Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 6(68), No. 1 - 2026, 113-130
https://doi.org/10.31926/but.mif.2026.6.68.1.9

ON CAUCHY-RIEMANN STRUCTURE AND INTEGRABILITY CONDITIONS OF F-STRUCTURES SATISFYING $F^3 - F^2 + F = 0$

Fethi LATTI[1] **and Abderrahim ZAGANE**[*,2]

Abstract

In this paper, we present the F-structures satisfying $F^3 - F^2 + F = 0$ and then we study the Cauchy-Riemann structure and the relationship between it and the F-structure. We also discuss some problem of integrability, partial integrability and complete integrability for this structure. Finally, we also provide some examples of the F-structure.

2020 Mathematics Subject Classification: 53C15, 58A30, 32G07.
Key words: F-structures, Nijenhuis tensor, Cauchy-Riemann structure, integrability, partial integrability, complete integrability.

1 Introduction

The theory of F-structure satisfying $F^3 \pm F = 0$, where F is a nonzero tensor field of type $(1, 1)$ on a differentiable manifold, was developed by Yano [20, 21], Ishihara and Yano [8] , Nakagawa [12], Vohra [14], Matsumoto [11] and Baik [2]. There are papers that include other forms of structures that are generalizations of the F-structure, and their authors study some properties related to the structures such as the integration conditions of the F- structure, the CR-structures, the parallelism of the distributions, and the submanifolds of the F- manifold, and have been published in research papers, most notably: [4, 5, 6, 9, 19].There are also some recent studies related to the F-structure, such as [10, 13, 17, 18, 15, 16]. For this reason, the field of F-structures on a differentiable manifold remains a rich field of research in differential geometry, and is of particular interest to this day.

[1]Department of Mathematics, SALHI Ahmed Naama University Center, Naama, Algeria, e-mail: etafati@hotmail.fr

[2*]*Corresponding author*, Department of Mathematics, Ahmed Zabana University-Relizane, Laboratory of Analysis, Geometry and its Applications (LAGA), Relizane, Algeria, e-mail: zaganeabr2018@gmail.com

The main purpose of the present paper is to study the F-structure satisfying $F^3 - F^2 + F = 0$. After stating introduction, we study in section 2 some basic properties of operators l and m defined by F-structure. In section 3, we prove some properties of Nijenhuis tensor of F, l and m, while in Section 4 we study the Cauchy-Riemann structure and the relationship between the F structure. In section 5 we establish a necessary and sufficient conditions for integrability of the distributions induced by operators. In section 6, we investigate the Partial integrability and complete integrability conditions of the F-structure, while in last section, we provide some examples of the F-structure.

2 The F-structure satisfying $F^3 - F^2 + F = 0$

Let M^n be an n-dimensional manifold. A distribution D of dimension k on M is a subbundle of TM such that, for all point x of M, D_x is a k-dimensional subspace of T_xM. A vector field X on M is said to belong (tangent) to D if $X_x \in D_x$ for all $x \in M$. The set of vector field belong to D is also denoted by D. D is said to be involutive if $[X, Y]$ belongs to D for every vector fields X, Y belonging to D i.e. $[X, Y] \in D$, for every vector fields $X, Y \in D$. A submanifold N of M is called an integral manifold of D, if $T_xN = D_x$ for any point $x \in N$. We say the distribution D is integrable if through each point of M there exists an integral manifold of D. We need the classical theorem of Frobenius, which we formulate as follows ([3, p.197]). A distribution is integrable if and only if it is involutive (see [7] for more details).

Let M^n be an n-dimensional manifold and F be a nonzero $(1,1)$-tensor field on M of rank $rank(F) = r$ satisfying the polynomial equation:

$$F^3 - F^2 + F = 0, \tag{1}$$

such a structure on M is called an F-structure of rank r and of degree 3. If the rank of F, $rank(F) = r =$ constant, then M is called an F-structure manifold of degree 3.

We define two operators l and m on M respectively by

$$l = -F^3, \tag{2}$$
$$m = I + F^3, \tag{3}$$

where I denotes the identity operator on M, then we get

Lemma 1. *Let M be an F-structure manifold, then we have*

$$l + m = I, \tag{4}$$
$$l^2 = l, \tag{5}$$
$$m^2 = m, \tag{6}$$
$$Fl = lF = F, \tag{7}$$
$$Fm = mF = 0, \tag{8}$$
$$lm = ml = 0. \tag{9}$$

Proof. *i*) Combining (2) and (3) we get (4).

$$
\begin{aligned}
ii)\ l^2 &= (F^2 - F)^2 \\
&= F^4 - 2F^3 + F^2 \\
&= -F^3 + F^4 - F^3 + F^2 \\
&= l + F(F^3 - F^2 + F) \\
&= l.
\end{aligned}
$$

iii) $m^2 = (I - l)^2 = I - 2l + l^2 = I - 2l + l = I - l = m.$

iv) $Fl = -FF^3 = -F(F^2 - F) = -F^3 + F^2 = F.$

v) $Fm = F(I - l) = F - Fl = 0.$

vi) $lm = l(I - l) = l - l^2 = l - l = 0.$

□

Lemma 2. *Let M be an F-structure manifold, then we have*

$$
\begin{aligned}
F^{3/2} l &= l F^{3/2} = F^{3/2}, \qquad (10)\\
F^{3/2} m &= m F^{3/2} = 0. \qquad (11)
\end{aligned}
$$

Proof. In consequence of (7) and (8), we get (10) and (11) . □

Proposition 1. *Let M^n be an F-structure manifold, the following identities hold*

$$
\begin{aligned}
\operatorname{Im} l_x &= \ker m_x, \qquad (12)\\
\operatorname{Im} m_x &= \ker l_x, \qquad (13)\\
\operatorname{Im} l_x &= \operatorname{Im} F_x, \qquad (14)\\
\ker l_x &= \ker F_x, \qquad (15)\\
T_x M &= \operatorname{Im} l_x \oplus \operatorname{Im} m_x, \qquad (16)\\
\dim(\operatorname{Im} l_x) &= r\,,\ \dim(\operatorname{Im} m_x) \ = \ n - r, \qquad (17)\\
\ker l_x &= \ker F_x^{3/2}, \qquad (18)
\end{aligned}
$$

for all $x \in M$.

Proof. For all $x \in M$ and $X \in T_x M$.

(*i*) If $X \in \operatorname{Im} l_x$, There is $Z \in T_x M,\ X = lZ$, using (9), we have $mX = mlZ = 0$, then $X \in \ker m_x$.
Conversely, If $X \in \ker m_x$, so $mX = 0$, using (4), we have $lX = X$, and from it $X \in \operatorname{Im} l_x$. Therefore $\operatorname{Im} l_x = \ker m_x$.

(*ii*) The formula (13) is obtained by a proof similar to that of the formula (12).

(*iii*) If $X \in \operatorname{Im} l_x$, There is $Z \in T_xM,\ X = lZ$, using (2), we have
$X = -F^3Z = FY$, where $Y = -F^2Z \in T_xM$, then $X \in \operatorname{Im} F_x$.
Conversely, If $X \in \operatorname{Im} F_x$, There is $Z \in T_xM,\ X = FZ$, using (7), we have $X = lFZ = lY$, where $Y = FZ \in T_xM$, then $X \in \operatorname{Im} l_x$.

(*vi*) The formula (15) is obtained by a proof similar to that of the formula (14).

(*v*) By applying the well-known rank theorem in linear algebra on T_xM, we find $T_xM = \operatorname{Im} l_x \oplus \ker l_x$, using (13), we get $T_xM = \operatorname{Im} l_x \oplus \operatorname{Im} m_x$.

(*vi*) By (14) and (16), we find $\dim(\operatorname{Im} l_x) = \dim(\operatorname{Im} F_x) = rank(F) = r$ and $\dim(\operatorname{Im} m_x) = n - r$.

(*vii*) The formula (18) is obtained by a proof similar to that of the formula (14). □

Thus the operators l and m acting in the tangent space at each point of M are therefore complementary projection operators and there exist two complementary distributions $D_l = \operatorname{Im} l$ and $D_m = \operatorname{Im} m$ corresponding to the projection operators l and m respectively. From (17), the dimensions of D_l and D_m are r and $n - r$ respectively.

From (10), we find

$$(F^{3/2})^2 l = (F^{3/2})^2 = F^3 = -l,$$

it is clear that $F^{3/2}$ acts on D_l as an almost complex structure and on D_m as a null operator. Hence r must be even, say $r = 2k$.

3 Nijenhuis tensor

The Nijenhuis tensor N_F of F is expressed as follows

$$N_F(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y], \tag{19}$$

for any vector fields X and Y on M.

The integrability of F-structure is equivalent to the vanishing of the Nijenhuis tensor [7, 8, 22].

The Nijenhuis tensor N_F satisfies the following relations:

$$N_F(mX,mY) = F^2[mX,mY], \tag{20}$$
$$lN_F(mX,mY) = F^2[mX,mY], \tag{21}$$
$$mN_F(X,Y) = m[FX,FY], \tag{22}$$
$$mN_F(FX,FY) = m[F^2X,F^2Y], \tag{23}$$
$$mN_F(lX,lY) = m[FX,FY], \tag{24}$$

for any vector fields X and Y on M.

Proposition 2. *Let M be an F-structure manifold, we have the following equivalences*

$$mN_F(X,Y)=0 \;\Leftrightarrow\; mN_F(FX,FY)=0 \Leftrightarrow mN_F(lX,lY)=0, \tag{25}$$

for any vector fields X and Y on M.

Proof. The proof follows from (22), (23) and (24). □

Proposition 3. *Let M be an F-structure manifold, we have the following equivalence*

$$N_F(FX,FY)=0 \;\Leftrightarrow\; N_F(lX,lY)=0, \tag{26}$$

for any vector fields X and Y on M.

Proof. (i) Assume that $N_F(FX,FY)=0$, we replace X,Y with $-F^2X$, $-F^2Y$, respectively, we obtain $N_F(lX,lY)=0$.

(ii) Conversely, assume that $N_F(lX,lY)=0$, we replace X,Y with FX, FY, respectively, we obtain $N_F(FX,FY)=0$. □

Proposition 4. *Let M be an F-structure manifold. If F is integrable, then we have*

$$\begin{aligned}
(i)&\quad -F^2[FX,FY]+F[X,Y]=l\big([FX,Y]+[X,FY]\big),\\
(ii)&\quad [FX,FY]=l[FX,FY],\\
(iii)&\quad m[FX,FY]=0,
\end{aligned}$$

for any vector fields X and Y on M.

Proof. (i) Since $N_F(X,Y)=0$ we obtain

$$[FX,FY]+F^2[X,Y]=F\big([FX,Y]+[X,FY]\big),$$

operating on it by $-F^2$ we get

$$-F^2\big([FX,FY]+F^2[X,Y]\big)=-F^3\big([FX,Y]+[X,FY]\big).$$

Using (2) and (7), we find

$$-F^2[FX,FY]+F[X,Y]=l\big([FX,Y]+[X,FY]\big).$$

(ii) From (7), we find

$$N_F(X,Y)-lN_F(X,Y)=[FX,FY]-l[FX,FY], \tag{27}$$

since $N_F(X,Y)=0$ we obtain, $[FX,FY]=l[FX,FY]$.

(iii) Using (4), we find $m[FX,FY]=0$. □

Let N_l and N_m denote the Nijenhuis tensors corresponding to the operators l and m respectively, then

$$\begin{aligned} N_l(X,Y) &= [lX,lY]-l[lX,Y]-l[X,lY]+l[X,Y], \\ N_m(X,Y) &= [mX,mY]-m[mX,Y]-m[X,mY]+m[X,Y], \end{aligned}$$

for any vector fields X and Y on M.

Proposition 5. *Let M be an F-structure manifold, then we have*

$$N_l(X,Y)=N_m(X,Y)=m[lX,lY]+l[mX,mY], \tag{28}$$

for any vector fields X and Y on M.

Proof. Using (4), we have, $lX+mX=X$, then

$$\begin{aligned} N_l(X,Y) &= [lX,lY]-l[lX,lY+mY]-l[lX+mX,lY] \\ &\quad +l[lX+mX,lY+mY], \\ &= [lX,lY]-l[lX,lY]-l[lX,mY]-l[lX,lY]-l[mX,lY] \\ &\quad +l[lX,lY]+l[lX,mY]+l[mX,lY]+l[mX,mY] \\ &= [lX,lY]-l[lX,lY]+l[mX,mY] \\ &= m[lX,lY]+l[mX,mY]. \end{aligned}$$

$$\begin{aligned} N_m(X,Y) &= [mX,mY]-m[mX,lY+mY] \\ &\quad +m[lX+mX,lY+mY], \\ &= [mX,mY]-m[mX,lY]-m[mX,mY]-m[lX,mY] \\ &\quad -m[mX,mY]+m[lX,lY]+m[lX,mY]+m[mX,lY] \\ &\quad +m[mX,mY] \\ &= [mX,mY]-m[mX,mY]+m[lX,lY] \\ &= m[lX,lY]+l[mX,mY]. \end{aligned}$$

□

By virtue of Proposition 5, we get the following proposition.

Proposition 6. *Let M be an F-structure manifold, the both operators l and m are integrable if and only if*

$$N_l(X,Y)=0,$$

for any vector fields X and Y on M.

Proposition 7. *Let M be an F-structure manifold, the following identities hold*

$$\begin{aligned} N_l(lX, lY) &= m[lX, lY], & (29)\\ N_l(mX, mY) &= l[mX, mY], & (30)\\ N_l(X, Y) &= N_l(lX, lY) + N_l(mX, mY), & (31)\\ mN_F(X, Y) &= N_l(FX, FY), & (32)\\ N_l(lX, mY) &= 0,\\ N_l(mX, lY) &= 0, \end{aligned}$$

for any vector fields X and Y on M.

Proof. By virtue of (5), (6), (9) and (28), we get

$$\begin{aligned} (i) \quad & N_l(lX, lY) = m[l^2X, l^2Y] + l[mlX, mlY] = m[lX, lY],\\ (ii) \quad & N_l(mX, mY) = m[lmX, lmY] + l[m^2X, m^2Y] = l[mX, mY],\\ (iii) \quad & N_l(lX, mY) = m[l^2X, lmY] + l[mlX, m^2Y] = 0,\\ (iv) \quad & N_l(mX, lY) = m[lmX, l^2Y] + l[m^2X, mlY] = 0. \end{aligned}$$

(v) By virtue of (28), (29) and (30) we get (31).
(vi) In (29), replacing X, Y with FX, FY, respectively, we find

$$N_l(FX, FY) = m[FX, FY].$$

By (22), we obtain (32). □

Proposition 8. *Let M be an F-structure manifold, the following identity hold*

$$N_F(mX, mY) = F^2N_l(mX, mY),$$

for any vector fields X and Y on M.

Proof. By virtue of (30), we have $N_l(mX, mY) = l[mX, mY]$, operating on it by F^2, we find $F^2N_l(mX, mY) = F^2[mX, mY]$.
On the other hand by (20), we have $N_F(mX, mY) = F^2[mX, mY]$.
Hence $N_F(mX, mY) = F^2N_l(mX, mY)$. □

4 Cauchy-Riemann structure

Let $T^{\mathbb{C}}M$ denotes the complexified tangent bundle of differentiable manifold M defined by

$$T^{\mathbb{C}}M = \{X + jY : \ X, Y \in TM\} = TM \otimes_{\mathbb{R}} \mathbb{C},$$

where j is the imaginary unit.

A CR-structure on M is a complex subbundle H of $T^{\mathbb{C}}M$ such that H is involutive and $H \cap \overline{H} = \{0\}$, where $\overline{H}$ denotes the complex conjugate of H. In this case, we say M is a CR-manifold.

Let F-structure on M of rank $r = 2k$ satisfying the equation (1). We define complex subbundle H of $T^{\mathbb{C}}M$ by

$$H = \left\{ X - jF^{3/2}X,\ X \in D_l \right\}. \tag{33}$$

Then, we have

$$Real(H) = D_l \ \ and \ \ H \cap \overline{H} = \{0\}. \tag{34}$$

Indeed

$$\begin{aligned} Z \in H \cap \overline{H} &\Rightarrow Z = X - jF^{3/2}X = X + jF^{3/2}X,\ X \in D_l \\ &\Rightarrow F^{3/2}X = 0 \\ &\Rightarrow Z = X \in \ker F^{3/2}, \end{aligned}$$

from, (13), (16) and (18), we have $Z \in \ker F^{3/2} = \ker l = D_m$
$Z \in D_l \cap D_m = \{0\}$.

Lemma 3. *Let M be an F-structure manifold, the following identity hold*

$$[P, Q] = [X, Y] - [F^{3/2}X, F^{3/2}Y] - j([F^{3/2}X, Y] + [X, F^{3/2}Y]), \tag{35}$$

for any $P = X - jF^{3/2}X, Q = Y - jF^{3/2}Y \in H$, where $X, Y \in D_l$.

Proof.

$$\begin{aligned} [P, Q] &= [X - jF^{3/2}X, Y - jF^{3/2}Y] \\ &= [X, Y] + [X, -jF^{3/2}Y] + [-jF^{3/2}X, Y] + [-jF^{3/2}X, -jF^{3/2}Y]) \\ &= [X, Y] - [F^{3/2}X, F^{3/2}Y] - j([F^{3/2}X, Y] + [X, F^{3/2}Y]). \end{aligned}$$

□

Lemma 4. *Let M be an F-structure manifold, the following identity hold*

$$l([F^{3/2}X, Y] + [X, F^{3/2}Y]) = [F^{3/2}X, Y] + [X, F^{3/2}Y], \tag{36}$$

$$l[F^{3/2}X, F^{3/2}Y] = [F^{3/2}X, F^{3/2}Y], \tag{37}$$

for any $X, Y \in D_l$.

Proof.

$$\begin{aligned} l([F^{3/2}X, Y] + [X, F^{3/2}Y]) &= l\big(F^{3/2}X.Y - Y.F^{3/2}X + X.F^{3/2}Y - F^{3/2}Y.X\big) \\ &= lF^{3/2}X.Y - lY.F^{3/2}X + lX.F^{3/2}Y - lF^{3/2}Y.X \end{aligned}$$

As $X, Y \in D_l$, we have $lX = X, lY = Y$ and using (10), we get

$$\begin{aligned} l([F^{3/2}X, Y] + [X, F^{3/2}Y]) &= F^{3/2}X.Y - Y.F^{3/2}X + X.F^{3/2}Y - F^{3/2}Y.X \\ &= [F^{3/2}X, Y] + [X, F^{3/2}Y]. \end{aligned}$$

The formula (37) is obtained by a similar calculation. □

Theorem 1. *Let M be an F-structure manifold. If $F^{3/2}$ is integrable, then the complex subbundle H defined by* (33) *is a CR-structure on M.*

Proof. From (34), we have $Real(H) = D_l$ and $H \cap \overline{H} = \{0\}$. It remains to show that H is involutive, let $P = X - jF^{3/2}X,\ Q = Y - jF^{3/2}Y \in H$, such that $X, Y \in D_l$. Using using (35), (36) and (37), we get

$$[P, Q] = [X, Y] - [F^{3/2}X, F^{3/2}Y] - j([F^{3/2}X, Y] + [X, F^{3/2}Y]).$$

Since $F^{3/2}$ is integrable, then $N_{F^{3/2}}(X, Y) = 0$ i.e.

$$[F^{3/2}X, F^{3/2}Y] - [X, Y] = F^{3/2}\big([F^{3/2}X, Y] + [X, F^{3/2}Y]\big),$$

operating on it by $-F^{3/2}$ we get

$$-F^{3/2}\big([F^{3/2}X, F^{3/2}Y] - [X, Y]\big) = l[F^{3/2}X, Y] + [X, F^{3/2}Y],$$

hence,

$$F^{3/2}\big([X, Y] - [F^{3/2}X, F^{3/2}Y]\big) = [F^{3/2}X, Y] + [X, F^{3/2}Y],$$

from that we find,

$$[P, Q] = [X, Y] - [F^{3/2}X, F^{3/2}Y] - jF^{3/2}\big([X, Y] - [F^{3/2}X, F^{3/2}Y]\big) \in H.$$

□

5 Integrability conditions of distributions induced of F-structure

Theorem 2. *Let M be an F-structure manifold. The distribution D_l is integrable if and only if*

$$N_l(lX, lY) = 0, \tag{38}$$

for any vector fields X and Y on M.

Proof. The distribution D_l is integrable if and only if for any vector fields X and Y on M we have

$$[lX, lY] \in D_l.$$

By virtue of (12) and (29) we get,

$$[lX, lY] \in D_l \Leftrightarrow m[lX, lY] = 0 \Leftrightarrow N_l(lX, lY) = 0.$$

□

Theorem 3. *Let M be an F-structure manifold. The distribution D_m is integrable if and only if*

$$N_l(mX, mY) = 0, \tag{39}$$

for any vector fields X and Y on M.

Proof. The distribution D_m is integrable if and only if for any vector fields X and Y on M we have

$$[mX, mY] \in D_m.$$

By virtue of (13) and (30) we get,

$$[mX, mY] \in D_m \quad \Leftrightarrow \quad l[mX, mY] = 0 \Leftrightarrow N_l(mX, mY) = 0.$$

□

Theorem 4. *Let M be an F-structure manifold. The distributions D_l and D_m are both integrable if and only if*

$$N_l(X, Y) = 0,$$

for any vector fields X and Y on M.

Proof. (i) Suppose that D_l and D_m are both integrable. It follows from (38) and (39)

$$N_l(lX, lY) = 0, \quad N_l(mX, mY) = 0.$$

By virtue of (31) we have,

$$N_l(X, Y) = N_l(lX, lY) + N_l(mX, mY) = 0.$$

(ii) Conversely, assume that $N_l(X, Y) = 0$. It follows from (31) that

$$N_l(lX, lY) + N_l(mX, mY) = 0.$$

We replace in him X, Y by lX, lY (resp. by mX, mY), we get

$$N_l(lX, lY) = 0, (\textit{resp. } N_l(mX, mY) = 0.$$

Then, D_l and D_m are both integrable. □

Theorem 5. *Let M be an F-structure manifold. The distribution D_l is integrable if and only if*

$$N_F(X, Y) = lN_F(X, Y), \tag{40}$$

for any vector fields X and Y on M.

Proof. Suppose that, D_l is integrable, then for any vector fields X and Y on M we have

$$[lX, lY] \in D_l.$$

Using (4) and (12), we get

$$\begin{aligned} [lX, lY] \in D_l &\Leftrightarrow m[lX, lY] = 0 \\ &\Leftrightarrow [lX, lY] - l[lX, lY] = 0. \end{aligned} \tag{41}$$

In the last equation we replace X, Y with FX, FY, respectively and using (7) we obtain

$$[FX, FY] - l[FX, FY] = 0. \tag{42}$$

Using (27), we find $N_F(X, Y) = lN_F(X, Y)$.
Conversely, suppose that $N_F(X, Y) = lN_F(X, Y)$, then from (27) we obtain (42). Replacing X and Y with $-F^2X$ and $-F^2Y$, respectively and using (2) we obtain (41), which implies $[lX, lY] \in D_l$ i.e D_l is integrable. □

Theorem 6. *Let M be an F-structure manifold, the following conditions are equivalent*

$$\begin{aligned} (i) &\quad D_l \text{ is integrable}, \\ (ii) &\quad N_l(lX, lY) = 0, \\ (iii) &\quad N_F(X, Y) = lN_F(X, Y), \\ (iv) &\quad mN_F(X, Y) = 0, \\ (v) &\quad mN_F(FX, FY) = 0, \\ (vi) &\quad mN_F(lX, lY) = 0, \\ (vii) &\quad N_l(FX, FY) = 0, \end{aligned}$$

for any vector fields X and Y on M.

Proof.

(1) From Theorem 2, we have $(i) \Leftrightarrow (ii)$.

(2) From Theorem 5, we have $(i) \Leftrightarrow (iii)$, hence $(ii) \Leftrightarrow (iii)$.

(3) From (4), we get $(iii) \Leftrightarrow (iv)$.

(4) From (25), we get $(iv) \Leftrightarrow (v) \Leftrightarrow (vi)$.

(5) From (32), we get $(iv) \Leftrightarrow (vii)$, hence $(vi) \Leftrightarrow (vii)$.

(6) Suppose that $N_l(FX, FY) = 0$. Comparing it with (22) and (32), we obtain $m[FX, FY] = 0$. In this, we replace X by $-F^2X$ and Y by $-F^2Y$, respectively, we obtain

$$m[lX, lY] = 0 \quad \Leftrightarrow \quad [lX, lY] \in D_l.$$

Hence, $(vii) \Leftrightarrow (i)$. □

Theorem 7. *Let M be an F-structure manifold. The distribution D_m is integrable if and only if*

$$N_F(mX, mY) = 0, \tag{43}$$

for any vector fields X and Y on M.

Proof. The distribution D_m is integrable if and only if $[mX, mY] \in D_m$ for any vector fields X and Y on M. Using (13), we get

$$\begin{aligned} [mX, mY] \in D_m &\Rightarrow l[mX, mY] = 0 \\ &\Rightarrow F^2 l[mX, mY] = 0 \\ &\Rightarrow F^2[mX, mY] = 0. \end{aligned}$$

From (20), we get $N_F(mX, mY) = 0$.
Conversely, assume that $N_F(mX, mY) = 0$, from (20), we find $F^2[mX, mY] = 0$. Operating on it by $-F$ we get $l[mX, mY] = 0$, i.e. $[mX, mY] \in D_m$, hence the distribution D_m is integrable.

□

By virtue of Theorem 3, Theorem 7 and Proposition 8, we get the following Theorem.

Theorem 8. *Let M be an F-structure manifold, the following conditions are equivalent*

$$\begin{aligned} (i) &\quad D_m \text{ is integrable}, \\ (ii) &\quad N_l(mX, mY) = 0, \\ (iii) &\quad N_F(mX, mY) = 0, \\ (iv) &\quad lN_F(mX, mY) = 0, \\ (v) &\quad F^2 N_l(mX, mY) = 0, \end{aligned}$$

for any vector fields X and Y on M.

From Theorem 5 and Theorem 7 we deduce:

Corollary 1. *Let M be an F-structure manifold. If F is an integrable structure, then both distributions D_l and D_m are integrable.*

Remark 1. *Let M be an F-structure manifold. If both distributions D_l and D_m are integrable, F not necessary integrable. see (Example 3).*

Theorem 9. *Let M be an F-structure manifold. The distributions D_l and D_m are both integrable if and only if*

$$N_F(X, Y) = lN_F(lX, lY) + N_F(lX, mY) + N_F(mX, lY), \tag{44}$$

for any vector fields X and Y on M.

Proof. i) Suppose that D_l and D_m are both integrable. Using (4), we get

$$\begin{aligned} N_F(X,Y) &= N_F(lX+mX, lY+mY) \\ &= N_F(lX,lY)+N_F(lX,mY)+N_F(mX,lY)+N_F(mX,mY). \quad (45) \end{aligned}$$

Then from (40) and (43), we have

$$N_F(X,Y) = lN_F(X,Y) \text{ and } N_F(mX,mY) = 0.$$

By virtue of (45), we get (44).
ii) Conversely, assume that (44) is satisfied. Using (45), we find

$$lN_F(lX,lY) = N_F(lX,lY)+N_F(mX,mY).$$

In this relation we replace X,Y with mX, mY respectively, we get $N_F(mX,mY) = 0$, as well $lN_F(lX,lY) = N_F(lX,lY)$. i.e. D_l and D_m are both integrable. □

By virtue of Proposition 6, Theorem 4 and Theorem 9, we get the following Theorem.

Theorem 10. *Let M be an F-structure manifold, the following conditions are equivalent*

(i) *l and m are integrable,*

(ii) *D_l and D_m are integrable,*

(iii) $N_l(X,Y) = 0$,

(iv) $N_F(X,Y) = lN_F(lX,lY)+N_F(lX,mY)+N_F(mX,lY)$,

for any vector fields X and Y on M.

6 Partial integrability and complete integrability of F-structure

Suppose that the distribution D_l is integrable and take an arbitrary vector field U in an integral manifold of D_l. We define an operator $\widetilde{F}$ by

$$\widetilde{F}U = FU,$$

then $\widetilde{F}$ leaves invariant tangent spaces of every integral manifolds of D_l. Also, $(\widetilde{F})^{3/2}$ acts as an almost product structure on each integral manifold of D_l.

For any vector fields U and V tangent to integral manifold of D_l, we denote by

$$N_{\widetilde{F}}(U,V) = [\widetilde{F}U,\widetilde{F}V] - \widetilde{F}[\widetilde{F}U,V] - \widetilde{F}[U,\widetilde{F}V] + (\widetilde{F})^2[U,V],$$

the Nijenhuis tensor of the structure $\widetilde{F}$ induced on each integral manifold of D_l from the structure F. Then we have

$$N_{\widetilde{F}}(lX,lY) = N_F(lX,lY), \quad (46)$$

for any vector fields X and Y on M.

Definition 1. *[22] We call an F-structure to be partially integrable if the distribution D_l is integrable and the structure $\widetilde{F}$ induced from F on each integral manifold of D_l is integrable. see[7, 14].*

Theorem 11. *Let M be an F-structure manifold. A necessary and sufficient condition for an F-structure to be partially integrable is that one of the following equivalent conditions be satisfied:*

$$N_F(lX, lY) = 0, \tag{47}$$

or

$$N_F(FX, FY) = 0,$$

for any vector fields X and Y on M.

Proof. Suppose that F-structure is partially integrable, then from (26) and (46), we find $N_{\widetilde{F}}(lX, lY) = 0 \Leftrightarrow N_F(lX, lY) = 0 \Leftrightarrow N_F(FX, FY) = 0$.
Conversely, from (26), we have $N_F(lX, lY) = 0 \Leftrightarrow N_F(FX, FY) = 0$, then by (46), the structure $\widetilde{F}$ is integrable. Also $N_F(lX, lY) = 0$, implies $mN_F(lX, lY) = 0$, by Theorem 6, we find, D_l is integrable. Thus F-struoture is partially integrable. □

Definition 2. *[1] Let M be an F-structure manifold. An F-structure is said to be completely integrable if the distribution D_l and D_m are both integrable, and the structure $\widetilde{F}$ induced from F on each integral manifold of D_l is integrable.*

From Definition 1 and Definition 2, we have the following theorem.

Theorem 12. *Let M be an F-structure manifold. A necessary and sufficient condition for an F-structure to be completely integrable is that the distribution D_m is integrable and that the F-structure is partially integrable.*

Theorem 13. *Let M be an F-structure manifold. In order that the F-structure to be completely integrable, it is necessary and sufficient that*

$$N_F(X, Y) = N_F(lX, mY) + N_F(mX, lY). \tag{48}$$

for any vector fields X and Y on M.

Proof. i) Suppose that the F-structure is a completely integrable, i.e. D_m is integrable and F-structure is partially integrable. Using (43), (45) and (47), we get (48).
ii) Conversely, assume that (48) is satisfied. Using (45), we find

$$N_F(lX, lY) + N_F(mX, mY) = 0.$$

In this relation we replace X, Y with mX, mY respectively, we get (43), as well (47), i.e. D_m is integrable and F-structure is partially integrable, hence the F-structureis is completely integrable. □

Theorem 14. *Let M be an F-structure manifold. In order that the F-structure to be integrable, it is necessary and sufficient that the F-structure is completely integrable and*

$$N_F(lX, mY) = -N_F(mX, lY),$$

for any vector fields X and Y on M.

7 Examples

Example 1. *Let (z_1, z_2) be the Cartesian coordinates of $\mathbb{C}^2$ and F be a tensor of type $(1,1)$, defined by*

$$F = \begin{pmatrix} 1 & j \\ j & 0 \end{pmatrix}.$$

It is easy to find out that rank$(F) = 2$ and $F^3 - F^2 + F = 0$. Then we have

$$l = -F^3 = I, \quad m = I - l = 0,$$

$$(D_l)_z = T_z\mathbb{C}^2, \quad (D_m)_z = \{0\},$$

where $z = (z_1, z_2) \in \mathbb{C}^2$, it's easily verified that

$$N_F(\partial_{z_k}, \partial_{z_h}) = [F\partial_{z_k}, F\partial_{z_h}] - F[F\partial_{z_k}, \partial_{z_h}] - F[\partial_{z_k}, F\partial_{z_h}] + F^2[\partial_{z_k}, \partial_{z_h}] = 0,$$

for all $k, h = 1, 2$, i.e. F is integrable (partially, completely) integrable, as well D_l and D_m are integrable.

Example 2. *Let (x_1, x_2) be the Cartesian coordinates of $\mathbb{R}^2$ and F be a tensor of type $(1,1)$, defined by*

$$F = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to find out that rank$(F) = 2$ and $F^3 - F^2 + F = 0$. Then we have

$$l = -F^3 = I, \quad m = I - l = 0,$$

$$(D_l)_x = T_x\mathbb{R}^2, \quad (D_m)_x = \{0\},$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, it's easily verified that

$$N_F(\partial_{x_k}, \partial_{x_h}) = [F\partial_{x_k}, F\partial_{x_h}] - F[F\partial_{x_k}, \partial_{x_h}] - F[\partial_{x_k}, F\partial_{x_h}] + F^2[\partial_{x_k}, \partial_{x_h}] = 0,$$

for all $k, h = 1, 2$, i.e. F is integrable (partially, completely) integrable, as well D_l and D_m are integrable.

We put $G^2 = F^3 = -I$, i.e. $G = F^{3/2} = (-I)^{1/2}$, then for $G = \begin{pmatrix} -a & -b \\ \dfrac{1+a^2}{b} & a \end{pmatrix}$, *we get* $G^2 = F^3 = -I$ *where* a, b *are real constants and* $b \neq 0$. *Because* $F^{3/2}$ *is integrable, then*

$$\begin{aligned} H &= \left\{X - jF^{3/2}X,\ X \in D_l\right\} = \left\{X - j\begin{pmatrix} -a & -b \\ \frac{1+a^2}{b} & a \end{pmatrix} X,\ X \in T\mathbb{R}^2\right\} \\ &= \left\{\begin{pmatrix} x + j(ax+by) \\ y - j(\frac{1+a^2}{b}x + ay) \end{pmatrix},\ x, y \in \mathbb{R}\right\} \end{aligned}$$

is CR*-structure.*

Example 3. *On* $M = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4,\ x_1 \neq 0\}$ *(4-dimensional manifold), we define the tensor* F *of type* $(1,1)$*, by*

$$F = \begin{pmatrix} 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & \frac{1}{x_1} \\ -\frac{1}{x_1} & 0 & 1 & 0 \\ 0 & -x_1 & 0 & 1 \end{pmatrix}.$$

It is easy to find out that $rank(F) = 4$ *and* $F^3 - F^2 + F = 0$. *Then we have*

$$l = -F^3 = I, \quad m = I - l = 0,$$

$$(D_l)_x = T_xM, \quad (D_m)_x = \{0\}.$$

For all vector fields X *and* Y *on* M*, we have,* $mN_F(X,Y) = lN_F(mX, mY) = 0$*, i.e.* D_l *and* D_m *are both integrable.*

$$\begin{aligned} N_F(\partial_{x_1}, \partial_{x_2}) &= [F\partial_{x_1}, F\partial_{x_2}] - F[F\partial_{x_1}, \partial_{x_2}] - F[\partial_{x_1}, F\partial_{x_2}] + F^2[\partial_{x_1}, \partial_{x_2}] \\ &= [\frac{-1}{x_1}\partial_{x_3}, -x_1\partial_{x_4}] - F[\frac{-1}{x_1}\partial_{x_3}, \partial_{x_2}] - F[\partial_{x_1}, -x_1\partial_{x_4}] + 0 \\ &= \frac{1}{x_1}\partial_{x_2} + \partial_{x_4} \neq 0. \end{aligned}$$

Hence F *is not integrable. We also have*
$N_F(l\partial_{x_1}, l\partial_{x_2}) = N_F(\partial_{x_1}, \partial_{x_2}) \neq 0$*, then* F *is not partially (completely) integrable.*

References

[1] AI-Aqeel, A., *Integrability conditions of a structure satisfying* $F^5 - F = 0$, Arab Gulf J. Sci. Res. **6** (1988), no. 2, 163-171.

[2] Baik, Y. B., *A certain polynomial structure*, J. Korean Math. Soc. **16** (1980), no. 2, 167-175.

[3] Brickell, F. and Clark, R. S., *Differentiable manifolds*, Van Nostrand Reinhold Co, 1970.

[4] Das, L. S., *Submanifolds of F-structure satisfying $F^K + (-)^{K+1}F = 0$*, Int. J. Math. Math. Sci. **26** (2001), no. 3, 167-172.

[5] Das, L. S., *On CR-structures and F-structure satisfying $F^K + (-)^{K+1}F = 0$*, Rocky Mountain J. Math. **36** (2006), no. 3, 885-892.

[6] Das, L. S., Nikíc, J. and Nivas, R., *Parallelism of distributions and geodesics on $F(a_1, a_2, \ldots, a_n)$-structure Lagrangian manifolds*, Differ. Geom. Dyn. Syst. **8** (2006), 82-89.

[7] De León, M. and Rodrigues, P. R., *Methods of differential geometry in analytical mechanics*, North-Holland Mathematics Studies, 1989.

[8] Ishihara, S. and Yano, K., *On integrability conditions of a structure F satisfying $F^3 + F = 0$*, Q. J. Math. **15** (1964), no. 2, 217-222.

[9] Kankarej, M. M. and Srivastava, S. K., *On $a_nF^n + a_{n-1}F^{n-1} + \ldots + a_2F^2 + a_1F = 0$ structure manifolds and its integrability condition*, J. Tensor Soc. India **22** (2004), 61-80.

[10] Khana, M. N. I., *On Cauchy-Riemann structures and the general even order structure*, J. Sci. Arts **53** (2020), no. 4, 801-808.

[11] Matsumoto, K., *On a structure defined by a tensor field f of type $(1,1)$ satisfying $F^3 - F = 0$*, Bull. Yamagata Univ. **1** (1976), 33-47.

[12] Nakagawa, H., *On framed f-structure induced on submanifolds in space, almost Hermitian or Kählerian*, Kôdai Math. Semin. Rep. **18** (1966), 161-183.

[13] Singh, A., Pandey, R. K. and Khare, S., *Parallelism of distributions and geodesics on $F(2K + S, S)$-structure Lagrangian manifolds*, Int. J. Contemp. Math. Sci. **9** (2014), no. 11, 514-522.

[14] Singh, K. D. and Vohra, R. K., *Integrability conditions of $(1,1)$ tensor field F satisfying $F^3 - F = 0$*, Demonstr. Math. **7** (1974), no. 1, 85-92.

[15] Singh, L., *On integrability of F-structure satisfying $F^{2K+1} + F = 0$*, IJES **5** (2016), no. 9, 64-66.

[16] Singh, L., *On the structure equation $F^6 + F^4 + F^2 = 0$*, Int. J. Appl. Res. **8** (2021), no. 1, 422-423.

[17] Singh, L. and Gautam, S. K., *On CR-structure and F-structure satisfying $F^{p_1p_2} + F = 0$*, Int. J. Eng. Appl. Sci. **2** (2015), no. 10, 1-2.

[18] Singh, L. and Gautam, S. K., *On CR-structure and F-structure satisfying $F^{p^2+2} + F = 0$*, Int. J. Res. Math. Comput. **3** (2015), no. 3, 15-18.

[19] Upadhyay, M. D. and Gupta, V. C., *Integrability conditions of a structure* F_c *satisfying* $F^3 + c^2F = 0$, Publ. Math. **24** (1977), no. 3-4, 249-255.

[20] Yano, K., *On a structure* f *satisfying* $f^3 + f = 0$, Tech. Rep. Univ. Washington, No. 12, June 20, 1961.

[21] Yano, K., *On a structure defined by a tensor field* f *of type* $(1, 1)$ *satisfying* $F^3 + F = 0$, Tensor (N.S.) **14** (1963), 99-109.

[22] Yano, K. and Kon, M., *Structures on manifolds*, Series in Pure Math., vol. 3, World Scientific, Singapore, 1984.