $j \in \mathbb{N}$. *The covariant derivative of order* j *of* $\boldsymbol{Z}$ *is denoted by* $D^j\boldsymbol{Z}$ *and defined as:*

$$D^j\boldsymbol{Z} : \underbrace{C^k(T\boldsymbol{A}) \times C^k(T\boldsymbol{A}) \times \cdots \times C^k(T\boldsymbol{A})}_{j\text{-times}} \to C^{k-j}(T\boldsymbol{A})$$

where

$$\begin{aligned} D^j\boldsymbol{Z}(A_1, A_2, \ldots, A_{j-1}, A) &= \nabla_A D^{j-1}\boldsymbol{Z}(A_1, A_2, \ldots, A_{j-1}) \\ &\quad - \sum_{i=1}^{j-1} D^{j-1}\boldsymbol{Z}(A_1, A_2, \ldots, \nabla_A A_i, \ldots, A_{j-1}) \end{aligned} \tag{2}$$

for all $A_1, A_2, \ldots, A_{j-1} \in C^k(T\boldsymbol{A})$.

Definition 2. *Let* $\mho \subseteq \boldsymbol{A}$ *be an open convex set and* $\boldsymbol{Z} \in \mathfrak{X}(\boldsymbol{A})$. *The covariant derivative* $D\boldsymbol{Z} = \nabla_{(.)}\boldsymbol{Z}$ *is Lipschitz with contant* $\mathfrak{E} > 0$, *if for any geodesic* ψ *and* $x, y \in \mathbb{R}$ *such that* $\psi[x, y] \subseteq \mho$, *it holds the inequality*

$$\|\mathbb{M}_{\psi,y,x} D\boldsymbol{Z}(\psi(y))\mathbb{M}_{\psi,x,y} - D\boldsymbol{Z}(\psi(x))\| \leq \mathfrak{E}\int_x^y \|\psi'(t)\|dt,$$

and we write $D\boldsymbol{Z} \in Lip_{\mathfrak{E}}(\mho)$. *If* $\boldsymbol{A}$ *is finite dimensional Euclidean space, then it coincides with Lipschitz condition for* $D\boldsymbol{Z} : \boldsymbol{A} \to \boldsymbol{A}$.

Definition 3. *Let* $\mho \subseteq \boldsymbol{A}$ *be an open convex set,* ψ *be a curve in* $\boldsymbol{A}$, $[t, t + \delta e] \subset Dom(\psi)$ *and* $\boldsymbol{Z} \in \mathfrak{X}(\boldsymbol{A})$ *of class* C^0 *on* $\boldsymbol{A}$. *The divided difference of first order for* $\boldsymbol{Z}$ *on the points* $\psi(t)$ *and* $\psi(t + \delta e)$ *in the direction* $\psi'(t)$ *is defined by*

$$[\psi(t + \delta x), \psi(t); \boldsymbol{Z}]\psi'(t) = \frac{1}{\delta e}(\mathbb{M}_{\psi,t+\delta e,t}\boldsymbol{Z}(\psi(t + \delta e)) - \boldsymbol{Z}(\psi(t))). \tag{3}$$

The case when $\boldsymbol{A}$ *is a Banach space, if* ψ *is the geodesic joining* u_1 *and* u_2 *such that*

$$\psi(t) = u_1 + t(u_2 - u_1), \qquad t \in \mathbb{R},$$

then from (3), we obtain

$$[u_2, u_1; \boldsymbol{Z}](u_2 - u_1) = \boldsymbol{Z}(u_2) - \boldsymbol{Z}(u_1).$$

Also if $D\boldsymbol{Z}(u)$ *exists, then* $D\boldsymbol{Z}(u) = [u, u; \boldsymbol{Z}]$.

Next, we take a theorem from [1] which will be used to prove the convergence of our iterative method.

Theorem 1. *Let* ψ *be a geodesic in* $\boldsymbol{A}$ *and let* $\boldsymbol{Z} \in \mathfrak{X}(\boldsymbol{A})$ *of class* C^1 *on* $\boldsymbol{A}$. *Then,*

$$\mathbb{M}_{\psi,t,0}\boldsymbol{Z}(\psi(t)) = \boldsymbol{Z}(\psi(0)) + \int_0^t \mathbb{M}_{\psi,a,0} D\boldsymbol{Z}(\psi(a))\psi'(a)da. \tag{4}$$

3 Steffensen's method in Riemannian manifolds

In this section, we will study the semilocal convergence analysis of the Steffensen's method in Riemannian manifolds. The method (1) to solve a vector field $\mathbf{Z}(u) = 0,\ u \in \mathbf{A}$ in Riemannian manifolds has the form

$$\left.\begin{aligned} l_n &= \mathbf{Z}(u_n), \quad u_0 \in \mho \\ v_n &= \exp_{u_n}(l_n), \\ m_n &= -[u_n, v_n; \mathbf{Z}]^{-1}\mathbf{Z}(u_n), \\ u_{n+1} &= \exp_{u_n}(m_n), \ \text{for each } n = 0, 1, 2, \ldots \end{aligned}\right\} \tag{5}$$

Suppose that $\mathbf{Z}(u)$ satisfies the conditions:

1. $\|\mathbf{Z}(u_0)\| \le \xi, \quad \xi > 0,$
2. $\|D\mathbf{Z}(u_0)^{-1}\| \le \zeta_0, \quad \zeta_0 > 0,$
3. $\|\mathbb{M}_{\Phi,b,a} D\mathbf{Z}(\Phi(b))\mathbb{M}_{\Phi,a,b} - D\mathbf{Z}(\Phi(a))\| \le K\int_a^b \|\Phi'(x)\|dx,$
   where Φ is the geodesic such that $\Phi[a, b] \subseteq \mho,\ K > 0.$

First, we shall show that the operator $[u_0, v_0; \mathbf{Z}]^{-1}$ is bounded. Let $I_{u_0} : T_{u_0}\mathbf{A} \to T_{u_0}\mathbf{A}$ be the identity operator and let α_n be the family of minimizing geodesics joining the points u_n and v_n such that $\alpha_n(0) = u_n$ and $\alpha_n(1) = v_n$ for each $n = 0, 1, 2, \ldots$ Then, we have

$$\begin{aligned} &\|I_{u_0} - D\mathbf{Z}(u_0)^{-1}[u_0, v_0; \mathbf{Z}]\| \\ &= \|D\mathbf{Z}(u_0)^{-1}\| \int_0^1 \|\mathbb{M}_{\alpha_0,1,0} D\mathbf{Z}(\alpha_0(t))\mathbb{M}_{\alpha_0,0,1} - D\mathbf{Z}(u_0)\|dt \\ &\le \frac{1}{2}\zeta_0 K\|l_0\| = \frac{1}{2}\zeta_0\xi K < 1, \end{aligned}$$

if $K\xi\zeta_0 < 2$, then the operator $[u_0, v_0; \mathbf{Z}]$ is invertible and

$$\|[u_0, v_0; \mathbf{Z}]^{-1}\| \le \frac{2\zeta_0}{2 - K\xi\zeta_0} = c.$$

Now, we define the polynomial

$$z(f) = \frac{L}{2}f^2 - \frac{f}{c} + \xi, \quad L = K\left(1 + \frac{1}{c}\right), \quad f \in [0, f'] \tag{6}$$

and we denote f^* and f^{**} the two positive roots of $z(f)$ such that $0 < f^* \le f^{**} < f'$, if $L\xi c^2 \le \frac{1}{2}$.

Also we define the sequences $\{f_n\}$ and $\{\zeta_{n+1}\}$ for $n \ge 0$, by

$$f_{n+1} = f_n - \frac{z(f_n)}{z'(f_n)}, \quad f_0 = 0, \tag{7}$$

$$\zeta_{n+1} = \frac{\zeta_0}{1 - \zeta_0 K d(u_{n+1}, u_0)}.$$

Now, we will prove some lemmas to prove the convergence of our method.

Lemma 1. *Let $\boldsymbol{Z} \in \mathfrak{X}(\boldsymbol{A})$ of class C^1 on $\boldsymbol{A}$. Then, for all $n \in \mathbb{N}$, we have*

$$\mathbb{M}_{\phi_{n-1},1,0}\boldsymbol{Z}(u_n) = \Big(\int_0^1 \big(\mathbb{M}_{\phi_{n-1},1,0}D\boldsymbol{Z}(\phi_{n-1}(t))\mathbb{M}_{\phi_{n-1},0,1} - D\boldsymbol{Z}(u_{n-1})\big)dt \\ + \int_0^1 \big(D\boldsymbol{Z}(u_{n-1}) - \mathbb{M}_{\alpha_{n-1},1,0}D\boldsymbol{Z}(\alpha_{n-1}(t))\mathbb{M}_{\alpha_{n-1},0,1}\big)dt\Big)m_{n-1}, \tag{8}$$

where ϕ_{n-1} is the family of minimizing geodesics joining the points u_{n-1} and u_n such that $\phi_{n-1}(0) = u_{n-1}$ and $\phi_{n-1}(1) = u_n$, α_{n-1} is as given above.

Proof. We know that

$$[\phi_{n-1}(s+h), \phi_{n-1}(s); \mathbf{Z}]\phi'_{n-1}(s) = \frac{1}{h}\Big(\mathbb{M}_{\phi_{n-1},s+h,s}\mathbf{Z}(\phi_{n-1}(s+h)) - \mathbf{Z}(\phi_{n-1}(s))\Big),$$

put s=0 and h=1 in above equality, we get

$$[u_n, u_{n-1}; \mathbf{Z}]\phi'_{n-1}(0) = \mathbb{M}_{\phi_{n-1},1,0}\mathbf{Z}(u_n) - \mathbf{Z}(u_{n-1}).$$

Since $\phi_{n-1}(t) = \exp_{u_{n-1}}(tm_{n-1})$, we have $\phi'_{n-1}(0) = m_{n-1}$,

putting this value in above equality, we get

$$[u_n, u_{n-1}; \mathbf{Z}]m_{n-1} = \mathbb{M}_{\phi_{n-1},1,0}\mathbf{Z}(u_n) - \mathbf{Z}(u_{n-1}). \tag{9}$$

By (5), we have

$$m_{n-1} = -[u_{n-1}, v_{n-1}; \mathbf{Z}]^{-1}\mathbf{Z}(u_{n-1}),$$

therefore

$$\mathbf{Z}(u_{n-1}) = -[u_{n-1}, v_{n-1}; \mathbf{Z}]m_{n-1}. \tag{10}$$

By (9) and (10), we have

$$\begin{aligned}\mathbb{M}_{\phi_{n-1},1,0}\mathbf{Z}(u_n) &= \Big([u_{n-1}, u_n; \mathbf{Z}] - [u_{n-1}, v_{n-1}; \mathbf{Z}]\Big)m_{n-1} \\ &= \Big(\int_0^1 \big(\mathbb{M}_{\phi_{n-1},1,0}D\mathbf{Z}(\phi_{n-1}(t))\mathbb{M}_{\phi_{n-1},0,1} - D\mathbf{Z}(u_{n-1})\big)dt \\ &+ \int_0^1 \big(D\mathbf{Z}(u_{n-1}) - \mathbb{M}_{\alpha_{n-1},1,0}D\mathbf{Z}(\alpha_{n-1}(t))\mathbb{M}_{\alpha_{n-1},0,1}\big)dt\Big)m_{n-1}.\end{aligned}$$

□

Lemma 2. *Suppose the sequence $\{f_n\}$ is generated by (7). If $L\xi c^2 \leq \frac{1}{2}$ and $f \in [0, f^*]$, then the sequence $\{f_n\}$ is increasing and bounded above. Hence converges to f^*.*

Proof. We define the function h by

$$h(f) = f - \frac{z(f)}{z'(f)}; \qquad h'(f) = \frac{z(f)z''(f)}{\big(z'(f)\big)^2},$$

as $z(f) \geq 0,\ z''(f) > 0,\ z'(f) < 0$ in $[0, f^*]$. We have

$$h'(f) = \frac{z(f)z''(f)}{\big(z'(f)\big)^2} \geq 0 \ \ \forall \ \ f \in [0, f^*].$$

Therefore the function h is increasing on $[0, f^*]$. So, if $f_k \in [0, f^*]$ for some $k \in \mathbb{N}$, then

$$f_k \leq f_k - \frac{z(f_k)}{z'(f_k)} = f_{k+1}$$

and

$$f_{k+1} = f_k - \frac{z(f_k)}{z'(f_k)} \leq f^* - \frac{z(f^*)}{z'(f^*)} = f^*.$$

□

Now, we can prove the convergence of the our method.

Theorem 2. *Let $\boldsymbol{A}$ be a complete Riemannian manifold, $\mho \subseteq \boldsymbol{A}$ be an open convex set and $\boldsymbol{Z} \in \mathfrak{X}(\boldsymbol{A})$ satisfies the conditions* (1) – (3) *with:*

$$\xi K \zeta_0 < 2, \quad L\xi c^2 \leq \frac{1}{2}, \quad \zeta_0 K f^* < 1, \quad K\zeta_0(3f^* + \xi + f^{**}) < 2, \quad V[u_0, f^*] \subseteq \mho.$$

Then, the method given by (5) *converges to a singular point u^* of the vector field $\boldsymbol{Z}$ in $V[u_0, f^*]$ and the singular point u^* is unique in $V[u_0, f^{**} + \xi] \cap \mho$.*

Proof. To prove the theorem, first we shall prove some conditions for $i \geq 0$.

$(C1)$ $v_i \in V[u_0, f^*]$,
$(C2)$ $u_i \in V[u_0, f^*]$,
$(C3)$ $\|D\mathbf{Z}(u_i)^{-1}\| \leq \zeta_i$.

For $i = 0$, $(C1)$, $(C2)$, and $(C3)$ are trivial. Now, we will prove for $i \in \mathbb{N}$. Since

$$d(u_1, u_0) = \|m_0\| = \|[u_0, v_0; \mathbf{Z}]^{-1}\mathbf{Z}(u_0)\| \leq c\xi = f_1 - f_0 \leq f^*,$$

therefore $u_1 \in V[u_0, f^*] \subseteq \mho$.

By Lemma 1, we have

$$\mathbb{M}_{\phi_{n-1},1,0}\mathbf{Z}(u_n) = \Big(\int_0^1 \big(\mathbb{M}_{\phi_{n-1},1,0}D\mathbf{Z}(\phi_{n-1}(t))\mathbb{M}_{\phi_{n-1},0,1} - D\mathbf{Z}(u_{n-1})\big)dt + \int_0^1 \big(D\mathbf{Z}(u_{n-1}) - \mathbb{M}_{\alpha_{n-1},1,0}D\mathbf{Z}(\alpha_{n-1}(t))\mathbb{M}_{\alpha_{n-1},0,1}\big)dt\Big)m_{n-1}.$$

Put $n = 1$ and taking norm both sides, we get

$$\begin{aligned}
\|\mathbf{Z}(u_1)\| &= \|\mathbb{M}_{\phi_0,1,0}\mathbf{Z}(u_1)\| \\
&= \bigg\| \bigg(\int_0^1 \big(\mathbb{M}_{\phi_0,1,0} D\mathbf{Z}(\phi_0(t))\mathbb{M}_{\phi_0,0,1} - D\mathbf{Z}(u_0)\big)dt \\
&+ \int_0^1 \big(D\mathbf{Z}(u_0) - \mathbb{M}_{\alpha_0,1,0} D\mathbf{Z}(\alpha_0(t))\mathbb{M}_{\alpha_0,0,1}\big)dt \bigg) m_0 \bigg\| \\
&\leq \left(\frac{K}{2} d(u_1, u_0) + \frac{K}{2} d(v_0, u_0)\right)\|m_0\| \\
&= \left(\frac{K}{2} d(u_1, u_0) + \frac{K}{2}\|l_0\|\right) d(u_1, u_0) \\
&= \frac{K}{2} d(u_1, u_0)^2 + \frac{K}{2}\|\mathbf{Z}(u_0)\| d(u_1, u_0) \\
&\leq \frac{K}{2}(f_1 - f_0)^2 + \frac{K}{2} z(f_0)(f_1 - f_0) = z(f_1).
\end{aligned}$$

As the sequence (7) is increasing and the polynomial (6) is decreasing in $[0, f^*]$, we have

$$d(v_1, u_0) \leq d(u_1, u_0) + d(v_1, u_1) = d(u_1, u_0) + \|l_1\| = d(u_1, u_0) + \|\mathbf{Z}(u_1)\| \leq f^*,$$

therefore $v_1 \in V[u_0, f^*] \subseteq \mho$. Thus $(C1)$ and $(C2)$ hold for $i = 1$.

We assume that $u_i, v_{i-1} \in V[u_0, f^*] \subseteq \mho$, for $i = 2, 3, 4, \ldots, n$. Then, we will prove for $i = n + 1$. Since

$$\begin{aligned}
z(f_n) &= \int_0^1 \Big(z'\big(f_{n-1} + x(f_n - f_{n-1})\big) - z'(f_{n-1})\Big) dx (f_n - f_{n-1}) \\
&= L \int_0^1 x(f_n - f_{n-1})^2 dx = \frac{L}{2}(f_n - f_{n-1})^2,
\end{aligned}$$

we have $\|\mathbf{Z}(u_n)\| \leq z(f_n)$, for all $n \in \mathbb{N}$,
as

$$\begin{aligned}
\|\mathbf{Z}(u_n)\| &= \|\mathbb{M}_{\phi_{n-1},1,0}\mathbf{Z}(u_n)\| \\
&= \bigg\| \bigg(\int_0^1 \big(\mathbb{M}_{\phi_{n-1},1,0} D\mathbf{Z}(\phi_{n-1}(t))\mathbb{M}_{\phi_{n-1},0,1} - D\mathbf{Z}(u_{n-1})\big)dt \\
&+ \int_0^1 \big(D\mathbf{Z}(u_{n-1}) - \mathbb{M}_{\alpha_{n-1},1,0} D\mathbf{Z}(\alpha_{n-1}(t))\mathbb{M}_{\alpha_{n-1},0,1}\big)dt \bigg) m_{n-1} \bigg\| \leq
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\frac{K}{2} d(u_n, u_{n-1}) + \frac{K}{2} d(v_{n-1}, u_{n-1})\right) \|m_{n-1}\| \\
&= \left(\frac{K}{2} d(u_n, u_{n-1}) + \frac{K}{2} \|l_{n-1}\|\right) d(u_n, u_{n-1}) \\
&= \frac{K}{2} d(u_n, u_{n-1})^2 + \frac{K}{2} \|\mathbf{Z}(u_{n-1})\| d(u_n, u_{n-1}) \\
&\leq \frac{K}{2} (f_n - f_{n-1})^2 + \frac{K}{2} z(f_{n-1})(f_n - f_{n-1}) \\
&\leq \frac{L}{2} (f_n - f_{n-1})^2 = z(f_n).
\end{aligned}
$$

We have

$$
d(v_n, u_0) \leq d(u_n, u_0) + d(v_n, u_n) = d(u_n, u_0) + \|l_n\| = d(u_n, u_0) + \|\mathbf{Z}(u_n)\| \leq f^*,
$$

so that $v_n \in V[u_0, f^*] \subseteq \mho$. Now, we will show that the operator $[u_n, v_n; \mathbf{Z}]^{-1}$ is bounded. Let $\psi : [0,1] \to \mathbf{A}$ be the minimizing geodesic with $\psi(0) = u_0$, $\psi(1) = u_n$, and $\|\psi'(0)\| = d(u_0, u_n)$. Then, we have

$$
\begin{aligned}
&\|D\mathbf{Z}(u_0)^{-1} \mathbb{M}_{\psi,1,0} [u_n, v_n; \mathbf{Z}] \mathbb{M}_{\psi,0,1} - I_{u_0}\| \\
&\leq \|D\mathbf{Z}(u_0)^{-1} \mathbb{M}_{\psi,1,0} ([u_n, v_n; \mathbf{Z}] - D\mathbf{Z}(u_n)) \mathbb{M}_{\psi,0,1}\| \\
&+ \|D\mathbf{Z}(u_0)^{-1} (\mathbb{M}_{\psi,1,0} D\mathbf{Z}(u_n) \mathbb{M}_{\psi,0,1} - D\mathbf{Z}(u_0))\| \\
&\leq \|D\mathbf{Z}(u_0)^{-1}\| \int_0^1 \|\mathbb{M}_{\alpha_n,1,0} D\mathbf{Z}(\alpha_n(t)) \mathbb{M}_{\alpha_n,0,1} - D\mathbf{Z}(u_n)\| dt \\
&+ \|D\mathbf{Z}(u_0)^{-1}\| \|\mathbb{M}_{\psi,1,0} D\mathbf{Z}(u_n) \mathbb{M}_{\psi,0,1} - D\mathbf{Z}(u_0)\| \\
&\leq \zeta_0 \left(K d(u_n, u_0) + \int_0^1 \|\mathbb{M}_{\alpha_n,1,0} D\mathbf{Z}(\alpha_n(t)) \mathbb{M}_{\alpha_n,0,1} - D\mathbf{Z}(u_n)\| dt \right) \\
&\leq \frac{K\zeta_0}{2} \big(2(f_n - f_0) + z(f_n)\big) \leq \zeta_0 \left(z'(f_n) + \frac{1}{c} \right) < 1.
\end{aligned}
$$

Therefore $\mathbb{M}_{\psi,1,0}[u_n, v_n; \mathbf{Z}]\mathbb{M}_{\psi,0,1}$ is invertible by Banach's lemma and

$$
\begin{aligned}
\|[u_n, v_n; \mathbf{Z}]^{-1}\| &= \|\mathbb{M}_{\psi,1,0} [u_n, v_n; \mathbf{Z}]^{-1} \mathbb{M}_{\psi,0,1}\| \\
&\leq \frac{\|D\mathbf{Z}(u_0)^{-1}\|}{1 - \|D\mathbf{Z}(u_0)^{-1}\| \|\mathbb{M}_{\psi,1,0} [u_n, v_n; \mathbf{Z}]^{-1} \mathbb{M}_{\psi,0,1} - D\mathbf{Z}(u_0)\|} \\
&\leq \frac{-1}{z'(f_n)}.
\end{aligned}
$$

We obtain

$$
d(u_{n+1}, u_n) \leq \|[u_n, v_n; \mathbf{Z}]^{-1}\| \|\mathbf{Z}(u_n)\| \leq \frac{-z(f_n)}{z'(f_n)} = f_{n+1} - f_n \tag{11}
$$

and

$$
d(u_{n+1}, u_0) \leq d(u_{n+1}, u_n) + d(u_n, u_0) \leq f_{n+1} - f_n + f_n - f_0 = f_{n+1} - f_0 \leq f^*.
$$

Therefore $u_{n+1} \in V[u_0, f^*] \subseteq \mho$.

Suppose $(C3)$ holds for $i = 1, 2, \ldots, n$ and we shall prove for $i = n+1$. Let $\delta : [0,1] \to \mathbf{A}$ be the minimizing geodesic with $\delta(0) = u_0, \delta(1) = u_{n+1}$, and $\|\delta'(0)\| = d(u_{n+1}, u_0)$.

We obtain that

$$\begin{aligned}&\|\mathbb{M}_{\delta,1,0} D\mathbf{Z}(u_{n+1})\mathbb{M}_{\delta,0,1} - D\mathbf{Z}(u_0)\| \\ \leq & K \int_0^1 \|\delta'(0))\| ds = K d(u_{n+1}, u_0) \leq K f^*\end{aligned}$$

and

$$\|D\mathbf{Z}(u_0)^{-1}\| \|\mathbb{M}_{\delta,1,0} D\mathbf{Z}(u_{n+1})\mathbb{M}_{\delta,0,1} - D\mathbf{Z}(u_0)\| \leq \zeta_0 K f^* < 1,$$

as $\zeta_0 K f^* < 1$.

Therefore $\mathbb{M}_{\delta,1,0} D\mathbf{Z}(u_{n+1})\mathbb{M}_{\delta,0,1}$ is invertible by Banach's lemma and

$$\begin{aligned}\|D\mathbf{Z}(u_{n+1})^{-1}\| =& \|\mathbb{M}_{\delta,1,0} D\mathbf{Z}(u_{n+1})^{-1}\mathbb{M}_{\delta,0,1}\| \\ \leq & \frac{\|D\mathbf{Z}(u_0)^{-1}\|}{1 - \|D\mathbf{Z}(u_0)^{-1}\| \|\mathbb{M}_{\delta,1,0} D\mathbf{Z}(u_{n+1})\mathbb{M}_{\delta,0,1} - D\mathbf{Z}(u_0)\|} \\ \leq & \frac{\zeta_0}{1 - \zeta_0 K d(u_{n+1}, u_0)} = \zeta_{n+1},\end{aligned}$$

therefore it holds for $i = n+1$. Thus $(C1) - (C3)$ hold for $i \geq 0$.

Now, we can prove the convergence theorem. Since $\{f_n\}$ is a convergent sequence and hence it is a Cauchy sequence therefore from (11) the sequence $\{u_n\}$ is also a convergent sequence and let the sequence $\{u_n\}$ converges to $u^* \in V[u_0, f^*]$. Now we will show that u^* is a singularity of $\mathbf{Z}$. As for all $n \in \mathbb{N}$,

$$\|\mathbf{Z}(u_n)\| \leq z(f_n),$$

taking $n \to \infty$ both sides, we get

$$\|\mathbf{Z}(u^*)\| \leq z(f^*) = 0.$$

Then, we have $\mathbf{Z}(u^*) = 0$. Finally, we will show that the singularity is unique in $V[u_0, f^{**} + \xi] \cap \mho$. Let v^* be the another singularity of $\mathbf{Z}$ in $V[u_0, f^{**} + \xi] \cap \mho$. Let $\rho : [0,1] \to \mathbf{A}$ be the minimizing geodesic with $\rho(0) = u^*$, $\rho(1) = v^*$, and $\|\rho'(0)\| = d(u^*, v^*)$.

We obtain

$$\begin{aligned}&\|\mathbb{M}_{\rho,t,0} D\mathbf{Z}(\rho(t))\mathbb{M}_{\rho,0,t} - D\mathbf{Z}(u^*)\| \\ \leq & K \int_0^t \|\rho'(0)\| ds = K t d(u^*, v^*) \leq K t \Big(d(u_0, u^*) + d(u_0, v^*) \Big)\end{aligned}$$

and

$$
\begin{aligned}
&\|D\mathbf{Z}(u^*)^{-1}\| \int_0^1 \|\mathbb{M}_{\rho,t,0} D\mathbf{Z}(\rho(t))\mathbb{M}_{\rho,0,t} - D\mathbf{Z}(u^*)\| dt \\
\leq &\Big(\frac{1}{\zeta_0} - Kf^*\Big)^{-1} \int_0^1 Kt\Big(d(u_0, u^*) + d(u_0, v^*)\Big) dt \\
\leq &\Big(\frac{1}{\zeta_0} - Kf^*\Big)^{-1} \frac{K}{2}(f^* + f^{**} + \xi) < 1.
\end{aligned}
$$

It shows that the operator

$$
T = \int_0^1 \mathbb{M}_{\rho,t,0} D\mathbf{Z}(\rho(t))\mathbb{M}_{\rho,0,t} dt
$$

is invertible by Banach's lemma and we have

$$
0 = \mathbb{M}_{\rho,1,0}\mathbf{Z}(v^*) - \mathbf{Z}(u^*) = \int_0^1 \mathbb{M}_{\rho,t,0} D\mathbf{Z}(\rho(t))\mathbb{M}_{\rho,0,t}(\rho'(0)) dt.
$$

So that $\rho'(0) = 0$. We have $0 = \|\rho'(0)\| = d(u^*, v^*)$, implies that $u^* = v^*$. This completes the proof. □

Theorem 3. *Suppose that u^* is a singular point of $\boldsymbol{Z}$ in $V[u_0, f^*]$. If $V(u_0, f^{**}) \subseteq \mho$, then the only singular point of $\boldsymbol{Z}$ in $V[u_0, r]$ is u^*, where $f^* < r \leq f^{**}$.*

Proof. Let v^* be the singular point of $\mathbf{Z}$ in $V[u_0, r]$ and let $\Lambda : [0, 1] \to \mathbf{A}$ be the minimizing geodesic with $\Lambda(0) = u_0$, $\Lambda(1) = v^*$, and $\|\Lambda'(0)\| = d(u_0, v^*)$. Then, by (4), we have

$$
\begin{aligned}
\mathbb{M}_{\Lambda,1,0}\mathbf{Z}(v^*) &= \mathbb{M}_{\Lambda,1,0}\mathbf{Z}(v^*) - \mathbf{Z}(u_0) + \mathbf{Z}(u_0) + D\mathbf{Z}(u_0)\Lambda'(0) - D\mathbf{Z}(u_0)\Lambda'(0) \\
&= \int_0^1 \mathbb{M}_{\Lambda,t,0} D\mathbf{Z}(\Lambda(t))\mathbb{M}_{\Lambda,0,t}\Lambda'(0) dt - D\mathbf{Z}(u_0)\Lambda'(0) + \mathbf{Z}(u_0) \\
&\quad + D\mathbf{Z}(u_0)\Lambda'(0) \\
&= \int_0^1 \big(\mathbb{M}_{\Lambda,t,0} D\mathbf{Z}(\Lambda(t))\mathbb{M}_{\Lambda,0,t} - D\mathbf{Z}(u_0)\big)\Lambda'(0) dt + \mathbf{Z}(u_0) \\
&\quad + D\mathbf{Z}(u_0)\Lambda'(0).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\frac{Ld(u_0, v^*)^2}{2} \geq &\frac{Kd(u_0, v^*)^2}{2} \geq \|\mathbf{Z}(u_0) + D\mathbf{Z}(u_0)\Lambda'(0)\| \\
\geq &\frac{1}{\|D\mathbf{Z}(u_0)^{-1}\|}\|D\mathbf{Z}(u_0)^{-1}\mathbf{Z}(u_0) + \Lambda'(0)\| \\
\geq &\frac{1}{\zeta_0}\Big(\|\Lambda'(0)\| - \|D\mathbf{Z}(u_0)^{-1}\mathbf{Z}(u_0)\|\Big) \\
\geq &\Big(\frac{d(u_0, v^*)}{\zeta_0} - \xi\Big) \geq \Big(\frac{d(u_0, v^*)}{c} - \xi\Big).
\end{aligned}
$$

Therefore

$$z(d(u_0, v^*)) = \frac{Ld(u_0, v^*)^2}{2} - \frac{d(u_0, v^*)}{c} + \xi \geq 0.$$

Since $d(u_0, v^*) \leq r \leq f^{**}$, we have $d(u_0, v^*) \leq f^*$, hence by Theorem 2, $u^* = v^*$. □

Theorem 4. *Suppose that all the assumptions of Theorem 2 are satisfied. If* $f^* \leq f^{**}$*, then, we obtain the following error bound:*

(a) *If* $f^* < f^{**}$*, then*

$$d(u^*, u_n) \leq f^* - f_n = \frac{(f^{**} - f^*)\Lambda^{2^n}}{1 - \Lambda^{2^n}}, \quad \textit{where} \quad \Lambda = \frac{f^*}{f^{**}}. \tag{12}$$

(b) *If* $f^* = f^{**}$*, then*

$$d(u^*, u_n) \leq f^* - f_n = \frac{f^*}{2^n}. \tag{13}$$

Proof. First, from (11), we have $\{f_n\}$ is a majorizing sequence of $\{u_n\}$. Then, for $j \geq 1$ and $i \geq 1$, we have

$$d(u_{i+j}, u_i) \leq \sum_{n=i}^{i+j-1} d(u_{n+1}, u_n) \leq \sum_{n=i}^{i+j-1} (f_{n+1} - f_n) = f_{i+j} - f_i,$$

so that, if $j \to \infty$, then by the convergence of $\{u_n\}$ and $\{f_n\}$, it follws that

$$d(u^*, u_n) \leq f^* - f_n.$$

Second, we prove (a). As $f^* < f^{**}$, then

$$\begin{aligned} z(f) &= \frac{L}{2}(f^* - f)(f^{**} - f) \\ &= \frac{L}{2} a_n b_n, \end{aligned}$$

where $a_n = f^* - f_n$ and $b_n = f^{**} - f_n$, for all $n \geq 0$. As $z'(f_n) = -\frac{L}{2}(a_n + b_n)$, then

$$a_{n+1} = f^* - f_{n+1} = \frac{a_n^2}{a_n + b_n}, \quad b_{n+1} = f^{**} - f_{n+1} = \frac{b_n^2}{a_n + b_n}, \quad n \geq 0.$$

Also,

$$\frac{a_{n+1}}{b_{n+1}} = \frac{a_n^2}{b_n^2} = \cdots = \Lambda^{2^{n+1}}, \quad n \geq 0.$$

Finally, from $b_{n+1} = (f^{**} - f^*) + a_{n+1}$, we have (12). Third, we prove (b). As $f^* = f^{**}$, then $a_n = b_n,\ z(f_n) = \frac{L}{2}a_n^2$,

$$a_{n+1} = \frac{a_n}{2} = \cdots = \frac{f^*}{2^n}, \quad n \geq 0,$$

and we have (13). □

4 Numerical examples

In this section, we provide two numerical examples.

Example 1. *Consider the integral equation*

$$\boldsymbol{Z}(u)(v) = -1 + u(v) + \frac{1}{4}u(v)\int_0^1 \frac{v}{v+w}u(w)dw,\ u(v) \subset \boldsymbol{A} = C[0,1]$$

and we define the norm $\|u\| = max_{0\le v\le 1}|u(v)|$. *Initially for* $u_0 = u_0(v) = 0$, *we have*

$$\|\boldsymbol{Z}(u_0)\| = 1 = \xi, \|D\boldsymbol{Z}(u_0)^{-1}\| = 1 = \zeta_0, \|D^2\boldsymbol{Z}(u)\| = 0.150514997 = K.$$

Hence all the assumptions for convergence are satisfied and the Steffensen's method can be applied to get the desired singular point.

Example 2. *Consider the vector field* $\boldsymbol{Z}$ *from* $\Omega = (-1,1)^3 \subseteq \boldsymbol{A} = \mathbb{R}^3$ *to* $\Omega = (-1,1)^3$ *given by*

$$\boldsymbol{Z}(\boldsymbol{u}) = \boldsymbol{Z}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} e^{u_1} - 1 \\ u_2^2 + u_2 \\ u_3 \end{pmatrix}$$

with the norm $\|.\|_\infty$. *For the point* $\boldsymbol{u} = (u_1, u_2, u_3)^T$, *the first and second Fréchet derivatives of* $\boldsymbol{Z}$ *are:*

$$D\boldsymbol{Z}(\boldsymbol{u}) = \begin{bmatrix} e^{u_1} & 0 & 0 \\ 0 & 2u_2+1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$D^2\boldsymbol{Z}(\boldsymbol{u}) = \left[\begin{array}{ccc|ccc|ccc} e^{u_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Initially for $\boldsymbol{u}_0 = (0.11, 0.11, 0.11)^T$, *we have*

$$\|\boldsymbol{Z}(\boldsymbol{u}_0)\| = \max(0.12, 0.12, 0.11) = 0.12 = \xi,$$
$$\|D\boldsymbol{Z}(\boldsymbol{u}_0)^{-1}\| = 1 = \zeta_0, \|D^2\boldsymbol{Z}(\boldsymbol{u})\| = \max(1.12, 2, 0) = 2 = K.$$

Hence all the assumptions for convergence are satisfied and the Steffensen's method can be applied to get the desired singular point.

5 Conclusion

In this article, we have studied the semilocal convergence analysis of the Steffensen's method in Riemannian manifolds. We have presented the semilocal convergence analysis of the Steffensen's method under Lipschitz continuity condition on first order covariant derivative of a vector field and by using majorant principle. Finally, two numerical examples are given.

References

[1] Amat, S., Argyros, I. K., Busquier, S., Castro, R., Hilout, S. and Plaza, S., *Traub-type high order iterative procedures on Riemannian manifolds*, SeMA J. **63** (2014), no. 1, 27-52.

[2] Argyros, I. K., *Convergence and applications of Newton-type iterations*, Springer, New York, 2008.

[3] Argyros, I. K., Cho, Y. J. and Hilout, S., *Numerical methods for equations and variational inclusions*, CRC Press, New York, 2012.

[4] Argyros, I. K., Hilout, S. and Tabatabai, M. A., *Mathematical modelling with applications in biosciences and engineering*, Nova Publishers, New York, 2011.

[5] Ezquerro, J. A., Hernandez, M. A., Romero, N. and Velasco, A. I., *On Steffensen's method on Banach spaces*, J. Comput. Appl. Math. **249** (2013), 9-23.

[6] Lang, S., *Differential and Riemannian manifolds*, Graduate Texts in Mathematics, Vol. 160, Springer-Verlag, New York, 1995.

[7] Li, W., Szidarovszky, F. and Kuang, Y., *Notes on the stability of dynamic economic systems*, Appl. Math. Comput. **108** (2000), 85-89.

[8] Sakai, T., *Riemannian geometry*, American Mathematical Society, Providence, RI, 1996.

[9] Tabatabai, M. A., Eby, W. M. and Singh, K. P., *Hyperbolastic modeling of wound healing*, Math. Comput. Modell. **53** (2011), 755-768.

[10] Tu, L. W., *An introduction to manifolds*, Universitext, Springer-Verlag, New York, 2011.