

## INTRODUCTION TO GENERALIZED LEONARDO-ALWYN HYBRID NUMBERS

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### Abstract

In [6], Gökbaşı defined a new type of number sequence called Leonardo-Alwyn sequence. In this paper, we consider the generalized Leonardo-Alwyn hybrid numbers and investigate some of their properties. We also give some applications related to the generalized Leonardo-Alwyn hybrid numbers in matrices.

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## 1 Introduction

Let  $p, q$  and  $r$  be integers such that  $p^2 + 4q > 0$  the generalized Leonardo-Alwyn sequence  $\{LA_n^{(r)}\}_{n \geq 0}$  is described by

$$LA_{n+3}^{(r)} = (1+p)LA_{n+2}^{(r)} + (q-p)LA_{n+1}^{(r)} - qLA_n^{(r)}, \quad (1)$$

with initial conditions  $LA_0^{(r)} = a$ ,  $LA_1^{(r)} = b$  and  $LA_3^{(r)} = pb + qa + r$ , where  $a$  and  $b$  are arbitrary real numbers.

The solutions of the equation  $x^3 - (1+p)x^2 - (q-p)x + q = 0$  associated with the recurrence relation (1) are

$$\varphi = 1, \quad \psi_1 = \frac{p + \sqrt{p^2 + 4q}}{2}, \quad \psi_2 = \frac{p - \sqrt{p^2 + 4q}}{2}.$$

Note that

$$\psi_1 + \psi_2 = p, \quad \psi_1\psi_2 = -q, \quad \psi_1 - \psi_2 = \sqrt{p^2 + 4q}. \quad (2)$$

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So the Binet formula for the generalized Leonardo-Alwyn sequence is given by

$$L\mathcal{A}_n^{(r)} = \frac{1}{1-p-q} \left[ r + \left( \frac{\Phi_1 \psi_1^n - \Phi_2 \psi_2^n}{\psi_1 - \psi_2} \right) \right], \quad (3)$$

where  $\Phi_j = \Phi(\psi_j) = ((1-p-q)a-r)\psi_j + (1-p-q)b + (p^2 + pq - p)a + (p-1)r$  (with  $j = 1, 2$ ).

Generalized Leonardo-Alwyn sequence is a generalization of some sequences such as the Leonardo, John-Edouard and Ernst sequences. These sequences have applications in number theory, geometry and algebra. Hence, these sequences have been studied by Alp and Koçer [2, 1], Catarino and Borges [3, 4], Gökbaşı [6], Kuhapatanakul and Chobsorn [7], Shannon [10], Soykan [11, 12] and Tan et al. [13, 14].

**Remark 1.** *If  $r = 1$ , some particular cases of Eq. (1) are:*

- *If  $a = b = 1$  and  $p = q = 1$ , we get the Leonardo sequence*

$$Le_{n+2} = Le_{n+1} + Le_n + 1, \quad Le_0 = Le_1 = 1.$$

*Note that  $Le_n = 2F_{n+1} - 1$ , where  $F_n$  is the  $n$ -th Fibonacci number.*

- *If  $a = b = 1$ ,  $p = 1$  and  $q = 2$ , we have Ernst sequence*

$$Er_{n+2} = Er_{n+1} + 2Er_n + 1, \quad Er_0 = Er_1 = 1.$$

*Further, we can write  $Er_n = \frac{1}{2}[3J_{n+1} - 1]$ , where  $J_n$  is the  $n$ -th Jacobsthal number.*

Recently, Özdemir [9] defined a new generalization of complex ( $i^2 = -1$ ), hyperbolic ( $h^2 = 1$ ) and dual ( $\varepsilon^2 = 0$ ) numbers different from above generalizations. In this generalization, the author gave a system of such numbers that consists of all three number systems together. This set was called hybrid numbers, denoted by  $\mathbb{T}$ , is defined as

$$\mathbb{T} = \left\{ Z = z_0 + z_1i + z_2\varepsilon + z_3h : \begin{array}{l} i^2 = -1, \varepsilon^2 = 0, h^2 = 1, \\ ih = -hi = \varepsilon + i \end{array} \right\}, \quad (4)$$

where  $z_0, z_1, z_2$  and  $z_3$  are real numbers.

Two hybrid numbers  $Z = z_0 + z_1i + z_2\varepsilon + z_3h$  and  $W = w_0 + w_1i + w_2\varepsilon + w_3h$  are equal if all their components are equal. Further,  $Z = W$  if only if  $z_0 = w_0, z_1 = w_1, z_2 = w_2$  and  $z_3 = w_3$ . We can give operations and properties with the hybrid numbers:

- $Z + W = (z_0 + w_0) + (z_1 + w_1)i + (z_2 + w_2)\varepsilon + (z_3 + w_3)h$  (addition),
- $Z - W = (z_0 - w_0) + (z_1 - w_1)i + (z_2 - w_2)\varepsilon + (z_3 - w_3)h$  (subtraction),
- $kZ = kz_0 + kz_1i + kz_2\varepsilon + kz_3h$  (multiplication by scalar  $k \in \mathbb{R}$ ).

With respect to the addition operation, the inverse element of  $Z = z_0 + z_1i + z_2\varepsilon + z_3h$  is  $-Z = -z_0 - z_1i - z_2\varepsilon - z_3h$ . This implies that,  $(\mathbb{T}, +)$  is an abelian group.

The hybrid product is obtained by distributing the terms to the right, preserving the order of multiplication of the units and then writing the values of the following substituting each product of units by the equalities  $i^2 = -1$ ,  $h^2 = 1$ ,  $\varepsilon^2 = 0$  and  $ih = -hi = \varepsilon + i$ . The table 1 shows us that the multiplication operation with the hybrid numbers is not commutative.

Table 1: The multiplication table for the basis of  $\mathbb{T}$ .

$\times$	1	$i$	$\varepsilon$	$h$
1	1	$i$	$\varepsilon$	$h$
$i$	$i$	-1	$1 - h$	$\varepsilon + i$
$\varepsilon$	$\varepsilon$	$1 + h$	0	$-\varepsilon$
$h$	$h$	$-(\varepsilon + i)$	$\varepsilon$	1

The multiplication operation with hybrid numbers has the property of associativity. The conjugate of a hybrid number  $Z = z_0 + z_1i + z_2\varepsilon + z_3h$ , denoted by  $\overline{Z}$ , is defined as  $\overline{Z} = z_0 - z_1i - z_2\varepsilon - z_3h$ . The real number

$$\mathcal{C}(Z) = Z\overline{Z} = z_0^2 + (z_1 - z_2)^2 - z_2^2 - z_3^2$$

is called the character of the hybrid number  $Z = z_0 + z_1i + z_2\varepsilon + z_3h$ . The real number  $\sqrt{|\mathcal{C}(Z)|}$  will be called the norm of the hybrid number  $Z$  and will be denoted by  $\|Z\|_{\mathbb{T}}$ . In [8], Morales studied hybrid numbers and applied it to many number sequences of second-order.

Now, we introduce the generalized Leonardo-Alwyn hybrid numbers and investigate some of their properties. We also give some applications related to the generalized Leonardo-Alwyn hybrid numbers in matrices.

## 2 Generalized Leonardo-Alwyn hybrid numbers

For  $m \geq 0$ , the generalized Leonardo-Alwyn hybrid numbers are defined by

$$La\mathcal{H}_m^{(r)} = LA_m^{(r)} + iLA_{m+1}^{(r)} + \varepsilon LA_{m+2}^{(r)} + hLA_{m+3}^{(r)}, \tag{5}$$

where  $LA_m^{(r)}$  is the  $m$ -th generalized Leonardo-Alwyn number, and  $i, \varepsilon, h$  are hybrid units.

**Remark 2.** For  $a = b = 1$  and  $p = q = r = 1$ , we obtain  $\psi_1 = \frac{1+\sqrt{5}}{2}$ ,  $\psi_2 = \frac{1-\sqrt{5}}{2}$ ,  $\Phi_1 = -2\psi_1$  and  $\Phi_2 = -2\psi_2$ . Then, we obtain the Leonardo hybrid number defined by

$$HLe_m = Le_m + Le_{m+1}i + Le_{m+2}\varepsilon + Le_{m+3}h,$$

where  $Le_m$  is the  $m$ -th Leonardo number (see, e.g., [1]).

**Theorem 1.** *Let  $m \geq 0$  be an integer. Then, we have*

$$La\mathcal{H}_{m+2}^{(r)} = pLa\mathcal{H}_{m+1}^{(r)} + qLa\mathcal{H}_m^{(r)} + r\Psi, \quad (6)$$

with  $\Psi = 1+i+\varepsilon+h$ ,  $La\mathcal{H}_0^{(r)} = a+bi+(pb+qa+r)\varepsilon+((p^2+q)b+(pq+q)a+(p+1)r)h$  and  $La\mathcal{H}_1^{(r)} = b+(pb+qa+r)i+((p^2+q)b+(pq+q)a+(p+1)r)\varepsilon+((p^2+2pq)b+(p^2q+pq+q^2)a+(p^2+p+q+1)r)h$ .

*Proof.* If  $m = 0$  and  $\Psi = 1 + i + \varepsilon + h$ , then we get

$$\begin{aligned} pLa\mathcal{H}_1^{(r)} + qLa\mathcal{H}_0^{(r)} + r\Psi &= pb + [p^2b + pqa + pr]i \\ &\quad + [p(p^2 + q)b + (p^2q + pq)a + (p^2 + p)r]\varepsilon \\ &\quad + [p^2(p + 2p)b + p(p^2q + pq + q^2)a + p(p^2 + p + q + 1)r]h \\ &\quad + qa + qbi + [pqb + q^2a + qr]\varepsilon \\ &\quad + [(p^2q + q^2)b + (pq^2 + q^2)a + (pq + q)r]h \\ &\quad + r + ri + r\varepsilon + rh \\ &= LA_2^{(r)} + iLA_3^{(r)} + \varepsilon LA_4^{(r)} + hLA_5^{(r)} \\ &= La\mathcal{H}_2^{(r)}. \end{aligned}$$

If  $m \geq 0$  and using Eqs. (1) and (5), we obtain

$$\begin{aligned} La\mathcal{H}_{m+2}^{(r)} &= LA_{m+2}^{(r)} + iLA_{m+3}^{(r)} + \varepsilon LA_{m+4}^{(r)} + hLA_{m+5}^{(r)} \\ &= pLA_{m+1}^{(r)} + qLA_m^{(r)} + r + [pLA_{m+2}^{(r)} + qLA_{m+1}^{(r)} + r]i \\ &\quad + [pLA_{m+3}^{(r)} + qLA_{m+2}^{(r)} + r]\varepsilon + [pLA_{m+4}^{(r)} + qLA_{m+3}^{(r)} + r]h \\ &= p[LA_{m+1}^{(r)} + iLA_{m+2}^{(r)} + \varepsilon LA_{m+3}^{(r)} + hLA_{m+4}^{(r)}] \\ &\quad + q[LA_m^{(r)} + iLA_{m+1}^{(r)} + \varepsilon LA_{m+2}^{(r)} + hLA_{m+3}^{(r)}] \\ &\quad + r + ri + r\varepsilon + rh \\ &= pLa\mathcal{H}_{m+1}^{(r)} + qLa\mathcal{H}_m^{(r)} + r\Psi, \end{aligned}$$

where  $\Psi = 1 + i + \varepsilon + h$  which completes the proof.  $\square$

**Theorem 2** (Binet formula of  $La\mathcal{H}_m^{(r)}$ ). *For  $m \geq 0$ , the Binet formula for generalized Leonardo-Alwyn hybrid numbers is*

$$La\mathcal{H}_m^{(r)} = \frac{1}{1-p-q} \left[ r\Psi + \left( \frac{\Phi_1\psi_1^m\Psi_1 - \Phi_2\psi_2^m\Psi_2}{\psi_1 - \psi_2} \right) \right], \quad (7)$$

where  $\Psi$  as in Eq. (6),  $\Psi_1 = 1 + \psi_1i + \psi_1^2\varepsilon + \psi_1^3h$  and  $\Psi_2 = 1 + \psi_2i + \psi_2^2\varepsilon + \psi_2^3h$ .

*Proof.* Using Eqs. (5) and (3), we can write the following expression

$$\begin{aligned} La\mathcal{H}_m^{(r)} &= LA_m^{(r)} + iLA_{m+1}^{(r)} + \varepsilon LA_{m+2}^{(r)} + hLA_{m+3}^{(r)} \\ &= \frac{1}{1-p-q} \left[ r + \left( \frac{\Phi_1\psi_1^m - \Phi_2\psi_2^m}{\psi_1 - \psi_2} \right) \right] \\ &\quad + \frac{1}{1-p-q} \left[ r + \left( \frac{\Phi_1\psi_1^{m+1} - \Phi_2\psi_2^{m+1}}{\psi_1 - \psi_2} \right) \right] i \\ &\quad + \frac{1}{1-p-q} \left[ r + \left( \frac{\Phi_1\psi_1^{m+2} - \Phi_2\psi_2^{m+2}}{\psi_1 - \psi_2} \right) \right] \varepsilon \\ &\quad + \frac{1}{1-p-q} \left[ r + \left( \frac{\Phi_1\psi_1^{m+3} - \Phi_2\psi_2^{m+3}}{\psi_1 - \psi_2} \right) \right] h \\ &= \frac{1}{1-p-q} \left[ r(1+i+\varepsilon+h) + \left( \frac{\Phi_1\psi_1^m\Psi_1 - \Phi_2\psi_2^m\Psi_2}{\psi_1 - \psi_2} \right) \right], \end{aligned}$$

where  $\Psi_1 = 1 + \psi_1i + \psi_1^2\varepsilon + \psi_1^3h$  and  $\Psi_2 = 1 + \psi_2i + \psi_2^2\varepsilon + \psi_2^3h$ . □

**Remark 3.** For  $a = b = 1$  and  $p = q = r = 1$ , we obtain  $\psi_1 = \frac{1+\sqrt{5}}{2}$ ,  $\psi_2 = \frac{1-\sqrt{5}}{2}$ ,  $\Phi_1 = -2\psi_1$  and  $\Phi_2 = -2\psi_2$ . Then, the Binet formula for the Leonardo hybrid number has the form

$$HLe_m = 2 \left( \frac{\psi_1^{m+1}\Psi_1 - \psi_2^{m+1}\Psi_2}{\psi_1 - \psi_2} \right) - (1 + i + \varepsilon + h).$$

**Proposition 1.** Let  $m \geq 0$  be an integer. Then, the character of the hybrid number  $La\mathcal{H}_m^{(r)}$  is given by

$$La\mathcal{H}_m^{(r)} \overline{La\mathcal{H}_m^{(r)}} = \frac{1}{\rho^2} \left\{ \begin{array}{l} 2r(1-q-pq)H_m - 2r(p^2+p+q)H_{m+1} \\ +(1-p^2q^2)H_m^2 + (1-2p-(p^2+q)^2)H_{m+1}^2 \\ -2q(1+pq+p^3)H_{m+1}H_m - r^2 \end{array} \right\}, \quad (8)$$

where  $\rho = 1 - p - q$  and  $H_m = \frac{\Phi_1\psi_1^m - \Phi_2\psi_2^m}{\psi_1 - \psi_2}$ .

*Proof.* By definition  $\mathfrak{C} \left( La\mathcal{H}_m^{(r)} \right) = La\mathcal{H}_m^{(r)} \overline{La\mathcal{H}_m^{(r)}}$  and Eq. (5), we have

$$\mathfrak{C} \left( La\mathcal{H}_m^{(r)} \right) = \left( LA_m^{(r)} \right)^2 + \left( LA_{m+1}^{(r)} - LA_{m+2}^{(r)} \right)^2 - \left( LA_{m+2}^{(r)} \right)^2 - \left( LA_{m+3}^{(r)} \right)^2.$$

By Eq. (2),  $\rho = 1 - p - q$  and some elementary calculations we obtain

$$\begin{aligned} \rho^2 \mathfrak{C} \left( La\mathcal{H}_m^{(r)} \right) &= \left[ r + \left( \frac{\Phi_1\psi_1^m - \Phi_2\psi_2^m}{\psi_1 - \psi_2} \right) \right]^2 + \left[ \frac{\Phi_1\psi_1^{m+1}(1-\psi_1) - \Phi_2\psi_2^{m+1}(1-\psi_2)}{\psi_1 - \psi_2} \right]^2 \\ &\quad - \left[ r + \left( \frac{\Phi_1\psi_1^{m+2} - \Phi_2\psi_2^{m+2}}{\psi_1 - \psi_2} \right) \right]^2 - \left[ r + \left( \frac{\Phi_1\psi_1^{m+3} - \Phi_2\psi_2^{m+3}}{\psi_1 - \psi_2} \right) \right]^2. \end{aligned}$$

Then, we have

$$\begin{aligned} \rho^2 \mathcal{C} \left( La\mathcal{H}_m^{(r)} \right) &= 2r(1 - q - pq)H_m - 2r(p^2 + p + q)H_{m+1} \\ &\quad + (1 - p^2q^2)H_m^2 + (1 - 2p - (p^2 + q)^2)H_{m+1}^2 \\ &\quad - 2q(1 + pq + p^3)H_{m+1}H_m - r^2, \end{aligned}$$

where  $H_m = \frac{\Phi_1\psi_1^m - \Phi_2\psi_2^m}{\psi_1 - \psi_2}$  and  $H_{m+2} = pH_{m+1} + qH_m$ . □

**Theorem 3.** *The generating function for the generalized Leonardo-Alwyn hybrid number sequence  $\{La\mathcal{H}_m^{(r)}\}_{m \geq 0}$  is*

$$\sum_{m=0}^{\infty} La\mathcal{H}_m^{(r)} t^m = \frac{\left\{ \begin{array}{l} La\mathcal{H}_0^{(r)} + \left( La\mathcal{H}_1^{(r)} - (1+p)La\mathcal{H}_0^{(r)} \right) t \\ + \left( La\mathcal{H}_2^{(r)} - (1+p)La\mathcal{H}_1^{(r)} - (q-p)La\mathcal{H}_0^{(r)} \right) t^2 \end{array} \right\}}{1 - (1+p)t - (q-p)t^2 + qt^3}. \quad (9)$$

*Proof.* Let  $g(t) = \sum_{m=0}^{\infty} La\mathcal{H}_m^{(r)} t^m$ . Then, we obtain

$$g(t) = La\mathcal{H}_0^{(r)} + La\mathcal{H}_1^{(r)}t + La\mathcal{H}_2^{(r)}t^2 + \dots. \quad (10)$$

Multiply Eq. (10) on both sides by  $-(1+p)t$ ,  $-(q-p)t^2$  and  $+qt^3$ . Then, we have

$$-(1+p)t g(t) = -(1+p)La\mathcal{H}_0^{(r)}t - (1+p)La\mathcal{H}_1^{(r)}t^2 - (1+p)La\mathcal{H}_2^{(r)}t^3 + \dots. \quad (11)$$

$$-(q-p)t^2 g(t) = -(q-p)La\mathcal{H}_0^{(r)}t^2 - (q-p)La\mathcal{H}_1^{(r)}t^3 - (q-p)La\mathcal{H}_2^{(r)}t^4 + \dots. \quad (12)$$

$$qt^3 g(t) = qLa\mathcal{H}_0^{(r)}t^3 + qLa\mathcal{H}_1^{(r)}t^4 + qLa\mathcal{H}_2^{(r)}t^5 + \dots. \quad (13)$$

By adding Eqs. (10), (11), (12) and (13), we have

$$g(t) = \frac{\left\{ \begin{array}{l} La\mathcal{H}_0^{(r)} + \left( La\mathcal{H}_1^{(r)} - (1+p)La\mathcal{H}_0^{(r)} \right) t \\ + \left( La\mathcal{H}_2^{(r)} - (1+p)La\mathcal{H}_1^{(r)} - (q-p)La\mathcal{H}_0^{(r)} \right) t^2 \end{array} \right\}}{1 - (1+p)t - (q-p)t^2 + qt^3}.$$

□

**Remark 4.** *For  $a = b = 1$  and  $p = q = r = 1$ . The generating function for the Leonardo hybrid numbers  $HLe_m$*

$$g(t) = \frac{1 + i + 3\varepsilon + 5h - (1 - i + \varepsilon + h)t + (1 - i - \varepsilon - 3h)t^2}{1 - 2t + t^3}$$

(see, e.g., [1]).

**Theorem 4.** *Let  $m \geq 0$  be an integer. Then*

$$\sum_{j=0}^m La\mathcal{H}_j^{(r)} = \frac{1}{\rho} [r\Psi(m + 2p + q)] + (1-p)La\mathcal{H}_0^{(r)} + La\mathcal{H}_1^{(r)} - La\mathcal{H}_{m+1}^{(r)} - (p+q)La\mathcal{H}_m^{(r)}.$$

*Proof.* By using Binet formula of the generalized Leonardo-Alwyn hybrid numbers  $La\mathcal{H}_m^{(r)} = \frac{1}{\rho} [r\Psi + H_m]$  and  $\rho = 1 - p - q$ , we find that

$$\begin{aligned} \sum_{j=0}^m La\mathcal{H}_j^{(r)} &= \frac{1}{\rho} \left[ r\Psi(m+1) + \sum_{j=0}^m H_j \right] \\ &= \frac{1}{\rho} [r\Psi(m+1) + (1-p)H_0 + H_1 - (p+q)H_m - H_{m+1}] \\ &= \frac{1}{\rho} [r\Psi(m+1) + (1-p)H_0 - (p+q)H_m] + La\mathcal{H}_1^{(r)} - La\mathcal{H}_{m+1}^{(r)}. \end{aligned}$$

By Eqs. (2), (5) and some elementary calculation, we obtain

$$\begin{aligned} \sum_{j=0}^m La\mathcal{H}_j^{(r)} &= \frac{1}{\rho} [r\Psi(m+p) - (p+q)H_m] + (1-p)La\mathcal{H}_0^{(r)} + La\mathcal{H}_1^{(r)} - La\mathcal{H}_{m+1}^{(r)} \\ &= \frac{1}{\rho} [r\Psi(m+2p+q)] + (1-p)La\mathcal{H}_0^{(r)} + La\mathcal{H}_1^{(r)} \\ &\quad - La\mathcal{H}_{m+1}^{(r)} - (p+q)La\mathcal{H}_m^{(r)} \end{aligned}$$

as desired. □

**Remark 5.** For  $a = b = 1$  and  $p = q = r = 1$ . The summation formula of the Leonardo hybrid numbers  $HLe_m$  is

$$\sum_{m=0}^n HLe_m = HLe_{n+2} - (n+2)\Psi - (2i + 4\varepsilon + 8h).$$

**Theorem 5.** The exponential generating function for the generalized Leonardo-Alwyn hybrid number  $La\mathcal{H}_m^{(r)}$  is

$$\sum_{m=0}^{\infty} La\mathcal{H}_m^{(r)} \frac{t^m}{m!} = \frac{1}{1-p-q} \left[ r\Psi e^t + \left( \frac{\Phi_1\Psi_1 e^{\psi_1 t} - \Phi_2\Psi_2 e^{\psi_2 t}}{\psi_1 - \psi_2} \right) \right].$$

*Proof.* By considering Theorem 2 and  $\rho = 1 - p - q$ , we have

$$\begin{aligned} \sum_{m=0}^{\infty} La\mathcal{H}_m^{(r)} \frac{t^m}{m!} &= \frac{1}{\rho} \left[ r\Psi \sum_{m=0}^{\infty} \frac{t^m}{m!} + \left( \frac{\Phi_1\Psi_1 \sum_{m=0}^{\infty} \frac{(\psi_1 t)^m}{m!} - \Phi_2\Psi_2 \sum_{m=0}^{\infty} \frac{(\psi_2 t)^m}{m!}}{\psi_1 - \psi_2} \right) \right] \\ &= \frac{1}{\rho} \left[ r\Psi e^t + \left( \frac{\Phi_1\Psi_1 e^{\psi_1 t} - \Phi_2\Psi_2 e^{\psi_2 t}}{\psi_1 - \psi_2} \right) \right]. \end{aligned}$$

□

For simplicity of notation, let

$$\mathcal{H}_n = \frac{\Phi_1 \psi_1^m \Psi_1 - \Phi_2 \psi_2^m \Psi_2}{\psi_1 - \psi_2}. \tag{14}$$

Then, if  $\rho = 1 - p - q$ , the Binet formula of the generalized Leonardo-Alwyn hybrid numbers is given by

$$La\mathcal{H}_n^{(r)} = \frac{1}{\rho} [r\Psi + \mathcal{H}_n]. \tag{15}$$

Note that  $\mathcal{H}_{n+2} = p\mathcal{H}_{n+1} + q\mathcal{H}_n$  and  $\psi_j^2 = p\psi_j + q$  for  $j = 1, 2$ .

The Vajda's identity for the sequence  $\{\mathcal{H}_n\}_{n \geq 0}$  and generalized Leonardo-Alwyn hybrid sequence  $\{La\mathcal{H}_n^{(r)}\}_{n \geq 0}$  are given in the next theorem.

**Theorem 6.** *Let  $n, u$  and  $v$  be integers such that  $n \geq 0, u \geq 0$  and  $n + v \geq 0$ . Then, we have*

$$\mathcal{H}_{n+u}\mathcal{H}_{n+v} - \mathcal{H}_n\mathcal{H}_{n+u+v} = \frac{1}{p^2 + 4q} (\Phi_1\Phi_2(-q)^n(\psi_1^u - \psi_2^u) [\psi_1^v\Psi_2\Psi_1 - \psi_2^v\Psi_1\Psi_2]), \tag{16}$$

$$\begin{aligned} La\mathcal{H}_{n+u}^{(r)}La\mathcal{H}_{n+v}^{(r)} - La\mathcal{H}_n^{(r)}La\mathcal{H}_{n+u+v}^{(r)} \\ = \frac{1}{\rho^2} \left\{ \frac{1}{p^2+4q}\Phi_1\Phi_2(-q)^n(\psi_1^u - \psi_2^u) [\psi_1^v\Psi_2\Psi_1 - \psi_2^v\Psi_1\Psi_2] \right. \\ \left. + r [\Psi\mathcal{K}_n(u) - \mathcal{K}_{n+v}(u)\Psi] \right\}, \end{aligned} \tag{17}$$

where  $\mathcal{K}_n(u) = \mathcal{H}_n - \mathcal{H}_{n+u}$ .

*Proof.* (16): Using Eq. (14),  $\Phi_1, \Phi_2, \Psi_1$  and  $\Psi_2$  as in Theorem 2, we have

$$\begin{aligned} (p^2 + 4q) [\mathcal{H}_{n+u}\mathcal{H}_{n+v} - \mathcal{H}_n\mathcal{H}_{n+u+v}] \\ = (\Phi_1\psi_1^{n+u}\Psi_1 - \Phi_2\psi_2^{n+u}\Psi_2) (\Phi_1\psi_1^{n+v}\Psi_1 - \Phi_2\psi_2^{n+v}\Psi_2) \\ - (\Phi_1\psi_1^n\Psi_1 - \Phi_2\psi_2^n\Psi_2) (\Phi_1\psi_1^{n+u+v}\Psi_1 - \Phi_2\psi_2^{n+u+v}\Psi_2) \\ = \Phi_1\Phi_2(-q)^n(\psi_1^u - \psi_2^u) [\psi_1^v\Psi_2\Psi_1 - \psi_2^v\Psi_1\Psi_2]. \end{aligned}$$

Note that  $\Psi_1\Psi_2 \neq \Psi_2\Psi_1$  in above equation.

(17): By formula (15) and Eq. (16), we get

$$\begin{aligned} \rho^2 [La\mathcal{H}_{n+u}^{(r)}La\mathcal{H}_{n+v}^{(r)} - La\mathcal{H}_n^{(r)}La\mathcal{H}_{n+u+v}^{(r)}] \\ = [r\Psi + \mathcal{H}_{n+u}] [r\Psi + \mathcal{H}_{n+v}] - [r\Psi + \mathcal{H}_n] [r\Psi + \mathcal{H}_{n+u+v}] \\ = \mathcal{H}_{n+u}\mathcal{H}_{n+v} - \mathcal{H}_n\mathcal{H}_{n+u+v} + r [\Psi\mathcal{K}_n(u) - \mathcal{K}_{n+v}(u)\Psi] \\ = \frac{1}{p^2+4q} (\Phi_1\Phi_2(-q)^n(\psi_1^u - \psi_2^u) [\psi_1^v\Psi_2\Psi_1 - \psi_2^v\Psi_1\Psi_2]) + r [\Psi\mathcal{K}_n(u) - \mathcal{K}_{n+v}(u)\Psi], \end{aligned}$$

where  $\mathcal{K}_n(u) = \mathcal{H}_n - \mathcal{H}_{n+u}$ . □

It is easily seen that for special values of  $u$  and  $v$  by Theorem 6, we get new identities for generalized Leonardo-Alwyn hybrid numbers:

- Catalan's identity:  $v = -u$  and  $n \geq u$ .
- Cassini's identity:  $u = 1, v = -1$  and  $n \geq 1$ .

- d'Ocagne's identity:  $u = 1$ ,  $v = m - n$  and  $m \geq n$ .

**Corollary 1.** *Catalan identity for generalized Leonardo-Alwyn hybrid numbers. Let  $n \geq 0$ ,  $u \geq 0$  be integers such that  $n \geq u$ . Then*

$$\begin{aligned} & La\mathcal{H}_{n+u}^{(r)} La\mathcal{H}_{n-u}^{(r)} - \left( La\mathcal{H}_n^{(r)} \right)^2 \\ &= \frac{1}{\rho^2} \left\{ \frac{1}{p^2+4q} \Phi_1 \Phi_2 (-q)^{n-u} (\psi_1^u - \psi_2^u) [\psi_2^u \Psi_2 \Psi_1 - \psi_1^u \Psi_1 \Psi_2] \right. \\ & \quad \left. + r [\Psi \mathcal{K}_n(u) - \mathcal{K}_{n-u}(u) \Psi] \right\}. \end{aligned} \quad (18)$$

**Corollary 2.** *Cassini identity for generalized Leonardo-Alwyn hybrid numbers. Let  $n \geq 1$  be an integer. Then*

$$\begin{aligned} & La\mathcal{H}_{n+1}^{(r)} La\mathcal{H}_{n-1}^{(r)} - \left( La\mathcal{H}_n^{(r)} \right)^2 \\ &= \frac{1}{\rho^2} \left\{ \frac{1}{p^2+4q} \Phi_1 \Phi_2 (-q)^{n-1} (\psi_1 - \psi_2) [\psi_2 \Psi_2 \Psi_1 - \psi_1 \Psi_1 \Psi_2] \right. \\ & \quad \left. + r [\Psi \mathcal{K}_n(1) - \mathcal{K}_{n-1}(1) \Psi] \right\}. \end{aligned} \quad (19)$$

**Corollary 3.** *d'Ocagne identity for generalized Leonardo-Alwyn hybrid numbers. Let  $n \geq 0$ ,  $m \geq 0$  be integers such that  $m \geq n$ . Then*

$$\begin{aligned} & La\mathcal{H}_{n+1}^{(r)} La\mathcal{H}_m^{(r)} - La\mathcal{H}_n^{(r)} La\mathcal{H}_{m+1}^{(r)} \\ &= \frac{1}{\rho^2} \left\{ \frac{1}{p^2+4q} \Phi_1 \Phi_2 (-q)^n (\psi_1 - \psi_2) [\psi_1^{m-n} \Psi_2 \Psi_1 - \psi_2^{m-n} \Psi_1 \Psi_2] \right. \\ & \quad \left. + r [\Psi \mathcal{K}_n(1) - \mathcal{K}_m(1) \Psi] \right\}. \end{aligned} \quad (20)$$

### 3 Matrix representation of generalized Leonardo-Alwyn hybrid numbers

Now, we will give the matrix representation of generalized Leonardo-Alwyn hybrid numbers. Also, we obtain a formula for generalized Leonardo-Alwyn hybrid number  $La\mathcal{H}_m^{(r)}$ , in terms of four-diagonal determinant, by using the same kind of approach that was used by Cereceda in [5].

**Theorem 7.** *Let  $m \geq 0$  be an integer. Then*

$$\begin{bmatrix} La\mathcal{H}_{m+3}^{(r)} \\ La\mathcal{H}_{m+2}^{(r)} \\ La\mathcal{H}_{m+1}^{(r)} \end{bmatrix} = \mathbf{Q}_{\mathcal{H}} \cdot \begin{bmatrix} La\mathcal{H}_{m+2}^{(r)} \\ La\mathcal{H}_{m+1}^{(r)} \\ La\mathcal{H}_m^{(r)} \end{bmatrix}, \quad (21)$$

where

$$\mathbf{Q}_{\mathcal{H}} = \begin{bmatrix} 1+p & q-p & -q \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

**Theorem 8.** *Let  $m \geq 0$  be an integer. Then*

$$\begin{bmatrix} La\mathcal{H}_{m+3}^{(r)} & La\mathcal{G}_{m+4}^{(r)} & -qLa\mathcal{H}_{m+2}^{(r)} \\ La\mathcal{H}_{m+2}^{(r)} & La\mathcal{G}_{m+3}^{(r)} & -qLa\mathcal{H}_{m+1}^{(r)} \\ La\mathcal{H}_{m+1}^{(r)} & La\mathcal{G}_{m+2}^{(r)} & -qLa\mathcal{H}_m^{(r)} \end{bmatrix} = \begin{bmatrix} La\mathcal{H}_3^{(r)} & La\mathcal{G}_4^{(r)} & -qLa\mathcal{H}_2^{(r)} \\ La\mathcal{H}_2^{(r)} & La\mathcal{G}_3^{(r)} & -qLa\mathcal{H}_1^{(r)} \\ La\mathcal{H}_1^{(r)} & La\mathcal{G}_2^{(r)} & -qLa\mathcal{H}_0^{(r)} \end{bmatrix} \cdot \mathbf{Q}_{\mathcal{H}}^m, \tag{22}$$

where  $La\mathcal{G}_{m+1}^{(r)} = La\mathcal{H}_{m+1}^{(r)} - (1 + p)La\mathcal{H}_m^{(r)}$ .

*Proof.* We use induction on  $m$ . If  $m = 0$  then the result is obvious. Assuming the formula (22) holds for  $m > 0$ , we shall prove it for  $m + 1$ . Using the induction's hypothesis and relation (1), we have

$$\begin{aligned} & \begin{bmatrix} La\mathcal{H}_3^{(r)} & La\mathcal{G}_4^{(r)} & -qLa\mathcal{H}_2^{(r)} \\ La\mathcal{H}_2^{(r)} & La\mathcal{G}_3^{(r)} & -qLa\mathcal{H}_1^{(r)} \\ La\mathcal{H}_1^{(r)} & La\mathcal{G}_2^{(r)} & -qLa\mathcal{H}_0^{(r)} \end{bmatrix} \cdot \mathbf{Q}_{\mathcal{H}}^{m+1} \\ &= \begin{bmatrix} La\mathcal{H}_3^{(r)} & La\mathcal{G}_4^{(r)} & -qLa\mathcal{H}_2^{(r)} \\ La\mathcal{H}_2^{(r)} & La\mathcal{G}_3^{(r)} & -qLa\mathcal{H}_1^{(r)} \\ La\mathcal{H}_1^{(r)} & La\mathcal{G}_2^{(r)} & -qLa\mathcal{H}_0^{(r)} \end{bmatrix} \cdot \mathbf{Q}_{\mathcal{H}}^m \cdot \begin{bmatrix} 1+p & q-p & -q \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} La\mathcal{H}_{m+3}^{(r)} & La\mathcal{G}_{m+4}^{(r)} & -qLa\mathcal{H}_{m+2}^{(r)} \\ La\mathcal{H}_{m+2}^{(r)} & La\mathcal{G}_{m+3}^{(r)} & -qLa\mathcal{H}_{m+1}^{(r)} \\ La\mathcal{H}_{m+1}^{(r)} & La\mathcal{G}_{m+2}^{(r)} & -qLa\mathcal{H}_m^{(r)} \end{bmatrix} \begin{bmatrix} 1+p & q-p & -p \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} La\mathcal{H}_{m+4}^{(r)} & (q-p)La\mathcal{G}_{m+3}^{(r)} - qLa\mathcal{G}_{m+2}^{(r)} & -qLa\mathcal{H}_{m+3}^{(r)} \\ La\mathcal{H}_{m+3}^{(r)} & (q-p)La\mathcal{G}_{m+2}^{(r)} - qLa\mathcal{G}_{m+1}^{(r)} & -qLa\mathcal{H}_{m+2}^{(r)} \\ La\mathcal{H}_{m+2}^{(r)} & (q-p)La\mathcal{G}_{m+1}^{(r)} - qLa\mathcal{G}_m^{(r)} & -qLa\mathcal{H}_{m+1}^{(r)} \end{bmatrix} \\ &= \begin{bmatrix} La\mathcal{H}_{m+4}^{(r)} & La\mathcal{G}_{m+5}^{(r)} & -qLa\mathcal{H}_{m+3}^{(r)} \\ La\mathcal{H}_{m+3}^{(r)} & La\mathcal{G}_{m+4}^{(r)} & -qLa\mathcal{H}_{m+2}^{(r)} \\ La\mathcal{H}_{m+2}^{(r)} & La\mathcal{G}_{m+3}^{(r)} & -qLa\mathcal{H}_{m+1}^{(r)} \end{bmatrix} \end{aligned}$$

which ends the proof. □

**Remark 6.** *For  $a = b = 1$  and  $p = q = r = 1$ , we have the matrix representation of hybrid Leonardo numbers  $HLe_m$  as follows*

$$\begin{bmatrix} HLe_{m+3} & -HLe_{m+1} & -HLe_{m+2} \\ HLe_{m+2} & -HLe_m & -HLe_{m+1} \\ HLe_{m+1} & -HLe_{m-1} & -HLe_m \end{bmatrix} = \begin{bmatrix} HLe_3 & -HLe_1 & -HLe_2 \\ HLe_2 & -HLe_0 & -HLe_1 \\ HLe_1 & -HLe_{-1} & -HLe_0 \end{bmatrix} \cdot \mathbf{Q}_{\mathcal{H}}^m,$$

where

$$\mathbf{Q}_{\mathcal{H}} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Now, we present another way to obtain the  $n$ -th term of the generalized Leonardo-Alwyn hybrid sequence as the computation of a four-diagonal matrix.

We shall adopt the ideas stated in [5, p. 277] using the following result which is stated in such work:

**Theorem 9.** *Let  $\{x_n\}_{n \geq 1}$  be any third-order linear sequence defined recursively by the following:*

$$x_{n+3} = ux_{n+2} + vx_{n+1} + wx_n, \quad n \geq 0,$$

with  $x_0 = A \neq 0$ ,  $x_1 = B$  and  $x_2 = C$ . Then, for all  $n \geq 0$ :

$$x_n = \begin{pmatrix} A & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ Au - B & u & \frac{1}{A} & 0 & \cdots & \cdots & 0 \\ 0 & Bu - C & u & w & \ddots & \cdots & 0 \\ 0 & A & -\frac{v}{w} & u & w & \ddots & 0 \\ 0 & 0 & \frac{1}{w} & -\frac{v}{w} & u & & \\ \vdots & \vdots & & \ddots & \ddots & \ddots & w \\ 0 & \cdots & & 0 & \frac{1}{w} & -\frac{v}{w} & u \end{pmatrix}_{(n+1) \times (n+1)} \quad (23)$$

In the case of the generalized Leonardo-Alwyn sequence and taking into account the recurrence relation (1), we have  $u = 1 + p$ ,  $v = q - p$ ,  $w = -q$ , and initial conditions  $La\mathcal{H}_0^{(r)}$ ,  $La\mathcal{H}_1^{(r)}$  and  $La\mathcal{H}_2^{(r)}$ . Then, by the use of the previous theorem, we have the following result which gives a different way to calculate the  $n$ -th term of this sequence:

**Proposition 2.** *For  $n \geq 0$ , we have*

$$La\mathcal{H}_n^{(r)} = \begin{pmatrix} La\mathcal{H}_0^{(r)} & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ Au - B & 1 + p & \left(La\mathcal{H}_0^{(r)}\right)^{-1} & 0 & \cdots & \cdots & 0 \\ 0 & Bu - C & 1 + p & -q & \ddots & \cdots & 0 \\ 0 & La\mathcal{H}_0^{(r)} & \frac{q-p}{q} & 1 + p & -q & \ddots & 0 \\ 0 & 0 & -\frac{1}{q} & \frac{q-p}{q} & 1 + p & & \\ \vdots & \vdots & & \ddots & \ddots & \ddots & -q \\ 0 & \cdots & & 0 & -\frac{1}{q} & \frac{q-p}{q} & 1 + p \end{pmatrix}_{(n+1) \times (n+1)}, \quad (24)$$

where  $Au - B = (1 + p)La\mathcal{H}_0^{(r)} - La\mathcal{H}_1^{(r)}$  and  $Bu - C = (1 + p)La\mathcal{H}_1^{(r)} - La\mathcal{H}_2^{(r)}$ .

## 4 Conclusions

In this paper, the sequence of generalized Leonardo-Alwyn hybrid numbers defined by a recurrence relation of third order was introduced. Some properties involving this sequence, including the Binet formula and the generating function, were presented. In the future, we intend to discuss the generalized Leonardo-Alwyn hybrid number with negative subscripts and the generalized Leonardo-Alwyn quaternion.

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