

## STUDY OF VECTOR FIELDS UNDER SOME SPECIAL TRANSFORMATION

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### Abstract

In this paper, we examine holomorphic conformal vector fields on almost contact metric manifolds subjected to special transformation. Furthermore, we establish the invariance of concircular, projective, and affine conformal vector fields on a Sasakian manifold under the special transformation.

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## 1 Introduction

Tanno [15] introduced D-homothetic deformation, which preserves the Sasakian structure. Blair [6] and Vaisman [17] introduced conformal deformation, while Ozdemid [13] generalized this notion. Blair [5] explored D-homothetic wrapping, and Beldjilali ([3], [1]) extended the study to D-homothetic biwrapping, D-isometric wrapping, and a new deformation [2] that deforms both the structure tensor and metric tensor simultaneously. They also demonstrated that this deformation preserves the Sasakian structure and established significant results regarding deformed Sasakian manifolds admitting Ricci solitons.

A smooth vector field  $V$  on a contact metric manifold is termed a Holomorphically planar conformal vector field (abbreviated as HPCV) if it satisfies the following condition

$$\nabla_Z V = \alpha Z + \beta \phi Z, \quad (1)$$

for any smooth vector field  $Z$ , and smooth functions  $\alpha$  and  $\beta$  on  $M$ . For  $\beta = 0$ , an HPCV field becomes a closed conformal vector field. Sharma [14] extended the concept of closed conformal vector fields to introduce HPCV fields on almost

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Hermitian manifolds. Ghosh and Sharma [10] characterized almost Hermitian manifolds that support HPCV fields. They demonstrated that a strictly non-geodesic, non-vanishing HPCV field on such a manifold is both homothetic and almost analytic.

Fialkow [9] defined a concircular vector field  $V$  on a Riemannian manifold by satisfying the following equation

$$\nabla_Z V = \rho Z, \quad (2)$$

where  $\nabla$  is a Levi-Civita connection and  $\rho$  is a smooth function on  $M$ .

A vector field  $V$  on  $M$  is called a projective vector field[19] if

$$(L_V \nabla)(Z, X) = P(Z)X + P(X)Z, \quad (3)$$

and affine conformal vector field[8] if

$$(L_V \nabla)(Z, X) = (Z\rho)X + (X\rho)Z - g(Z, X)D\rho, \quad (4)$$

for all vector fields  $Z, X$  on  $M$ , where  $L_V$  denotes Lie derivative along  $V$ ,  $P$  is a scalar 1-form,  $\rho$  being smooth function on  $M$ . The vector field  $V$  is affine when  $P = 0$  in equation (3) and  $\rho$  is constant in equation (4). Tashiro [16] demonstrated that on a compact orientable Riemannian manifold, every projective vector field is necessarily a Killing vector field. Nagaraja and Sharma [11] studied complete K-contact manifolds as Ricci almost solitons and proved that if  $V$  is projective, then  $V$  is Killing and the manifold is compact Einstein Sasakian. De and Sardar [7] investigated affine conformal vector fields on  $(k, \mu)$ '-almost Kenmots manifolds and established that a conformal vector field reduces to an affine vector field.

Motivating from above work on vector fields, in this paper we study the invariance of different types of vector fields on contact and Sasakian manifolds under new type of deformation introduced and studied by Beldjilali[2].

## 2 Preliminaries

An odd-dimensional Riemannian manifold  $(M, g)$  is termed an almost contact metric manifold if there exists a  $(1, 1)$ -tensor  $\phi$ , a vector field  $\xi$ , and a global 1-form  $\eta$  such that

$$\begin{aligned} \phi^2 Z &= -Z + \eta(Z)\xi, \quad \eta(\xi) = 1, \quad g(\phi Z, \phi X) = g(Z, X) - \eta(Z)\eta(X), \\ \phi \circ \xi &= 0, \quad \eta \circ \phi = 0, \end{aligned} \quad (5)$$

for any vector fields  $Z$  and  $X$  on  $M$ . Such a manifold is said to be a contact metric manifold if  $d\eta(Z, X) = \Phi(Z, X)$ , where  $\Phi(Z, X) = g(Z, \phi X)$ .

On a contact metric manifold we denote by  $l = R(\circ, \xi)\xi$  and  $h = \frac{1}{2}L_\xi \phi$ , where  $R$  is a Riemannian curvature tensor and  $L$  is Lie differentiation respectively, the quantities  $l$  and  $h$  are symmetric where  $h$  is trace free and anti-commutative with  $\phi$ . The following equation holds in a contact metric manifold:

$$\nabla_Z \xi = -\phi Z - \phi h Z. \quad (6)$$

A contact metric manifold  $M$  is  $K$ -contact if  $\xi$  is Killing. Also it is known that  $M$  is  $K$ -contact if  $h = 0$ . An almost contact metric manifold is Sasakian if

$$(\nabla_Z \phi)(X) = g(Z, X)\xi - \eta(X)Z, \tag{7}$$

for any vector fields  $Z, X$  on  $M$ . Moreover, from (7), one can easily obtain that

$$\nabla_Z \xi = -\phi Z, \quad (\nabla_Z \eta)(X) = g(Z, \phi X) = \Phi(Z, X) \tag{8}$$

$$d\eta(Z, X) = g(Z, \phi X). \tag{9}$$

In a Sasakian manifold the following hold:

$$R(Z, X)\xi = \eta(X)Z - \eta(Z)X \tag{10}$$

$$S(Z, \xi) = 2n\eta(Z), \tag{11}$$

where  $R$  is the Riemannian curvature tensor and  $S$  is the Ricci curvature tensor. Let  $(\phi, \xi, \eta, g)$  be an almost contact metric structure on a  $(2n+1)$ -dimensional manifold  $M$ . A new structure  $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$  was defined by Beldjilal and Akoyal [2] in the form

$$\begin{aligned} \bar{\phi}Z &= \phi Z + \theta(\phi Z)\xi, \quad \bar{\xi} = \xi, \quad \bar{\eta}(Z) = \eta(Z) - \theta(Z), \\ \bar{g}(Z, X) &= fg(Z, X) - f\eta(Z)\eta(X) + \bar{\eta}(Z)\bar{\eta}(X), \end{aligned} \tag{12}$$

where  $\theta$  is a closed 1-form orthogonal to  $\eta$  and  $f$  is a smooth function on  $M$ . This new deformation is referred to as **BA**-deformation.

Let  $\nabla$  and  $\bar{\nabla}$  denote Riemannian connections on almost contact metric structure with respect to  $g$  and  $\bar{g}$  respectively. Relation between  $\nabla$  and  $\bar{\nabla}$  is given by[2]

$$\begin{aligned} \bar{g}(\bar{\nabla}_Z X, W) &= \bar{g}(\nabla_Z X, W) + \frac{1}{2}\{(Zf)g(X, W) + (Xf)g(Z, W) - (Wf)g(Z, X)\} \\ &\quad - f\left\{\frac{1}{2}((\nabla_Z \eta)(X) + (\nabla_X \eta)(Z))\eta(W) + d\eta(Z, W)\eta(X)\right. \\ &\quad \left.+ d\eta(X, W)\eta(Z)\right\} + \frac{1}{2}\{(\nabla_Z \bar{\eta})(X) + (\nabla_X \bar{\eta})(Z)\}\bar{\eta}(W) \\ &\quad + d\bar{\eta}(Z, W)\bar{\eta}(X) + d\bar{\eta}(X, W)\bar{\eta}(Z), \end{aligned} \tag{13}$$

for all vector fields  $Z, X$  and  $W$  on  $M$ .

For Sasakian manifold,  $f = 1$ . The relations between  $\nabla$  and  $\bar{\nabla}$  and the corresponding Riemannian and Ricci curvature are respectively given by [4]

$$\bar{\nabla}_Z X = \nabla_Z X + \theta(Z)\bar{\phi}X + \theta(X)\bar{\phi}Z - (\nabla_Z \theta)(X)\xi. \tag{14}$$

$$\begin{aligned} \bar{R}(Z, X)W &= R(Z, X)W + \theta(R(Z, X)W)\xi + \theta(X)g(Z, W)\xi - \theta(Z)g(X, W)\xi \\ &\quad + \eta(W)(\theta(Z)X - \theta(X)Z) + \theta(W)(\bar{\eta}(Z)X - \bar{\eta}(X)Z). \end{aligned} \tag{15}$$

$$\bar{S}(Z, X) = S(Z, X) - 2n\eta(Z)\eta(X) + 2n\bar{\eta}(Z)\bar{\eta}(X). \tag{16}$$

From (14) it follows that

$$(\bar{\nabla}_Z \bar{\phi})(X) = \bar{g}(Z, X)\bar{\xi} - \bar{\eta}(X)Z, \tag{17}$$

for all vector fields  $Z, X$  and  $Y$  on  $M$ , where  $\bar{R}$  and  $\bar{S}$  denote Riemannian curvature tensor and Ricci curvature tensor for metric  $\bar{g}$ .

### 3 HPCV field on contact metric manifold under BA-deformation

In this section we discuss HPCV field on contact metric manifold under **BA**-deformation. Throughout this section we choose  $g$ -orthonormal basis  $\{e_i\}_{0 \leq i \leq 2n}$  of the tangent space  $T_x M$  at each point  $x \in M$ , where  $e_0 = \xi$ . Then  $\{\bar{e}_i = e_i + \theta(e_i)\xi\}_{0 \leq i \leq 2n}$  is a  $\bar{g}$  orthonormal basis of  $T_x M$ ,  $\bar{g}(e_0, e_0) = \bar{g}(\xi, \xi) = 1$ . When  $\{i = 1, \dots, 2n\}$ , then  $\bar{g}(e_i, \xi) = -\theta(e_i)$ .

**Theorem 1.** *If HPCV field on a contact metric manifold under **BA**-deformation satisfies  $\bar{\phi}V = 0$  then manifold becomes  $K$ -contact manifold provided function is non-zero constant.*

*Proof.* Let us assume that  $\bar{\phi}V = 0$ . Use first equation of (12) in foregoing equation and operate  $\bar{\phi}$  on obtained equation, we get

$$V = \bar{\eta}(V)\xi. \quad (18)$$

Taking covariant derivative along  $Z$  in the above equation, and taking inner product with  $W$  to obtained equation and in view of (1), we arrive at

$$\begin{aligned} \bar{g}(\bar{\alpha}Z + \bar{\beta}\bar{\phi}Z, W) &= \bar{g}(\bar{\alpha}Z + \bar{\beta}\bar{\phi}Z, \xi)\bar{\eta}(W) + \bar{g}(V, \bar{\nabla}_Z \xi)\bar{\eta}(W) \\ &+ \bar{g}(\bar{\nabla}_Z \xi, W)\bar{\eta}(V). \end{aligned} \quad (19)$$

Using  $d\eta(Z, X) = g(Z, \phi X)$ , equation (6) and (12) in (13), we get

$$\begin{aligned} \bar{g}(\bar{\nabla}_Z X, W) &= \bar{g}(\nabla_Z X, W) + \frac{1}{2}\{(Zf)g(X, W) + (Xf)g(Z, W) - (Wf)g(Z, X)\} \\ &- f\{-g(\phi hZ, X)\eta(W) + g(Z, \phi W)\eta(X) + g(X, \phi W)\eta(Z)\} \\ &- g(\phi hZ, X)\bar{\eta}(W) - \frac{1}{2}\{(\nabla_Z \theta)(X) + (\nabla_X \theta)(Z)\}\bar{\eta}(W) \\ &+ g(Z, \phi W)\bar{\eta}(X) + g(X, \phi W)\bar{\eta}(Z). \end{aligned} \quad (20)$$

Now taking  $X = \xi$  in last equation, we obtain

$$\begin{aligned} \bar{g}(\bar{\nabla}_Z \xi, W) &= -fg(\phi Z, W) - fg(\phi hZ, W) + \frac{1}{2}\{(Zf)\eta(W) + (\xi f)g(Z, W) \\ &- (Wf)\eta(Z)\} - fg(Z, \phi W) - \frac{1}{2}\{(\nabla_Z \theta)(\xi) + (\nabla_\xi \theta)(Z)\}\bar{\eta}(W) \\ &+ g(Z, \phi W). \end{aligned} \quad (21)$$

Replacing  $W$  by  $V$  in the foregoing equation, we obtain

$$\begin{aligned} \bar{g}(\bar{\nabla}_Z \xi, V) &= -fg(\phi Z, V) - fg(\phi hZ, V) + \frac{1}{2}\{(Zf)\eta(V) + (\xi f)g(Z, V) \\ &- (Vf)\eta(Z)\} - fg(Z, \phi V) - \frac{1}{2}\{(\nabla_Z \theta)(\xi) + (\nabla_\xi \theta)(Z)\}\bar{\eta}(V) + g(Z, \phi V). \end{aligned} \quad (22)$$

Using (21) and (22) in (19), one obtains that

$$\begin{aligned} \bar{\alpha} \bar{g}(Z, W) + \bar{\beta} \bar{g}(\phi Z, W) &= \bar{\alpha} \bar{\eta}(Z) \bar{\eta}(W) - \bar{g}(\phi h Z, V) \bar{\eta}(W) + g(Z, \phi V) \bar{\eta}(W) \\ &- \bar{g}(\phi h Z, W) \bar{\eta}(V) + g(Z, \phi W) \bar{\eta}(V) + \frac{1}{2} \{ (Zf) \eta(V) + (\xi f) g(Z, V) \\ &- (Vf) \eta(Z) \} \bar{\eta}(W) + \frac{1}{2} \{ (Zf) \eta(W) + (\xi f) g(Z, W) - (Xf) \eta(Z) \} \bar{\eta}(V) \\ &- \{ (\nabla_Z \theta)(\xi) + (\nabla_\xi \theta)(Z) \} \bar{\eta}(V) \bar{\eta}(W). \end{aligned} \quad (23)$$

Utilizing  $Z = \xi$  in above equation, we achieve

$$\{ 2(\xi f) \eta(V) - (Vf) \} \bar{\eta}(W) + \{ 2(\xi f) \eta(W) - (Wf) \} \bar{\eta}(V) = 0. \quad (24)$$

Substituting  $W = \xi$  in last equation, we arrive at

$$2(\xi f) \eta(V) + (\xi f) \bar{\eta}(V) = (Vf). \quad (25)$$

Using foregoing equation in (24), we infer that

$$\{ 2(\xi f) \eta(W) - (Wf) - (\xi f) \bar{\eta}(W) \} \bar{\eta}(V) = 0. \quad (26)$$

If  $\bar{\eta}(V) = 0$  the equation (18) yields  $V = 0$ . Therefore we assume that  $\bar{\eta}(V) \neq 0$ . Now, equation (26) entails that

$$2(\xi f) \eta(W) - (Wf) - (\xi f) \bar{\eta}(W) = 0. \quad (27)$$

Replace  $W$  by  $V$  in last equation, we acquire

$$2(\xi f) \eta(V) - (\xi f) \bar{\eta}(V) = (Vf). \quad (28)$$

Comparison of (25) and (28) gives

$$2(\xi f) \bar{\eta}(V) = 0. \quad (29)$$

Since  $\bar{\eta}(V) \neq 0$ , we have  $(\xi f) = 0$ . Making use of this in (27) gives  $(Wf) = 0$ , which gives  $Df = 0$  for all vector fields  $W$ . On integrating it,  $f$  becomes constant. Using this in (23), we infer

$$\begin{aligned} \bar{\alpha} \bar{g}(Z, W) + \bar{\beta} \bar{g}(\phi Z, W) &= \bar{\alpha} \bar{\eta}(Z) \bar{\eta}(W) - \bar{g}(\phi h Z, V) \bar{\eta}(W) + g(Z, \phi V) \bar{\eta}(W) \\ &- \bar{g}(\phi h Z, W) \bar{\eta}(V) + g(Z, \phi W) \bar{\eta}(V) - \{ (\nabla_Z \theta)(\xi) + (\nabla_\xi \theta)(Z) \} \bar{\eta}(V) \bar{\eta}(W). \end{aligned} \quad (30)$$

Operating  $\phi$  on (18), gives  $\phi V = 0$ . Using this in (30), we obtain

$$\begin{aligned} \bar{\alpha} \bar{g}(Z, W) + \bar{\beta} \bar{g}(\phi Z, W) &= \bar{\alpha} \bar{\eta}(Z) \bar{\eta}(W) - \bar{g}(\phi h Z, W) \bar{\eta}(V) + g(Z, \phi W) \bar{\eta}(V) \\ &- \{ (\nabla_Z \theta)(\xi) + (\nabla_\xi \theta)(W) \} \bar{\eta}(V) \bar{\eta}(W). \end{aligned} \quad (31)$$

Contraction of (31) with respect to  $Z$  and  $W$  yields  $\bar{\alpha} = 0$ . Use of last equation in (31) produce

$$\begin{aligned} -\bar{\beta} \bar{g}(\phi Z, W) &= -\bar{g}(\phi h Z, W) \bar{\eta}(V) + g(Z, \phi W) \bar{\eta}(V) - \{ (\nabla_Z \theta)(\xi) \\ &+ (\nabla_\xi \theta)(Z) \} \bar{\eta}(V) \bar{\eta}(W). \end{aligned} \quad (32)$$

Replacing  $W$  by  $\phi W$  in the above equation, for all vector fields  $W$ , we obtain

$$-f\bar{\beta}\{-Z + \eta(Z)\xi\} = \bar{\eta}(V)\{-Z + \eta(Z)\xi - fhZ\}. \quad (33)$$

Contracting with  $Z$  and  $X$ , we get  $\bar{\eta}(V) = -f\bar{\beta}$ . The condition  $\bar{\eta}(V) \neq 0$  implies that  $f \neq 0$ ,  $\bar{\beta} \neq 0$ . Using this equation in (33), we can write  $-f\bar{\beta}hZ = 0$ , which implies  $-fhZ = 0$ . We know that  $f \neq 0$ ,  $Z$  arbitrary vector field which is non zero, therefore  $h = 0$ . Using this in (6), we obtain  $\nabla_Z \xi = -\phi Z$ . Hence manifold is  $K$ -contact.  $\square$

**Theorem 2.** *Let  $M$  be a complete connected contact metric manifold and HPCV field  $V$  be pointwise collinear with  $\xi$  under  $\mathbf{BA}$ -deformation. Then the manifold reduces to  $K$ -contact provided arbitrary function is non-zero constant.*

*Proof.* Let  $V = \bar{f} \bar{\xi}$ , where  $\bar{f}$  is a smooth function on  $M$ . Taking covariant derivative along  $Z$  in previous equation, and take inner product with  $W$  in the obtained equation, we obtain

$$\bar{\alpha} \bar{g}(Z, W) + \bar{\beta} \bar{g}(\bar{\phi}Z, W) = (Z\bar{f})\bar{g}(\xi, W) + \bar{f}\bar{g}(\bar{\nabla}_Z \xi, W). \quad (34)$$

Substituting  $X = \xi$  in (13) and using (34), we get

$$\begin{aligned} \bar{\alpha} \bar{g}(Z, W) + \bar{\beta} \bar{g}(\bar{\phi}Z, W) &= (Z\bar{f})\bar{\eta}(W) + \frac{\bar{f}}{2}((Zf)\eta(W) + (\xi f)g(Z, W) - (Wf)\eta(Z)) \\ &\quad + \bar{f}(-\bar{g}(\phi Z, W) - \bar{g}(\phi hZ, W) - fg(Z, \phi W) + g(Z, \phi W)) \\ &\quad - \frac{\bar{f}}{2}((\nabla_Z \theta)(\xi) + (\nabla_\xi \theta)(Z))\bar{\eta}(X). \end{aligned} \quad (35)$$

Inserting  $Z = \xi$  in (35), we infer that

$$\bar{\alpha} \bar{\eta}(W) = (\xi \bar{f})\bar{\eta}(W) + \frac{\bar{f}}{2}(2(\xi f)\eta(W) - (Wf)). \quad (36)$$

Replacing  $W$  by  $\phi W$  in previous equation, one achieve  $\phi Df = 0$ . Operate  $\phi$  on both side of last equation, we write it as

$$Df = (\xi f)\xi. \quad (37)$$

Taking inner product with  $X$  in foregoing equation, we obtain  $(Xf) = (\xi f)\eta(X)$ , for all vector fields  $X$ . We can write it as  $df = (\xi f)\eta$  and operating  $d$  to the last equation. Using Poincare lemma ( $d^2 = 0$ ), we obtain  $(\xi f)(\eta \wedge d\eta) = 0$ . Since  $\eta \wedge d\eta \neq 0$ , we get  $(\xi f) = 0$ . Using this in (37), we get  $Df = 0$ . On integrating it,  $f$  becomes constant. Substituting  $f = \text{constant}$  in (36), we have  $\bar{\alpha} = (\xi \bar{f})$ . Utilizing aforementioned equation in (35), one obtains

$$\begin{aligned} (\xi \bar{f})\bar{g}(Z, W) + \bar{\beta} \bar{g}(\bar{\phi}Z, W) &= (Z\bar{f})\bar{\eta}(W) + \bar{f}\{-\bar{g}(\phi Z, W) - \bar{g}(\phi hZ, W) - fg(Z, \phi W) \\ &\quad + g(Z, \phi W)\} - \frac{\bar{f}}{2}\{(\nabla_Z \theta)(\xi) + (\nabla_\xi \theta)(Z)\}\bar{\eta}(W). \end{aligned} \quad (38)$$

Utilizing  $W = \xi$  in (38), gives

$$(Z\bar{f}) = (\xi\bar{f}) + \frac{\bar{f}}{2}((\nabla_Z\theta)(\xi) + (\nabla_\xi\theta)(Z)). \quad (39)$$

In view of (39) in (37), we obtain

$$(\xi\bar{f}) + \bar{\beta}\bar{g}(\bar{\phi}Z, W) = (\xi\bar{f})\bar{\eta}(Z)\bar{\eta}(W) + \bar{f}\{-\bar{g}(\phi hZ, W) + g(Z, \phi W)\}. \quad (40)$$

Use first equation of (12) and contracting with  $Z$ ,  $W$  we get  $(\xi\bar{f}) = 0$ . Using this in (38), we find  $\bar{\alpha} = 0$ . Using  $(\xi\bar{f}) = 0$  in (40), we obtain

$$\bar{\beta}\bar{g}(\phi Z, W) + \bar{\beta}\theta(\phi Z)\bar{\eta}(W) = \bar{f}\bar{\eta}(Z)\bar{\eta}(W) + \bar{f}(-\bar{g}(\phi hZ, W) + g(Z, \phi W)). \quad (41)$$

Replacing  $W$  by  $\phi W$  in (41) and using first equation of (5), it follows that for all vector fields  $W$ , we get

$$(\bar{f} + f\bar{\beta})(-Z + \eta(Z)\xi) = \bar{f}fhZ. \quad (42)$$

Taking inner product with  $X$  and contraction of  $Z$  and  $X$  gives  $\bar{\beta} = -\frac{\bar{f}}{f}$ , where  $f$  and  $\bar{f}$  are non-zero. Using  $\bar{\beta}$  value in (42), we get  $\bar{f}fhZ = 0$ , since  $f$  and  $\bar{f}$  are non-zero we have  $hZ = 0$ , where  $Z$  is arbitrary vector field which is non zero therefore  $h = 0$ . Using this in (6), we get  $\nabla_Z\xi = -\phi Z$ . Hence  $M$  reduces to  $K$ -contact manifold.  $\square$

## 4 Vector fields on Sasakian manifold under BA-deformation

In this section we study concircular, projective and affine conformal vector fields on Sasakian manifold under **BA**-deformation.

**Theorem 3.** *The **BA**-deformation on a  $(2n + 1)$ -dimensional Sasakian manifold keeps a concircular vector field fixed. Further the deformed Sasakian manifold is isometric to unit sphere, manifold reduces to Einstein with eigen vector  $QV$  and an eigen value  $2n$ .*

*Proof.* Apply **BA**-deformation to equation (2) to get

$$\bar{\nabla}_Z V = \bar{\rho}Z. \quad (43)$$

In view of (14) and (2), equation (43) takes the form

$$\rho Z + \theta(Z)\bar{\phi}V + \theta(V)\bar{\phi}Z - (\nabla_Z\theta)(V)\xi = \bar{\rho}Z, \quad (44)$$

where  $\theta$  is a  $g$ -dual of  $\psi$  i.e.,  $\theta(X) = g(X, \psi)$ . Taking inner product with  $X$  in previous equation, we obtain

$$\rho\bar{g}(Z, X) + \theta(Z)\bar{g}(\bar{\phi}V, X) + \theta(V)\bar{g}(\bar{\phi}Z, X) - (\nabla_Z\theta)(V)\bar{\eta}(X) = \bar{\rho}\bar{g}(Z, X). \quad (45)$$

Substituting  $X = \xi$  in (45), we get

$$(\nabla_Z \theta)(V) = \rho \bar{\eta}(Z) - \bar{\rho} \bar{\eta}(Z). \quad (46)$$

In view of (45), third equation of (12) and first equation of (5) it follows from (44), that

$$-\bar{\rho} \phi^2 Z + \bar{\rho} \theta(Z) \xi = -\rho \phi^2 Z + \rho \theta(Z) \xi + \theta(Z) \bar{\phi} V + \theta(V) \bar{\phi} Z, \quad (47)$$

for all vector fields  $X$ .

Taking inner product with  $\xi$  in (47), we infer that  $(\bar{\rho} - \rho)\theta(Z) = 0$ . Since,  $\theta(Z) \neq 0$ , it follows that  $\bar{\rho} = \rho$ . Utilizing last equation in (45), we obtain  $(\nabla_Z \theta)(V) = 0$ . Taking covariant derivative along  $X$  in (43) and using (14), we obtain

$$\bar{\nabla}_X \bar{\nabla}_Z V = (X\bar{\rho})Z + \bar{\rho} \nabla_Z X + \bar{\rho} \theta(Z) \bar{\phi} X + \bar{\rho} \theta(X) \bar{\phi} Z - \bar{\rho} (\nabla_X \theta)(Z), \quad (48)$$

and

$$\bar{\nabla}_{[Z,X]} V = \bar{\rho} \nabla_Z X - \bar{\rho} \nabla_X Z. \quad (49)$$

In view of (48), (49) and  $\bar{R}(Z, X)V = \bar{\nabla}_Z \bar{\nabla}_X V - \bar{\nabla}_X \bar{\nabla}_Z V - \bar{\nabla}_{[Z,X]} V$ , we have

$$\bar{R}(Z, X)V = (Z\bar{\rho})X - (X\bar{\rho})Z + \bar{\rho} (\nabla_Z \theta)(X) \xi - \bar{\rho} (\nabla_X \theta)(Z) \xi, \quad (50)$$

where  $(\nabla_X \theta)(Z) = g(\nabla_X \psi, Z) = g(X, \phi Z)$ . Suppose that  $x_0 \in M$ ,  $\nabla_{U_i} U_j = 0 \forall i, j \in \{1, 2, \dots, (2n+1)\}$ . With at  $x_0$ ,  $d\theta(X, Z) = g(X, \phi Z) = 0$ .

Taking inner product of (50) with  $F$ , we obtain

$$\bar{g}(\bar{R}(Z, X)V, F) = (X\bar{\rho})\bar{g}(Y, F) - (Y\bar{\rho})\bar{g}(X, F). \quad (51)$$

Contracting the foregoing equation with respect to  $Z$  and  $F$ , it is easy to see that

$$\bar{S}(X, V) = -2n\bar{g}(D\rho, X). \quad (52)$$

Utilizing  $F = \xi$  in (51), we obtain

$$\bar{g}(\bar{R}(Z, X)V, \xi) = (Z\bar{\rho})\bar{\eta}(X) - (X\bar{\rho})\bar{\eta}(Z). \quad (53)$$

Taking inner product of (15), and substituting  $W = \xi$ , and replace  $F = V$  in the obtained equaion, we acquire

$$\bar{g}(\bar{R}(Z, X)V, \xi) = \bar{\eta}(Z)\bar{g}(X, F) - \bar{\eta}(X)\bar{g}(Y, F). \quad (54)$$

In view of (12) and (16) equation (52), reduces to

$$S(X, V) - 2n\eta(X)\eta(V) + 2n\bar{\eta}(X)\bar{\eta}(V) = -2ng(D\bar{\rho}, X) + 2n\eta(D\bar{\rho})\eta(X) - 2n\bar{\eta}(D\bar{\rho})\bar{\eta}(X). \quad (55)$$

Taking  $X = \xi$  in (54), we obtain  $\bar{\eta}(V) = -\bar{\eta}(D\bar{\rho})$ . Utilization of  $Z = \xi$  in the reduced equation obtained, comparing (53) with (54), gives

$$(\xi\bar{\rho})\bar{\eta}(X) - (X\bar{\rho}) = \bar{g}(X, V) - \bar{\eta}(V)\bar{\eta}(X). \quad (56)$$

Applying  $\bar{\eta}(V) = -\bar{\eta}(D\bar{\rho})$  in (56), we infer that  $\bar{g}(D\bar{\rho}, X) = -\bar{g}(X, V)$ , for all vector fields  $X$ . This implies that  $-D\bar{\rho} = V$ . Utilizing this in (52) gives  $\bar{S}(X, V) = 2n\bar{g}(X, V)$ . In view of (12) and (16), the last equation becomes

$$S(X, V) = 2ng(X, V), \quad (57)$$

where  $QV$  is eigen vector with an eigen value  $2n$ . Use of  $\bar{\rho} = \rho$ , in  $-D\bar{\rho} = V$ , we achieve  $-D\rho = V$ . Taking covariant derivative along  $Z$ , in last equation and with the use of (2), we get  $\nabla_Z D\rho = -\rho Z$ . From the well known result of Obata [12] that “necessary and sufficient condition for a complete connected Riemannian manifold  $(M, g)$  to be isometric to Euclidean sphere of radius  $\frac{1}{c}$  is that it admits a non trivial solution  $f$  of the system of differential equation  $\nabla\nabla f = -c^2 fg$ ”, we state that **BA**-deformed Sasakian manifold admitting a concircular vector field is isometric to unit sphere of radius 1.  $\square$

**Theorem 4.** *In a BA-deformed Sasakian manifold, projective and affine conformal vector fields become affine vector fields.*

*Proof.* Let us consider

$$(\bar{L}_V \bar{\nabla})(Z, X) = \bar{L}_V \bar{\nabla}(Z, X) - \bar{\nabla}(\bar{L}_V Z, X) - \bar{\nabla}(Z, \bar{L}_V X). \quad (58)$$

Suppose that  $x_0 \in M$ ,  $\nabla_{U_i} U_j = 0 \forall i, j \in \{1, 2, \dots, (2n+1)\}$ .  $[\xi, Z] = 0$  at  $x_0$ , in view of (14) and (2.4) we get  $\nabla_\xi Z = -\phi Z$ . Using this in (58) gives

$$\begin{aligned} (\bar{L}_V \bar{\nabla})(Z, X) &= (L_V \nabla)(Z, X) + (V(\theta(X)) + \theta(\nabla_X V) - \theta(\nabla_V X) - (\nabla_V \theta)(X) \\ &\quad + (\nabla_X \theta)(V))\bar{\phi}Z + (V(\theta(Z)) + \theta(\nabla_Z V) - \theta(\nabla_V Z) - (\nabla_V \theta)(Z) \\ &\quad + (\nabla_Z \theta)(V))\bar{\phi}X + \theta(Z)(\bar{\phi}\nabla_X V - \bar{\phi}\nabla_V X + \bar{g}(V, X)\xi - \bar{\eta}(X)\xi \\ &\quad + \theta(X)\bar{\phi}\phi V - \nabla_{\bar{\phi}X} V - \theta(\bar{\phi}X)\bar{\phi}V + (\nabla_{\bar{\phi}X}\theta)(V)\xi) + \theta(X)(\bar{g}(V, Z)\xi \\ &\quad - \bar{\eta}(Z)\xi + \theta(Z)\bar{\phi}\phi V - \nabla_{\bar{\phi}Z} V - \theta(\bar{\phi}Z)\bar{\phi}V + (\nabla_{\bar{\phi}Z}\theta)(V)\xi + \bar{\phi}\nabla_Z V \\ &\quad - \bar{\phi}\nabla_V Z) + ((\nabla_{\nabla_Z X}\theta)(V) + (\nabla_{\nabla_V Z}\theta)(X) - (\nabla_{\nabla_Z V}\theta)(X)) \quad (59) \\ &\quad - (V(\nabla_Z \theta)(X)) - (Z(\nabla_X \theta)(V)) + (Z(\nabla_V \theta)(X)) - (\nabla_V \theta)(\nabla_Z X) \\ &\quad + (\nabla_Z \theta)(\nabla_V X) - (\nabla_Z \theta)(\nabla_X V) - (\nabla_Z X)\xi(\theta(V)) + (\nabla_Z V)\xi(\theta(X)) \\ &\quad - (\nabla_V Z)\xi(\theta(X))\xi. \end{aligned}$$

Suppose  $V$  is a projective vector field under **BA**-deformation. Then

$$(\bar{L}_V \bar{\nabla})(Z, X) = \bar{P}(Z)X + \bar{P}(Z)X. \quad (60)$$

In view of (3), (12) and (60) in (59), operating  $\bar{\phi}$  to obtained equation, we arrive

$$\begin{aligned}
\bar{P}(Z)\bar{\phi}X + \bar{P}(X)\bar{\phi}Z = & P(Z)\bar{\phi}X + P(X)\bar{\phi}Z + \theta(Z)\{-\theta(X)\bar{\phi}V - \phi\nabla_{\bar{\phi}X}V \\
& - \theta(\phi\nabla_{\bar{\phi}X}V)\xi - \theta(\bar{\phi}X)\bar{\phi}\phi V\} + \theta(X)\{-\theta(Z)\bar{\phi}V \\
& - \phi\nabla_{\bar{\phi}Z}V - \theta(\phi\nabla_{\bar{\phi}X}V)\xi - \theta(\bar{\phi}Z)\bar{\phi}\phi V\} - (\theta(\nabla_V X) \\
& - \theta(\nabla_X V) + (\nabla_V\theta)(X) - (\nabla_X\theta)(V) - V(\theta(Z)))\bar{\phi}\phi Z \\
& - (\theta(\nabla_V Z) - \theta(\nabla_Z V) + (\nabla_V\theta)(Z) - (\nabla_Z\theta)(V) \\
& - V(\theta(X)))\bar{\phi}\phi X - \theta(X)(\bar{\phi}\phi\nabla_V Z - \bar{\phi}\phi\nabla_Z V) \\
& - \theta(Z)(\bar{\phi}\phi\nabla_V Z - \bar{\phi}\phi\nabla_X V). \tag{61}
\end{aligned}$$

Contracting (61) with  $\xi$ , we acquire

$$-\theta(Z)\theta(\phi\nabla_{\phi X}V) - \theta(Z)\theta(\phi X)\theta(V) - \theta(X)\theta(\phi\nabla_{\phi Z}V) - \theta(X)\theta(\phi Z)\theta(V) = 0. \tag{62}$$

Utilizing first equation of (12) in (61) and taking inner product with  $\xi$  in the obtained equation entails that

$$\begin{aligned}
\bar{P}(Z)\theta(\phi X) + \bar{P}(X)\theta(\phi Z) = & \theta(\phi X)(P(Z) - \theta(Z)\theta(V)) + \theta(\phi Z)(P(X) - \theta(X)\theta(V)) \\
& - 2\theta(X)\theta(V)\theta(\phi Z) - \theta(X)\theta(\phi\nabla_{\phi Z}V) \\
& - \theta(Z)\theta(\phi\nabla_{\phi X}V). \tag{63}
\end{aligned}$$

In view of (62) in (63), we obtain

$$\bar{P}(Z)\theta(\phi X) + \bar{P}(X)\theta(\phi Z) = P(Z)\theta(\phi X) + P(X)\theta(\phi Z) - 2\theta(Z)\theta(X)\theta(\phi V). \tag{64}$$

Add and subtract  $2\eta(Z)\theta(X)\theta(\phi V)$ , and in view of third equation of (12), we infer

$$\begin{aligned}
\bar{P}(Z)\theta(\phi X) + \bar{P}(X)\theta(\phi Z) = & P(Z)\theta(\phi X) + P(X)\theta(\phi Z) + 2\bar{\eta}(Z)\theta(X)\theta(\phi V) \\
& - 2\eta(Z)\theta(X)\theta(\phi V). \tag{65}
\end{aligned}$$

Replacing  $Z$  by  $\phi Z$  and  $X$  by  $\phi X$  in (65) entails that

$$(\bar{P}(\phi Z) - P(\phi Z))\theta(X) + (\bar{P}(\phi X) - P(\phi X))\theta(Z) = 0. \tag{66}$$

Add and subtract  $(\bar{P}(\phi Z) - P(\phi Z))\eta(X)$  in foregoing equation, to obtain

$$(\bar{P}(\phi Z))\eta(X) - (\bar{P}(\phi Z) - P(\phi Z))\bar{\eta}(X) + (\bar{P}(\phi X) - P(\phi X))\theta(Z) = 0. \tag{67}$$

Replace  $X$  by  $\phi X$  in (67) and using first equation of (5) in the resulting equation, we acquire

$$(-\bar{P}(X) + P(X) + \eta(X)\bar{P}(\xi) - \eta(X)P(\xi))\theta(Z) = 0. \tag{68}$$

Utilizing  $X = \xi$  in (64), we obtain

$$\bar{P}(\xi) = P(\xi). \tag{69}$$

Using foregoing equation in (68), we infer

$$\bar{P}(X) = P(X), \quad (70)$$

where,  $\theta(Z) \neq 0$ . By use of (70) in (60), and on comparing the resulting equation with (3), we obtain

$$(\bar{L}_V \bar{\nabla})(Z, X) = (L_V \nabla)(Z, X) = P(Z)X + P(X)Z. \quad (71)$$

We have the formula (Yano[19])

$$(\nabla_U L_V g)(Z, X) = g((L_V \nabla)(U, Z), X) + g((L_V \nabla)(U, X), Z), \quad (72)$$

where  $U$  is an any arbitrary vector field on  $M$ . In view of (71) in (72), and for  $Z = X = \xi$ , we get

$$2P(U) + 2P(\xi)\eta(U) = 0. \quad (73)$$

Setting  $U = \xi$  in (73), we get  $P(\xi) = 0$ , and using this equation in (73), we obtain  $P(U) = 0$ . Last equation in (71) yields  $(\bar{L}_V \bar{\nabla})(Z, X) = (L_V \nabla)(Z, X) = 0$ , which shows that  $V$  is an affine vector field. This completes the proof.

Next we consider affine conformal vector field  $V$  on a Sasakian manifold under **BA**-deformation, for which we have

$$(\bar{L}_V \bar{\nabla})(Z, X) = \bar{g}(Z, D\bar{\rho})X + \bar{g}(X, D\bar{\rho})Z - \bar{g}(Z, X)D\bar{\rho}. \quad (74)$$

In view of (4) and (74) in (59), we obtain

$$\begin{aligned} & \bar{g}(Z, D\bar{\rho})X + \bar{g}(X, D\bar{\rho})Z - \bar{g}(Z, X)D\bar{\rho} \\ &= g(Z, D\rho)X + g(X, D\rho)Z - g(Z, X)D\rho + (V(\theta(X)) + \theta(\nabla_X V) - \theta(\nabla_V X) \\ & - (\nabla_V \theta)(X) + (\nabla_X \theta)(V))\bar{\phi}Z + (V(\theta(Z)) + \theta(\nabla_Z V) - \theta(\nabla_V Z) - (\nabla_V \theta)(Z) \\ & + (\nabla_Z \theta)(V))\bar{\phi}X + \theta(Z)(\bar{\phi}\nabla_X V - \bar{\phi}\nabla_V X + \bar{g}(V, X)\xi - \bar{\eta}(X)\xi + \theta(X)\bar{\phi}\phi V \\ & - \nabla_{\bar{\phi}X} V - \theta(\bar{\phi}X)\bar{\phi}V + (\nabla_{\bar{\phi}X} \theta)(V)\xi) + \theta(X)(\bar{g}(V, Z)\xi - \bar{\eta}(Z)\xi + \theta(Z)\bar{\phi}\phi V \\ & - \nabla_{\bar{\phi}Z} V - \theta(\bar{\phi}Z)\bar{\phi}V + (\nabla_{\bar{\phi}X} \theta)(V)\xi + \bar{\phi}\nabla_Z V - \bar{\phi}\nabla_V Z) + ((\nabla_{\nabla_Z X} \theta)(V) \quad (75) \\ & + (\nabla_{\nabla_V Z} \theta)(X) - (\nabla_{\nabla_Z V} \theta)(X) - (V(\nabla_Z \theta)(X)) - (Z(\nabla_X \theta)(V)) \\ & + (Z(\nabla_V \theta)(X)) - (\nabla_V \theta)(\nabla_Z X) + (\nabla_Z \theta)(\nabla_V X) - (\nabla_Z \theta)(\nabla_X V) \\ & - (\nabla_Z X)\xi(\theta(V)) + (\nabla_Z V)\xi(\theta(X)) - (\nabla_V Z)\xi(\theta(X)))\xi. \end{aligned}$$

Taking  $Z = \xi$  in (75), we obtain

$$\begin{aligned} & \bar{\eta}(D\bar{\rho})X + \bar{g}(X, D\bar{\rho})\xi - \bar{\eta}(X)D\bar{\rho} \\ &= \eta(D\rho)X + g(X, D\rho)\xi - \eta(X)D\rho + (\nabla_V \theta)(\phi X)\xi + (\nabla_{\phi X} \theta)(V)\xi - V((\nabla_\xi \theta)(X))\xi \\ & - (\nabla_\xi \theta)(X)\xi(\theta(V))\xi - (\nabla_V \theta)(\xi)\phi X + (\nabla_\xi \theta)(V)\phi X + (\nabla_\xi \theta)(\nabla_V X)\xi \\ & - (\nabla_\xi \theta)(\nabla_X V)\xi + \xi((\nabla_V \theta)(X))\xi - \xi((\nabla_X \theta)(V))\xi - (\nabla_V \theta)(\xi)\xi(\theta(X))\xi \quad (76) \\ & + (\nabla_\xi \theta)(V)\xi(\theta(X))\xi. \end{aligned}$$

Utilization of  $X = \xi$  in foregoing equation, gives us

$$2\bar{\eta}(D\bar{\rho})\xi - D\bar{\rho} = 2\eta(D\rho)\xi - D\rho + \xi(\nabla_V\theta(\xi))\xi - \xi((\nabla_\xi\theta)(V))\xi. \quad (77)$$

Operating  $\phi$  on both sides of (76) entails that

$$\begin{aligned} \bar{\eta}(D\bar{\rho})\phi X - \bar{\eta}(X)\phi D\bar{\rho} &= \eta(D\rho)\phi X - \eta(X)\phi D\rho + (\nabla_V\theta)(\xi)X - (\nabla_\xi\theta)(V)X \\ &\quad + (\nabla_\xi\theta)(V)\eta(X)\xi - (\nabla_V\theta)(\xi)\eta(X)\xi. \end{aligned} \quad (78)$$

Taking  $X = \xi$  in equation(78), we obtain

$$\phi D\bar{\rho} = \phi D\rho. \quad (79)$$

In view of (79), third equation of (12) in (78) yields

$$\begin{aligned} \eta(D\bar{\rho})\phi X - \theta(D\bar{\rho})\phi X + \theta(X)\phi D\rho &= \eta(D\rho)\phi X + (\nabla_V\theta)(\xi)X + (\nabla_V\theta)(\xi)\eta(X)\xi \\ &\quad - (\nabla_\xi\theta)(V)X + (\nabla_\xi\theta)(V)\eta(X)\xi. \end{aligned} \quad (80)$$

Operating  $\phi$  on (80), with the use of third equation of (12), gives

$$\begin{aligned} &-\bar{\eta}(D\bar{\rho})X + \bar{\eta}(D\bar{\rho})\eta(X)\xi - \theta(X)D\bar{\rho} + \theta(X)\eta(D\bar{\rho})\xi \\ &= -\eta(D\rho)X + \eta(D\rho)\eta(X)\xi + (\nabla_V\theta)(\xi)\phi X - (\nabla_\xi\theta)(V)\phi. \end{aligned} \quad (81)$$

Contracting (81) with  $X$  gives

$$\bar{\eta}(D\bar{\rho}) = \eta(D\rho). \quad (82)$$

Operating  $\phi$  on (79), and using first quation of (5) and (82), we obtain

$$D\bar{\rho} = D\rho. \quad (83)$$

In view of (83) in (74), we acquire

$$(\bar{L}_V\bar{\nabla})(Z, X) = \bar{g}(Z, D\rho)Z + \bar{g}(X, D\rho)X - \bar{g}(Z, X)D\rho. \quad (84)$$

Setting  $Z = X = \xi$  in (84), using (82), and taking inner product with  $Z$ , we obtain

$$D\rho = 2(\xi\rho)\xi, \quad (85)$$

for all vector fields  $Z$ .

Equation (85) can be written as  $d\rho = 2(\xi\rho)\eta$ . Applying poincare Lemma ( $d^2 = 0$ ), we get  $(\xi\rho)(\eta \wedge d\eta) = 0$ , Since  $(\eta \wedge d\eta) \neq 0$ , we have  $(\xi\rho) = 0$ . Using this in (85), we get  $D\rho = 0$ . On integrating it,  $\rho$  becomes constant. Further we have  $(\bar{L}_V\bar{\nabla})(Z, X) = 0$  and  $(L_V\nabla)(Z, X) = 0$ . Thus  $V$  reduces to affine vector field.  $\square$

**Example:** We consider the 3-dimensional manifold  $M = \{(x, y, z) \in R^3 \mid z \neq 0\}$ , with  $(x, y, z)$  as the standard co-ordinates in  $R^3$ . Let  $e_1, e_2$  and  $e_3$  be the vector fields on  $M$  given by

$$e_1 = 2(y \frac{\partial}{\partial z} - \frac{\partial}{\partial x}), e_2 = 2 \frac{\partial}{\partial y}, e_3 = -2 \frac{\partial}{\partial z} = \xi.$$

The vectors  $e_1, e_2$  and  $e_3$  are linearly independent at each point of  $M$ .  
Let  $g$  be a Riemannian metric defined by

$$g(e_1, e_1) = g(e_3, e_3) = 1 = g(e_2, e_2) = 1, g(e_i, e_j) = 0, i, j \in \{1, 2, 3\} \text{ and } i \neq j.$$

Let  $\eta$  be a 1-form on  $M$  defined by  $\eta(X) = g(X, e_3)$ , for all  $X$  on  $M$  and  $e_3 = \xi$ .  
Let  $\phi$  be a (1,1) tensor field on  $M$  defined by

$$\phi e_1 = e_2, \phi e_2 = -e_1, \phi e_3 = 0.$$

Using linearity property of  $\phi$  and  $g$ , we have

$$\phi^2 X = -X + \eta(X)\xi, g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Thus the structure  $(\phi, \xi, \eta, g)$  defines an almost contact structure on  $M$ . Let  $\nabla$  be the Levi-civita-connection with respect to the metric  $g$ . Then we have

$$[e_1, e_2] = 2e_3, [e_1, e_3] = 0, [e_2, e_3] = 0.$$

From Koszul formula

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y).$$

We can easily calculate

$$\begin{aligned} \nabla_{e_1} e_1 &= 0, \nabla_{e_1} e_2 = e_3, \nabla_{e_1} e_3 = -e_2, \\ \nabla_{e_2} e_1 &= -e_3, \nabla_{e_2} e_2 = 0, \nabla_{e_2} e_3 = e_1, \\ \nabla_{e_3} e_1 &= -e_2, \nabla_{e_3} e_2 = e_1, \nabla_{e_3} e_3 = 0. \end{aligned}$$

These equations show that  $\nabla_X \xi$  and  $(\nabla_X \phi)(Y)$  satisfy equations (6) and (7).  
Let us consider 1-form  $\theta = dx + dy$  which gives  $d\theta = 0$  and  $\theta(\xi) = 0$ .

$$\theta(e_1) = -2, \theta(e_2) = 2, \theta(e_3) = 0.$$

From 1-form  $\theta$ , we get the following

$$\bar{\phi} e_1 = e_2 + e_3, \bar{\phi} e_2 = -e_1 + 2e_3, \bar{\phi} e_3 = 0.$$

Using equation (14), we get

$$\begin{aligned} \bar{\nabla}_{e_1} e_1 &= -6e_2 - 8e_3, \bar{\nabla}_{e_1} e_2 = 2e_1 + 4e_2 + e_3, \bar{\nabla}_{e_1} e_3 = -e_2 - 2e_3, \\ \bar{\nabla}_{e_2} e_1 &= 4e_1 + 2e_2 - e_3, \bar{\nabla}_{e_2} e_2 = -6e_1 + 8e_3, \bar{\nabla}_{e_2} e_3 = e_1 - 2e_3, \\ \bar{\nabla}_{e_3} e_1 &= -e_2 - 2e_3, \bar{\nabla}_{e_3} e_2 = e_1 - 2e_3, \bar{\nabla}_{e_3} e_3 = 0. \end{aligned}$$

We have concircular vector field  $\nabla_X V = \bar{\rho}X$ , making use of (14), we get  $\bar{\rho} = \rho$  of Theorem 4.1.

## 5 Conclusion

We observed from the proved results that the **BA**-deformation on Sasakian manifold achieves the similar results or stronger results under weaker conditions or nil conditions when compare to **D**-homothetic deformation of Sasakian manifold. **BA**-deformation is relevant in the study of more complex structures like space-time manifolds and is advantageous over other transformations. Compared to **D**-homothetic or **D**-conformal deformations, in **BA**-deformation the structure tensor  $\phi$  and the contact form  $\eta$  are obtained not just by scaling by a constant or scalar function, but additional 1-form is required. Because of the presence of this 1-form, we could achieve stronger results with weaker conditions.

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