Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 6(68), No. 1 - 2026, 99-112
https://doi.org/10.31926/but.mif.2026.6.68.1.8

STARLIKE FUNCTIONS ASSOCIATED WITH Q-MULTIPLICATIVE DERIVATIVE

Kadhavoor R. KARTHIKEYAN[*,1]

Abstract

The rate of change of a function in the sense of multiplicative complex derivative expressed in terms of quantum derivative have not be used to study the various subclasses of univalent functions. Here we attempt to define a new family of analytic functions by replacing the ordinary derivative in the analytic characterization of the starlike function with a quantity influenced by the definition of Q-multiplicative derivative. We explore the properties of a class of functions involving such a restrictive calculus. We obtain the estimates for the initial coefficients and Fekete-Szegő inequalities of the defined function classes. Also, we have obtained the coefficient estimates for the inverse function.

2000 Mathematics Subject Classification: 30C45.
Key words: univalent function, multiplicative calculus, quantum calculus, convex function, starlike function, subordination.

1 Introduction and definition

Let $\mathbb{U}$, $\mathbb{R}$, $\mathcal{C}$ and $\mathcal{N}$ denote the unit disc, set of real numbers, set of complex numbers and set of natural numbers respectively. For $q \in \mathbb{C}$, a q-natural number $[k]_q$ is defined by

$$[k]_q = \sum_{\nu=1}^{k} q^{\nu-1}, \qquad [0]_q = 0, \quad (q \in \mathbb{C})$$

and

$$[k]_q! = \begin{cases} 1, & \text{if } k = 0, \\ [k]_q[k-1]_q[k-2]_q \dots [1]_q, & \text{if } k = 1, 2, 3, \dots. \end{cases}$$

[1*] *Corresponding author*, Department of Applied Mathematics and Science, College of Engineering, National University of Science & Technology, Muscat P.O. Box 620, Oman, e-mail: karthikeyan@nu.edu.om

The study of Univalent function theory involving quantum calculus was first introduced and studied by Srivastava in [25]. But the much needed attention to this dulaity theory was garnered after Ismail et al. [11] introduced the class of q-starlike functions. For the studies pertaining this duality theory, refer to the recent survey-cum-expository article of Srivastava [26].

The q-difference operator for a function $\chi \in \mathcal{A}$ is defined by

$$\Delta_q \chi(\omega) := \begin{cases} \chi'(0), & \text{if } z = 0, \\ \dfrac{\chi(\omega) - \chi(q\omega)}{(1-q)\omega}, & \text{if } \omega \neq 0. \end{cases} \tag{1}$$

From (1), if $\chi \in \mathcal{A}$ we can easily see that $\Delta_q \chi(\omega) = 1 + \sum_{\nu=2}^{\infty} [\nu]_q a_\nu \omega^{\nu-1}$, for $\omega \neq 0$ and note that $\lim_{q \to 1^-} \Delta_q \chi(\omega) = \chi'(\omega)$. The *q-Jackson integral* is defined by (see [12])

$$I_q [\chi(\omega)] := \int_0^\omega \chi(t) d_q t = \omega(1-q) \sum_{\nu=0}^{\infty} q^\nu \chi(\omega q^\nu) \tag{2}$$

provided the series converges. Note that

$$\Delta_q I_q \chi(\omega) = \chi(\omega) \qquad \text{and} \qquad I_q \Delta_q \chi(\omega) = \chi(\omega) - \chi(0),$$

provided χ is continuous at $\omega = 0$. A q-analogue of the exponential function e^ω is

$$e_q^\omega = \sum_{\nu=0}^{\infty} \frac{\omega^\nu}{[\nu]_q!}.$$

Bashirov, Kurpinar and Özyapıin [4] (also see [5, 6, 24]) studied the properties of *Multiplicative calculus* which is not adaptable to many application problems when compared with the calculus due to Newton and Euler. But it has been a useful mathematical tool in economics and finance. The multiplicative derivative denoted by χ^* whose definition is given explicitly by

$$\chi^*(x) = \lim_{h \to 0} \left(\frac{\chi(x+h)}{\chi(x)} \right)^{\frac{1}{h}} = e^{\frac{\chi'(x)}{\chi(x)}} = e^{[\ln \chi(x)]'} \quad (x \in \mathbb{R})$$

where $\chi'(x)$ is the ordinary derivative. The $*$-derivative of χ at ω belonging to a small neighbourhood of a domain in a complex plane where χ is non-vanishing differentiable, is given by

$$\chi^*(\omega) = e^{\chi'(\omega)/\chi(\omega)} \tag{3}$$

and $\chi^{*(n)}(\omega) = e^{[\chi'(\omega)/\chi(\omega)]^{(n)}}$, $n = 1, 2, \ldots$. The q-analogue of the multiplicative derivative of a real valued function $\chi : \mathbb{R} \longrightarrow \mathbb{R}$ is given by

$$\Delta_q^* \chi(x) = \left(\frac{\chi(qx)}{\chi(x)} \right)^{\frac{1}{(q-1)x}} = e_q^{\Delta_q [\ln \chi(x)]} \tag{4}$$

where e_q is the q-analogue of the exponential function and Δ_q is the quantum derivative.

Let $\mathcal{S}$ denote the class of functions $\chi \in \mathcal{A}$ which are univalent in $\mathbb{U}$. We call $\mathcal{P}$ to denote the class of functions with normalization $\eta(0) = 1$ which satisfies $\operatorname{Re}(\eta(\omega)) > 0$, $\omega \in \mathbb{U}$. The well-known subclasses of $\mathcal{S}$ namely starlike and convex functions have the following differential characterizations respectively

$$\frac{\omega\chi'(\omega)}{\chi(\omega)} \in \mathcal{P} \quad \text{and} \quad 1 + \frac{\omega\chi''(\omega)}{\chi'(\omega)} \in \mathcal{P}.$$

We denote the class of starlike and convex functions by $\mathcal{S}^*$ and $\mathcal{C}$ respectively. Ma-Minda [17]introduced and studied the following subclasses of using subordination of analytic functions:

$$\mathcal{S}^*(\varphi) := \left\{\chi \in \mathcal{A} : \frac{\omega\chi'(\omega)}{\chi(\omega)} \prec \varphi(\omega)\right\}$$

and

$$\mathcal{C}(\varphi) := \left\{\chi \in \mathcal{A} : \left(1 + \frac{\omega\chi''(\omega)}{\chi'(\omega)}\right) \prec \varphi(\omega)\right\},$$

where $\varphi(\omega) \in \mathcal{P}$ is starlike symmetric with respect to horizontal axis and is of the form

$$\varphi(\omega) = 1 + \Lambda_1\omega + \Lambda_2\omega^2 + \Lambda_3\omega^3 + \cdots, \qquad (\Lambda_1 > 0;\ \omega \in \mathbb{U}). \tag{5}$$

Letting the superordinate function φ to map unit disc on to some specific conic regions, lots of subclasses of $\mathcal{S}$ can be obtained as special cases. Refer to [3, 9, 7, 8, 16, 18, 21, 20, 23] for exhaustive studies pertaining to subclasses of $\mathcal{A}$ associated with conic region.

Given the fact that most of the geometrically defined subclasses of $\mathcal{S}$ satisfy certain differential inequality, we tried to replace the ordinary derivative with a multiplicative derivative. But multiplicative derivative exist only if the function is non-zero in the chosen domain. So we will only use the idea and motivation behind the definition of a multiplicative calculus. Influenced by the definition of multiplicative derivative, recently Karthikeyan and Murugusundaramoorthy in [14] introduced and studied a class of analytic functions $\mathcal{R}(\varphi)$ satisfying the subordination condition

$$\frac{\omega e^{\frac{\omega^2\chi'(\omega)}{\chi(\omega)}}}{\chi(\omega)} \prec \varphi(\omega), \tag{6}$$

where $\varphi \in \mathcal{P}$ is defined as in (5). The class is non-empty and possess good geometrical implications but it does not reduce to well-known subclasses of $\mathcal{S}$. For the detailed analysis and closure properties of the class $\mathcal{R}(\varphi)$, refer to [14, 15].

Remark 1. *Replacing the ordinary derivative in $\mathcal{S}^*$ with a multiplicative derivative* (3) *would make the class $\mathcal{S}^*$ redundant. That is why in $\mathcal{R}(\varphi)$, authors replaced $\chi'(\omega)$ in $\mathcal{S}^*(\varphi)$ with $e^{\frac{\omega^2\chi'(\omega)}{\chi(\omega)}}$ instead of* (3).

In [15], authors defined a class of functions $\chi \in \mathcal{RC}(g, \varphi)$, which satisfy the conditions

$$\frac{\omega e^{\frac{\omega\chi'(\omega)}{\chi(\omega)}}}{eg(\omega)} \prec \varphi(\omega), \tag{7}$$

where $\chi \neq g$, $e = \exp(1)$, $g(\omega) = \omega + \sum_{\nu=2}^{\infty} b_\nu \omega^\nu \in \mathcal{S}^*$, φ is defined as in (5).

Remark 2. *In* (7)*, authors replaced* $\chi'(\omega)$ *in the definition of close-convex function with* $e^{\frac{\omega\chi'(\omega)}{\chi(\omega)}}$ *which is closer to the definition of multiplicative derivative* (3)*. But still, the authors had to impose a stringent condition of* $\chi \neq g$*.*

Motivated by the definition of $*$-derivative and the class $\mathcal{R}(\varphi)$, we now define a new subclass of analytic functions as follows.

Definition 1. *A function* $\chi(\omega) = \omega + \sum_{\nu=2}^{\infty} a_\nu \omega^\nu$ *in* $\mathcal{A}$ *is said to be* $\mathcal{QM}_q(\varphi)$ *if it satisfies the condition*

$$\frac{\omega e^{\frac{\omega^2 \Delta_q \chi(\omega)}{\chi(\omega)}}}{\chi(\omega)} \prec \varphi(\omega), \tag{8}$$

where $q \in (0, 1)$ *and* φ *is defined as in* (5)*.*

Remark 3. *The function* $f(z) = z - z^2/5$, $|z| < 1$ *belongs to the* $\mathcal{QM}_q(\varphi)$ *for* $q \in (0, 1)$*. For details, refer to [14, Example 1].*

Motivated by the class $\mathcal{RC}(g, \varphi)$, we define the following

Definition 2. *Let* $g(\omega) = \omega + \sum_{\nu=2}^{\infty} b_\nu \omega^\nu$ *belong to* $\mathcal{S}^*$ *and* $\chi(\omega) \in \mathcal{A}$ *with* $a_2 \neq b_2$*. We let* $\mathcal{CC}_q(g, \varphi)$*, denote the class of functions satisfying the conditions*

$$\frac{\omega e^{\frac{\omega \Delta_q \chi(\omega)}{\chi(\omega)}}}{eg(\omega)} \prec \varphi(\omega), \tag{9}$$

where $e = \exp(1)$*,* $q \in (0, 1)$ *and* φ *is defined as in* (5)*.*

Remark 4. *In line with the definition of* $\mathcal{QM}_q(\varphi)$*, we can redefine the class* $\mathcal{RC}(g, \varphi)$ *in the following manner: Let* $g(\omega) = \omega + \sum_{\nu=2}^{\infty} b_\nu \omega^\nu$ *belong to* $\mathcal{S}^*$ *and* $\chi(\omega) \in \mathcal{A}$ *with* $b_2 \neq 1$*. We let* $\mathcal{KC}_q(g, \varphi)$*, denote the class of functions satisfying the conditions*

$$\frac{\omega e^{\frac{\omega^2 \Delta_q \chi(\omega)}{\chi(\omega)}}}{g(\omega)} \prec \varphi(\omega),$$

where $q \in (0, 1)$ *and* φ *is defined as in* (5)*. Note that* $\mathcal{KC}_q(g, \varphi)$ *is analogous to study of analytic functions with fixed second coefficient, for details refer to [1, 2, 19]. But here we will not study anything further about the class* $\mathcal{KC}_q(g, \varphi)$*, since it involves computations which are very much cumbersome.*

Taking the limit from q to 1 from the left in (8) and (9), the classes $\mathcal{QM}_q(\varphi)$ and $\mathcal{CC}_q(g, \varphi)$ reduces to the classes $\mathcal{R}(\varphi)$ and $\mathcal{RC}(g, \varphi)$ respectively.

2 Coefficients inequalities of functions in $\mathcal{QM}_q(\varphi)$ and $\mathcal{CC}_q(g,\, \varphi)$

Let $\eta \in \mathcal{P}$ be of the form $\eta(\omega) = 1 + \sum_{\nu=1}^{\infty} \eta_\nu \omega^\nu$ and defined by

$$\eta(\omega) = \frac{1 + w(\omega)}{1 - w(\omega)}, \quad \omega \in \mathbb{U}.$$

For $\varphi(\omega)$ be defined as in (5), the function $\varphi[w(\omega)]$ can be expanded of the form

$$\varphi[w(\omega)] = 1 + \frac{\eta_1 \Lambda_1}{2}\omega + \frac{\Lambda_1}{2}\left[\eta_2 - \frac{\eta_1^2}{2}\left(1 - \frac{\Lambda_2}{\Lambda_1}\right)\right]\omega^2 + \cdots . \tag{10}$$

We state the following result that we will need the following result to establish our main result.

Lemma 1. *[17] If $d(\omega) = 1 + \sum\limits_{\nu=1}^{\infty} d_\nu \omega^\nu \in \mathcal{P}$, and ρ is complex number, then*

$$\left|d_2 - \rho d_1^2\right| \leq 2 \max\left\{1; |2v - 1|\right\},$$

and the result is sharp.

Theorem 1. *If $\chi(\omega) \in \mathcal{QM}_q(\varphi)$, then we have*

$$|a_2| \leq |\Lambda_1| + 1 \tag{11}$$

$$|a_3| \leq |\Lambda_1|\left[\max\left\{1; \left|\frac{\Lambda_2}{\Lambda_1} - \Lambda_1\right|\right\} + |q+1| + \frac{3}{2|\Lambda_1|}\left|1 + \frac{2(q-1)}{3}\right|\right] \tag{12}$$

and for all $\rho \in \mathbb{C}$

$$\begin{aligned}\left|a_3 - \rho a_2^2\right| \leq |\Lambda_1| \max\left\{1; \left|\frac{\Lambda_2}{\Lambda_1} - \Lambda_1(1-\rho)\right|\right\} + |\Lambda_1\left[(q+1) - 2\rho\right]| \\ + \frac{1}{2}\left|\left[3 + 2(q - \rho - 1)\right]\right|.\end{aligned} \tag{13}$$

Proof. As $\chi \in \mathcal{QM}_q(\varphi)$, by (8) we have

$$\frac{\omega e^{\frac{\omega^2 \Delta_q \chi(\omega)}{\chi(\omega)}}}{\chi(\omega)} = \varphi\left[w(\omega)\right]. \tag{14}$$

Expanding (14) and simplifying, we get

$$\frac{\omega e^{\frac{\omega^2 \Delta_q \chi(\omega)}{\chi(\omega)}}}{\chi(\omega)} = 1 + \left[1 - a_2\right]\omega + \frac{1}{2}\left[1 + 2a_2^2 + 2a_2\left([2]_q - 2\right) - 2a_3\right]\omega^2 + \cdots . \tag{15}$$

From (10) and (15), we obtain

$$a_2 = \left[1 - \frac{\eta_1 \Lambda_1}{2}\right] \tag{16}$$

and

$$a_3 = -\frac{\Lambda_1}{2}\left[\eta_2 - \frac{\eta_1^2}{2}\left(1 - \frac{\Lambda_2}{\Lambda_1} + \Lambda_1\right)\right] - \frac{\Lambda_1 \eta_1}{2}(q+1) + \frac{3}{2}\left[1 + \frac{2(q-1)}{3}\right]. \tag{17}$$

Equations (11) can be obtained by applying $|\eta_1| \leq 2$ in (16). Applying Lemma 1 in (17), we get (12).

Now to establish the inequality (13), we consider

$$\left|a_3 - \rho a_2^2\right| = \Bigg| -\frac{\Lambda_1}{2}\left[\eta_2 - \frac{\eta_1^2}{2}\left(1 - \frac{\Lambda_2}{\Lambda_1} + \Lambda_1\right)\right] - \frac{\Lambda_1 \eta_1}{2}(q+1) + \frac{3}{2}\left[1 + \frac{2(q-1)}{3}\right] - \rho\left[1 + \frac{\eta_1^2 \Lambda_1^2}{4} - \eta_1 \Lambda_1\right]\Bigg|$$

$$= \Bigg|\frac{\Lambda_1}{2}\left[\eta_2 - \frac{\eta_1^2}{2}\left(1 - \frac{\Lambda_2}{\Lambda_1} + \Lambda_1(1-\rho)\right)\right] + \frac{\Lambda_1 \eta_1}{2}\left[(q+1) - 2\rho\right] - \frac{3}{2}\left[1 + \frac{2}{3}(q - \rho - 1)\right]\Bigg|.$$

Applying Lemma 1 in the above equality, we can obtain (13). □

Letting $q \to 1^-$, we can obtain the following result.

Corollary 1. *[14, Corollary 1] Let $\chi \in \mathcal{R}(\varphi)$. Then,*

$$|a_2| \leq (1 + |\Lambda_1|)\,,,$$

$$|a_3| \leq |\Lambda_1|\left[\max\left\{1; \left|\frac{\Lambda_2}{\Lambda_1} - \Lambda_1\right|\right\} + \frac{3}{2|\Lambda_1|} + 2\right]$$

and for a complex number ρ,

$$\left|a_3 - \rho a_2^2\right| \leq |\Lambda_1|\left[\max\left\{1; \left|\frac{\Lambda_2}{\Lambda_1} - \Lambda_1(1-\rho)\right|\right\} + 2\,|1-\rho| + \frac{1}{2|\Lambda_1|}\,|3 - 2\rho|\right].$$

Letting $\varphi(\omega) = \frac{1+\omega}{1-\omega} = 1 + \sum_{\nu=1}^{\infty} 2\omega^\nu$ in Corollary 1, we have the following

Corollary 2. *Let $\chi \in \mathcal{A}$ satisfy the condition*

$$\mathrm{Re}\left(\frac{ze^{\frac{\omega^2\chi'(\omega)}{\chi(\omega)}}}{\chi(\omega)}\right) > 0.$$

Then,

$$|a_2| \leq 3, \qquad |a_3| \leq \frac{15}{2}$$

and for a complex number ρ,

$$\left|a_3 - \rho a_2^2\right| \leq 2\left[\max\{1;\ |2\rho - 1|\} + 2\,|1-\rho| + \frac{1}{4}\,|3 - 2\rho|\right].$$

Theorem 2. *If $\chi(\omega) \in \mathcal{CC}_q(g,\, \varphi)$, then we have*

$$|a_2| \leq \frac{1}{([2]_q - 1)}\left[|\Lambda_1| + 2\right], \tag{18}$$

$$|a_3| \leq \frac{|\Lambda_1|}{([3]_q - 1)}\left[\max\left\{1;\ \left|\frac{\Lambda_2}{\Lambda_1} + \frac{\Lambda_1\,(3 - [2]_q)}{2\,([2]_q - 1)}\right|\right\} + \frac{4}{([2]_q - 1)} + \frac{1}{|\Lambda_1|}\max\left\{1,\ \left|\frac{([2]_q + 3)}{([2]_q - 1)}\right|\right\}\right], \tag{19}$$

and for all $\rho \in \mathbb{C}$

$$\left|a_3 - \rho a_2^2\right| \leq \frac{|\Lambda_1|}{([3]_q - 1)}\left[\max\left\{1;\ \left|\frac{\Lambda_2}{\Lambda_1} + \frac{\Lambda_1\,(3 - [2]_q)}{2\,([2]_q - 1)} - \frac{\Lambda_1\rho\,([3]_q - 1)}{([2]_q - 1)^2}\right|\right\} + 4\left|\frac{1}{([2]_q - 1)} - \frac{\rho\,([3]_q - 1)}{([2]_q - 1)^2}\right| + \frac{1}{|\Lambda_1|}\max\left\{1,\ \left|3 - 4\left(\frac{([2]_q - 3)}{2\,([2]_q - 1)} + \frac{\rho\,([3]_q - 1)}{([2]_q - 1)^2}\right)\right|\right\}\right]. \tag{20}$$

Proof. As $\chi \in \mathcal{CC}_q(g,\, \varphi)$, by (8) we have

$$e^{\frac{\omega\chi'(\omega)}{\chi(\omega)} - 1} = \varphi\left[w(\omega)\right]\frac{g(\omega)}{\omega}. \tag{21}$$

Using (10), the right hand side of (21) on computation will yield

$$\varphi[w(\omega)] = 1 + \left[\frac{\eta_1\Lambda_1}{2} + b_2\right]\omega + \left\{\frac{\Lambda_1}{2}\left[\eta_2 - \frac{\eta_1^2}{2}\left(1 - \frac{\Lambda_2}{\Lambda_1}\right)\right] + \frac{b_2\eta_1\Lambda_1}{2} + b_3\right\}\omega^2 + \cdots. \tag{22}$$

The left hand side of (21) will be of the form

$$e^{\frac{\omega\chi'(\omega)}{\chi(\omega)}-1} = 1 + a_2\left([2]_q - 1\right)\omega + \frac{1}{2}\left[2a_3\left([3]_q - 1\right) + a_2^2\left([2]_q^2 - 4[2]_q + 3\right)\right]\omega^2 + \cdots \tag{23}$$

From (23) and (22), we obtain

$$a_2 = \frac{1}{\left([2]_q - 1\right)}\left[\frac{\eta_1\Lambda_1}{2} + b_2\right] \tag{24}$$

and

$$a_3 = \frac{1}{2\left([3]_q - 1\right)}\left\{\Lambda_1\left[\eta_2 - \frac{\eta_1^2}{2}\left(1 - \frac{\Lambda_2}{\Lambda_1} + \frac{\Lambda_1\left([2]_q - 3\right)}{2\left([2]_q - 1\right)}\right)\right] + \frac{2b_2\eta_1\Lambda_1}{\left([2]_q - 1\right)} + 2b_3 - \frac{b_2^2\left([2]_q - 3\right)}{\left([2]_q - 1\right)}\right\}. \tag{25}$$

Since $g(\omega) = \omega + \sum_{\nu=2}^{\infty} b_\nu\omega^\nu \in \mathcal{S}^*$ implies that $|b_\nu| \leq \nu$, $\forall\nu \geq 2$. Also it is well-known that $|\eta_1| \leq 2$. Equations (18) can be obtained by applying the well-known bounds of b_2 and η_1 in (24). From (25), we have

$$|a_3| \leq \frac{|\Lambda_1|}{2\left([3]_q - 1\right)}\left|\eta_2 - \frac{\eta_1^2}{2}\left(1 - \frac{\Lambda_2}{\Lambda_1} + \frac{\Lambda_1\left([2]_q - 3\right)}{2\left([2]_q - 1\right)}\right)\right| + \frac{|b_2\eta_1\Lambda_1|}{\left([2]_q - 1\right)\left([3]_q - 1\right)} + \frac{1}{\left([3]_q - 1\right)}\left|b_3 - \left(\frac{\left([2]_q - 3\right)}{2\left([2]_q - 1\right)}\right)b_2^2\right|. \tag{26}$$

Applying Lemma 1 together with inequalities $|\eta_1| \leq 2$ and $|b_2| \leq 2$ in (26), we get (19). Following the steps as in Theorem 1, we can establish the inequality (20) after long computation. □

Letting $\varphi(\omega) = \frac{1+\omega}{1-\omega} = 1 + \sum_{\nu=1}^{\infty} 2\omega^\nu$ and $q \to 1^-$ in Theorem 2, we have the following.

Corollary 3. *Let $\chi \in \mathcal{A}$ satisfy the condition*

$$\operatorname{Re}\left(\frac{ze^{\frac{\omega\chi'(\omega)}{\chi(\omega)}}}{eg(\omega)}\right) > 0,$$

where $g(\omega) \in \mathcal{S}^*$ and $e = \exp(1)$. Then,

$$|a_2| \leq 4, \qquad |a_3| \leq \frac{13}{2}$$

and for a complex number ρ,

$$\left|a_3 - \rho a_2^2\right| \leq \max\{1;\ 2|1 - 2\rho|\} + 4|1 - 2\rho| + \frac{1}{2}\max\{1,\ |5 - 8\rho|\}.$$

3 Coefficient Bounds For The Inverse Functions In $\mathcal{QM}_q(\varphi)$ And $\mathcal{CC}_q(g, \varphi)$

In this section, we will find the coefficient estimates for the inverse functions of χ belonging to the classes $\mathcal{QM}_q(\varphi)$ and $\mathcal{CC}_q(g, \varphi)$. Refer to [13, 22] for its relevance and application in the field of univalent function theory. The following result would help us to obtain the coefficient estimates for χ^{-1} (provided it exists), form the coefficient estimates of χ.

Lemma 2. *[10, p. 56] If the function $\chi \in \mathcal{A}$ and $\chi^{-1} = g(w)$ given by*

$$g(w) = w + \sum_{\nu=2}^{\infty} b_\nu w^\nu \tag{27}$$

are inverse functions, then for $\nu \geq 2$

$$b_\nu = \frac{(-1)^{\nu+1}}{\nu!} \times$$
$$\times \begin{vmatrix} \nu a_2 & 1 & 0 & \cdots & 0 \\ 2\nu a_3 & (\nu+1)a_2 & 2 & \cdots & 0 \\ 3\nu a_4 & (2\nu+1)a_3 & (\nu+2)a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & (\nu-2) \\ (\nu-1)\nu a_\nu & [\nu(\nu-2)+1]\, a_{\nu-1} & [\nu(\nu-3)+2]\, a_{\nu-2} & \cdots & (2\nu-2)a_2 \end{vmatrix} \tag{28}$$

The elements of the determinant in (28) are given by

$$\Theta_{ij} = \begin{cases} [(i-j+1)\nu + j - 1]\, a_{i-j+2}, & \text{if } i+1 \geq j \\ 0, & \text{if } i+1 < j. \end{cases}$$

The functions in $\mathcal{QM}_q(\varphi)$ and $\mathcal{CC}_q(g, \varphi)$ need not be univalent, but since $\chi(0) = 0$, $\chi'(0) = 1 \neq 0$ for all χ $\mathcal{QM}_q(\varphi)$ and $\mathcal{CC}_q(g, \varphi)$, there exist an inverse function in some small disk with center at $w = 0$.

Theorem 3. *Let χ^{-1} be the inverse of χ which are defined by*

$$\chi(\omega) = \omega + \sum_{\nu=2}^{\infty} a_\nu \omega^\nu \text{ and} \chi^{-1}(w) = w + \sum_{\nu=2}^{\infty} b_\nu w^\nu, \quad (|w| < r;\, r \geq 1/4).$$

If $\chi \in \mathcal{QM}_q(\varphi)$ then

$$|b_2| \leq |\Lambda_1| + 1,$$

$$|b_3| \leq |\Lambda_1| \max\left\{1; \left|\frac{\Lambda_2}{\Lambda_1} + \Lambda_1\right|\right\} + |\Lambda_1 [q-3]| + \frac{1}{2} |[2q-3]|.$$

and for a complex number τ,

$$\left|b_3 - \tau a_2^2\right| \leq |\Lambda_1| \max\left\{1; \left|\frac{\Lambda_2}{\Lambda_1} - \Lambda_1(3-\tau)\right|\right\} + |\Lambda_1 [(q+5) - 2\tau]|$$
$$+ \frac{1}{2} |[3 + 2(q - \tau + 1)]|.$$

Proof. From $\chi(\omega) = \omega + \sum_{\nu=2}^{\infty} a_\nu \omega^\nu$ and the Lemma 2, we have

$$b_2 = -a_2 \quad \text{and} \quad b_3 = b_3 = \frac{(-1)^4}{3!} \begin{vmatrix} 3a_2 & 1 \\ 6a_3 & 4a_2 \end{vmatrix} = 2a_2^2 - a_3.$$

The estimate for $|b_2| = |a_2|$ follows immediately from (16). Letting $\rho = 2$ in (13), we get the estimate $|b_3|$.

To find the Fekete-Szegő inequality for χ^{-1}, consider

$$\left|b_3 - \tau b_2^2\right| = \left|2a_2^2 - a_3 - \tau a_2^2\right| = \left|a_3 - (\tau - 2)a_2^2\right|.$$

Changing $\rho = (\tau - 2)$ in the (13), we get the desired result. □

Remark 5. *Note that letting $q \to 1^-$ in Theorem 3, we get the corresponding result obtained by Karthikeyan and Murugusundaramoorthy [14].*

For completeness, we just state the following result.

Theorem 4. *Let χ^{-1} be the inverse of χ which are defined by*

$$\chi(\omega) = \omega + \sum_{\nu=2}^{\infty} a_\nu \omega^\nu \; and \, \chi^{-1}(w) = w + \sum_{\nu=2}^{\infty} b_\nu w^\nu, \quad (|w| < r;\, r \geq 1/4).$$

If $\chi \in \mathcal{CC}_q(g, \varphi)$ then

$$|b_2| \leq \frac{1}{([2]_q - 1)} \left[|\Lambda_1| + 2\right],$$

$$|b_3| \leq \frac{|\Lambda_1|}{([3]_q - 1)} \left[\max\left\{1; \left|\frac{\Lambda_2}{\Lambda_1} + \frac{\Lambda_1 (3 - [2]_q)}{2([2]_q - 1)} - \frac{\Lambda_1 2([3]_q - 1)}{([2]_q - 1)^2}\right|\right\} \right.$$
$$+4\left|\frac{1}{([2]_q - 1)} - \frac{2([3]_q - 1)}{([2]_q - 1)^2}\right|$$
$$\left. + \frac{1}{|\Lambda_1|} \max\left\{1, \left|3 - 4\left(\frac{([2]_q - 3)}{2([2]_q - 1)} + \frac{2([3]_q - 1)}{([2]_q - 1)^2}\right)\right|\right\}\right]$$

and for a complex number τ,

$$\left|b_3 - \tau a_2^2\right| \leq \frac{|\Lambda_1|}{([3]_q - 1)} \left[\max\left\{1; \left|\frac{\Lambda_2}{\Lambda_1} + \frac{\Lambda_1 (3 - [2]_q)}{2([2]_q - 1)} - \frac{\Lambda_1(\tau - 2)([3]_q - 1)}{([2]_q - 1)^2}\right|\right\} \right.$$
$$+4\left|\frac{1}{([2]_q - 1)} - \frac{(\tau - 2)([3]_q - 1)}{([2]_q - 1)^2}\right|$$
$$\left. + \frac{1}{|\Lambda_1|} \max\left\{1, \left|3 - 4\left(\frac{([2]_q - 3)}{2([2]_q - 1)} + \frac{(\tau - 2)([3]_q - 1)}{([2]_q - 1)^2}\right)\right|\right\}\right].$$

4 Conclusions

The primary of this study was to replace the classical derivative with a multiplicative quantum derivative in the well-known class of starlike functions. Unfortunately, the classes would not well-defined with such a exact replacement. Since the multiplicative derivative is defined on a domain which does not admit zero, we did a bit of manipulation in definition of multiplicative derivative. Replacing such a *influenced* multiplicative quantum derivative in place of the classical derivative involved in the classes of starlike and close-to convex function, we defined the function class $\mathcal{QM}_q(\varphi)$ and $\mathcal{CC}_q(g, \varphi)$.

The future scope of this study is enormous, since the defined function classes $\mathcal{QM}_q(\varphi)$ and $\mathcal{CC}_q(g, \varphi)$ involved lots of constraints. Further, the classes involving higher order derivatives have to explored, but are very complicated and computationally cumbersome.

References

[1] Ahuja, O. P. and Silverman, H., *Extreme points of families of univalent functions with fixed second coefficient*, Colloq. Math. **54** (1987), no. 1, 127–137.

[2] Ali, R. M., Nagpal, S. and Ravichandran, V., *Second-order differential subordination for analytic functions with fixed initial coefficient*, Bull. Malays. Math. Sci. Soc. **34** (2011), no. 3, 611–629.

[3] Araci, S., Karthikeyan, K. R., Murugusundaramoorthy, G. and Khan, B., *Φ-like analytic functions associated with a vertical domain*, Math. Inequal. Appl. **26** (2023), no. 4, 935–949.

[4] Bashirov, A. E., Kurpinar, E. M. and Özyapıcı, A., *Multiplicative calculus and its applications*, J. Math. Anal. Appl. **337** (2008), no. 1, 36–48.

[5] Bashirov, A. E., Mısırlı, E., Tandoğdu, Y. and Özyapıcı, A., *On modeling with multiplicative differential equations*, Appl. Math. J. Chin. Univ. **26** (2011), 425–438.

[6] Bashirov, A. E. and Riza, M., *On complex multiplicative differentiation*, TWMS J. Appl. Eng. Math. **1** (2011), no. 1, 75–85.

[7] Dziok, J., Raina, R. K. and Sokół, J., *On α-convex functions related to shell-like functions connected with Fibonacci numbers*, Appl. Math. Comput. **218** (2011), no. 3, 996–1002.

[8] Dziok, J., Raina, R. K. and Sokół, J., *Certain results for a class of convex functions related to a shell-like curve connected with Fibonacci numbers*, Comput. Math. Appl. **61** (2011), no. 9, 2605–2613.

[9] Dziok, J., Raina, R. K. and Sokół, J., *On a class of starlike functions related to a shell-like curve connected with Fibonacci numbers*, Math. Comput. Modelling **57** (2013), no. 5-6, 1203–1211.

[10] Goodman, A. W., *Univalent functions. Vol. II*, Mariner Publishing Co., Inc., Tampa, FL, 1983.

[11] Ismail, M. E. H., Merkes, E. and Styer, D., *A generalization of starlike functions*, Complex Variables Theory Appl. **14** (1990), no. 1-4, 77–84.

[12] Jackson, F. H., *On q-definite integrals*, Quart. J. Pure Appl. Math. **41** (1910), 193–203.

[13] Karthikeyan, K. R., Murugusundaramoorthy, G. and Cho, N. E., *Some inequalities on Bazilevič class of functions involving quasi-subordination*, AIMS Math. **6** (2021), no. 7, 7111–7124.

[14] Karthikeyan, K. R. and Murugusundaramoorthy, G., *Properties of a class of analytic functions influenced by multiplicative calculus*, Fractal Fract. **8** (2024), no. 3, 131.

[15] Karthikeyan, K. R., Umadevi, E. and Murugusundaramoorthy, G., *Subclasses of close-to-convex functions involving the multiplicative derivative*, preprint.

[16] Khan, M. G., Ahmad, B., Murugusundaramoorthy, G., Mashwani, W. K., Yalçin, S., Shaba, T. G. and Salleh, Z., *Third Hankel determinant and Zalcman functional for a class of starlike functions with respect to symmetric points related with sine function*, J. Math. Comput. Sci. **25** (2022), no. 1, 29–36.

[17] Ma, W. C. and Minda, D., *A unified treatment of some special classes of univalent functions*, Proc. Conf. Complex Analysis (Tianjin, 1992), Conf. Proc. Lecture Notes Anal., I, Int. Press, Cambridge, MA, 157–169.

[18] Mendiratta, R., Nagpal, S. and Ravichandran, V., *A subclass of starlike functions associated with left-half of the lemniscate of Bernoulli*, Internat. J. Math. **25** (2014), no. 9, 1450090, 17 pp.

[19] Mendiratta, R., Nagpal, S. and Ravichandran, V., *Second-order differential superordination for analytic functions with fixed initial coefficient*, Southeast Asian Bull. Math. **39** (2015), no. 6, 851–864.

[20] Mustafa, N. and Murugusundaramoorthy, N., *Second Hankel determinant for Mocanu type bi-starlike functions related to shell-shaped region*, Turkish J. Math. **45** (2021), no. 3, 1270–1286.

[21] Mustafa, N. and Korkmaz, S., *Coefficient bound estimates and Szegö problem for a certain subclass of analytic and bi-univalent functions*, Turkish J. Sci. **7** (2022), no. 3, 211–218.

[22] Ponnusamy, S., Sharma, N. L. and Wirths, K. J., *Logarithmic coefficients of the inverse of univalent functions*, Results Math. **73** (2018), 160.

[23] Raina, R. K. and Sokół, J., *On a class of analytic functions governed by subordination*, Acta Univ. Sapientiae Math. **11** (2019), no. 1, 144–155.

[24] Riza, M., Özyapici, A. and Mısırlı, E., *Multiplicative finite difference methods*, Quart. Appl. Math. **67** (2009), no. 4, 745–754.

[25] Srivastava, H. M., *Univalent functions, fractional calculus, and associated generalized hypergeometric functions*, Univalent functions, fractional calculus, and their applications (Koriyama, 1988), Ellis Horwood Ser. Math. Appl., Horwood, Chichester, 329–354.

[26] Srivastava, H. M., *Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis*, Iran. J. Sci. Technol. Trans. A Sci. **44** (2020), no. 1, 327–344.