

## ANALYSIS OF RESULTS FOR A QUASISTATIC FRICTIONAL CONTACT PROBLEM WITH NORMAL DAMPED RESPONSE IN THERMO-VISCOELASTICITY

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### Abstract

In this manuscript, we explore a mathematical model that describes the quasistatic contact process with friction between a body governed by a non-linear thermo-viscoelastic constitutive law and an obstacle. The contact is represented using a general condition for normal damped response. We then develop a variational formulation for the mechanical problem and prove the existence of a weak solution. The proof is based on techniques involving variational inequalities, parabolic variational equality, and fixed-point methods.

Finally, a perturbation of the data is introduced, and a convergence result is established as the perturbation parameter approaches zero.

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*Key words:* thermo-viscoelastic, fixed point, convergence result.

## 1 Introduction

Contact mechanics is an extensive field that encompasses a wide range of situations involving interactions between deformable solids or between an object and a rigid surface. These contact scenarios are ubiquitous in daily life and are particularly relevant in structural mechanics and various industries, including automotive (related to vehicle structures), aeronautics (concerning issues like cracks in composites and fiber/matrix interfaces), energy production (involving the assembly of structures and the occurrence of cracks in welded joints), and transmission systems. As a result, significant efforts have been made to develop models and numerical simulations in this area. For more information, see, for example, [12, 15, 17] and the references therein.

The quasistatic approximation of contact problems is applied when the forces in a system change slowly over time, allowing the neglect of inertial terms in

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the equation of motion. This approach has been extensively studied, as seen in numerous papers, including [1, 2, 14] and the associated references.

Models incorporating thermo-elastic and thermo-viscoplastic constitutive laws have been widely used to represent the effects of temperature on the behavior of real materials, such as metals, magmas, and polymers. This has led to a substantial body of literature on the subject, as highlighted in works like [7, 5, 8] and the related references.

In this work, we examine a model that describes the frictional contact between a thermo-viscoelastic body with long-term memory and an obstacle, referred to as the "foundation." The contact is characterized by a normal damped response. Many examples of such conditions found in [3, 6, 11, 9].

The structure of the article is as follows: Section 2 introduces the contact model and discusses the contact boundary conditions. Section 3 outlines the assumptions regarding the data and presents the variational formulation. In Section 4, we prove the existence and uniqueness of the solution. Finally, Section 5 analyzes how the solution is affected by the contact boundary conditions and provides the proof of convergence, as detailed in Theorem 2.

## 2 Problem statement

Consider a material body occupying a confined region  $\Omega \subset \mathbb{R}^d (d = 2, 3)$ , characterized by a smooth boundary surface  $\Gamma$ . This surface can be divided into three measurable portions, namely  $\Gamma_1$ ,  $\Gamma_2$ , and  $\Gamma_3$ , with  $\Gamma_1$  having a non-zero measure. We define the unit outward normal vector on  $\Gamma$  as  $\nu$ . The body is situated on  $\Gamma_1 \times (0, T)$  and is fixed in place. On the surface  $\Gamma_2$ , we have surface tractions represented by a density function  $f_2$ , while within the volume of  $\Omega$ , there are volumetric forces described by a density function  $f_0$  and a volume thermal force with a density of  $q_0$ . It is assumed that the functions  $f_2$  and  $f_0$  vary slowly with respect to time. Here,  $T > 0$  defines the time interval  $[0, T]$  under consideration.

In the region  $\Gamma_3 \times (0, T)$ , the body comes into contact with a thermally conductive foundation, exhibiting frictional interactions. Temperature variation is assumed to be negligible on the combined surface  $\Gamma_1 \cup \Gamma_2 \times (0, T)$ . We maintain a constant thermal potential at  $\theta_F$ .

The classical formulation of the mechanical problem for an viscoelastic body, including thermal effects, can be expressed as follows.

### Problem $P$

Find a displacement field  $\mathbf{u} : \Omega \times (0, T) \rightarrow \mathbb{R}^d$ , a stress field  $\boldsymbol{\sigma} : \Omega \times (0, T) \rightarrow \mathbb{S}^d$ , a temperature field  $\theta : \Omega \times (0, T) \rightarrow \mathbb{R}$  such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{B}(\varepsilon(\mathbf{u}(t))) + \int_0^t \mathcal{M}(t-s)\varepsilon(\mathbf{u}(s))ds - C_e\theta(t) \quad \text{in } \Omega \times (0, T), \quad (1)$$

$$q(t) = -\mathcal{K}\nabla\theta(t) \quad \text{in } \Omega \times (0, T), \quad (2)$$

$$\dot{\theta}(t) + \operatorname{div} q(t) - \mathcal{R}\varepsilon(u(t)) = q_0(t) \quad \text{in } \Omega \times (0, T), \quad (3)$$

$$\operatorname{Div} \boldsymbol{\sigma} = -f_0 \quad \text{in } \Omega \times (0, T), \quad (4)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1 \times (0, T), \quad (5)$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = f_2 \quad \text{on } \Gamma_2 \times (0, T), \quad (6)$$

$$-\sigma_\nu = p_\nu(\dot{u}_\nu), \quad \text{on } \Gamma_3 \times (0, T) \quad (7)$$

$$\begin{cases} \|\boldsymbol{\sigma}_\tau\| \leq p_\tau(\dot{u}_\nu) \\ \boldsymbol{\sigma}_\tau = -p_\tau(\dot{u}_\nu) \frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \text{ if } \dot{\mathbf{u}}_\tau \neq \mathbf{0} \end{cases} \quad \text{on } \Gamma_3 \times (0, T), \quad (8)$$

$$\theta = 0 \quad \text{on } \Gamma_a \times (0, T), \quad (9)$$

$$\frac{\partial q(t)}{\partial \nu} = k_c(u_\nu(t)) \phi_l(\theta(t) - \theta_F) \quad \text{on } \Gamma_3 \times (0, T), \quad (10)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \theta(0) = \theta_0 \quad \text{in } \Omega. \quad (11)$$

We now describe the problem (1)-(11). To begin, equation (1)-(2) formulate the constitutive law governing the thermo-viscoelastic constitutive law with long term-memory, where  $\mathcal{A}$  and  $\mathcal{B}$  are the viscosity and elasticity operators, respectively, and  $\mathcal{M}$  is the relaxation operator. The thermal expansion is represented by the tensor  $C_e = (C_{ij})$ , while the thermal conductivity is denoted as  $\mathcal{K}$ . The absolute temperature is denoted as  $\theta$ , and the heat flux vector is described as  $q = (q_i) : \Omega \times (0, T) \rightarrow \mathbb{R}^d$ .

Equation (3) represents the Fourier law of heat conduction where the function  $\mathcal{R} = (\mathcal{R}_{ij})$  is the influence of the displacement field.

Equations (4) represent the equilibrium equation for the stress field. Equations (5)-(6) are the displacement-traction conditions.

Frictional contact conditions with normal damped response, as described by equations (7) and (8), provide a suitable adaptation of Coulomb's law of friction. This condition specifies a general relationship between the normal stress  $\sigma_\nu$  and the normal velocity  $\dot{u}_\nu$ , which can represent the behavior of a lubricant layer on the contact surface. Here,  $p_\nu$  and  $p_\tau$  are predetermined contact functions. Additionally, the tangential shear stress cannot exceed the maximum frictional resistance  $p_\tau$ . If this resistance is not reached, the surface adheres to the foundation, known as the "stick" state. When the shear stress equals the maximum resistance, there is relative sliding between the surface and the foundation, referred to as the "slip" state.

This type of contact scenario has been studied by several authors (see, for example, [17]).

Equation (9) means that the temperature vanishes on  $\Gamma_1 \cup \Gamma_2 \times (0, T)$ . The relation (10) represents a regularized thermal contact condition where  $\frac{\partial q}{\partial \nu}$  is the normal derivative of  $q$  such that:

$$\phi_l(r) = \begin{cases} -l & \text{if } r < -l, \\ r & \text{if } -l \leq r \leq l, \\ l & \text{if } r > l. \end{cases} \quad \begin{cases} k_c(\xi) = 0 & \text{if } \xi < 0, \\ k_c(\xi) > 0 & \text{if } \xi \geq 0, \end{cases}$$

here  $l$  is a large positive constant. Finally, The functions  $\mathbf{u}_0$  and  $\theta_0$  in (11) are the initial data.

### 3 Variational formulation and preliminaries

The following notations and conventions will be consistently employed throughout this paper. For a positive integer  $d$ , we use the notation  $\mathbb{S}^d$  to represent the space of second-order symmetric tensors defined on  $\mathbb{R}^d$ . The inner products and their corresponding norms for elements in  $\mathbb{R}^d$  and  $\mathbb{S}^d$  are defined as follows

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= v_i w_i, \quad \forall \mathbf{v}, \mathbf{w} \in \mathbb{R}^d \quad \text{and} \quad \boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij} \tau_{ij} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d, \\ \|\mathbf{v}\| &= (\mathbf{v} \cdot \mathbf{v})^{\frac{1}{2}}, \quad \forall \mathbf{v} \in \mathbb{R}^d \quad \text{and} \quad \|\boldsymbol{\sigma}\| = (\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})^{\frac{1}{2}}, \quad \forall \boldsymbol{\sigma} \in \mathbb{S}^d. \end{aligned}$$

Throughout this paper, indices  $i, j$  range from 1 to  $d$ . We adopt the convention of summation over repeated indices, and a comma following an index indicates a partial derivative with respect to the corresponding component of the independent variable.

Consider a bounded domain  $\Omega$  in  $\mathbb{R}^d$  with a smooth boundary  $\Gamma$ . We introduce the following function spaces

$$\begin{aligned} H &= L^2(\Omega; \mathbb{R}^d), \quad H_1 = \{\mathbf{v} \in H \mid \varepsilon(\mathbf{v}) \in \mathcal{H}\} = H^1(\Omega; \mathbb{R}^d), \\ \mathcal{H} &= \{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\} = L^2(\Omega; \mathbb{S}^d), \quad \mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H} \mid \text{Div } \boldsymbol{\sigma} \in H\}. \end{aligned}$$

where  $\varepsilon : H^1(\Omega; \mathbb{R}^d) \rightarrow L^2(\Omega; \mathbb{S}^d)$  and  $\text{Div} : \mathcal{H}_1 \rightarrow L^2(\Omega; \mathbb{R}^d)$  denote the deformation and the divergence operators, respectively, given by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2} (\mathbf{u}_{i,j} + \mathbf{u}_{j,i}), \quad \text{Div}(\boldsymbol{\sigma}) = \sigma_{ij,j}.$$

The spaces  $H$ ,  $H_1$ ,  $\mathcal{H}$ , and  $\mathcal{H}_1$  are Hilbert spaces endowed with the following inner products

$$(\mathbf{v}, \mathbf{w})_H = \int_{\Omega} v_i w_i dx \quad \forall \mathbf{v}, \mathbf{w} \in H, \quad (\boldsymbol{\tau}, \boldsymbol{\varsigma})_{\mathcal{H}} = \int_{\Omega} \tau_{ij} \varsigma_{ij} dx \quad \forall \boldsymbol{\tau}, \boldsymbol{\varsigma} \in \mathcal{H},$$

$$\begin{aligned} (\mathbf{v}, \mathbf{w})_{H_1} &= (\mathbf{v}, \mathbf{w})_H + (\varepsilon(\mathbf{v}), \varepsilon(\mathbf{w}))_{\mathcal{H}}, \quad \forall \mathbf{v}, \mathbf{w} \in H_1, \\ (\boldsymbol{\tau}, \boldsymbol{\varsigma})_{\mathcal{H}_1} &= (\boldsymbol{\tau}, \boldsymbol{\varsigma})_{\mathcal{H}} + (\text{Div } \boldsymbol{\tau}, \text{Div } \boldsymbol{\varsigma})_H, \quad \boldsymbol{\tau}, \boldsymbol{\varsigma} \in \mathcal{H}_1. \end{aligned}$$

The associated norms in  $H$ ,  $H_1$ ,  $\mathcal{H}$  and  $\mathcal{H}_1$  are denoted by  $\|\cdot\|_H$ ,  $\|\cdot\|_{H_1}$ ,  $\|\cdot\|_{\mathcal{H}}$  and  $\|\cdot\|_{\mathcal{H}_1}$  respectively.

Given  $\mathbf{v} \in H^{\frac{1}{2}}(\Gamma; \mathbb{R}^d)$ , we denote the normal and tangential components of  $\mathbf{v}$  on the boundary as  $v_{\nu} = \mathbf{v} \cdot \boldsymbol{\nu}$  and  $\mathbf{v}_{\tau} = \mathbf{v} - v_{\nu} \boldsymbol{\nu}$ . Similarly, for a tensor field  $\boldsymbol{\sigma} : \Gamma \rightarrow \mathbb{S}^d$ , we define its normal and tangential components as  $\sigma_{\nu} = \boldsymbol{\sigma} \boldsymbol{\nu} \cdot \boldsymbol{\nu}$  and  $\boldsymbol{\sigma}_{\tau} = \boldsymbol{\sigma} \boldsymbol{\nu} - \sigma_{\nu} \boldsymbol{\nu}$ . Furthermore, we recall the following Green formula

$$(\boldsymbol{\sigma}, \varepsilon(\mathbf{w}))_{\mathcal{H}} + (\text{Div } \boldsymbol{\sigma}, \mathbf{w})_H = \int_{\Gamma} \boldsymbol{\sigma} \boldsymbol{\nu} \cdot \mathbf{w} da, \quad \forall \mathbf{w} \in H_1.$$

For the displacement field, we require the closed subspace of  $H^1(\Omega)^d$  defined by

$$V = \left\{ \mathbf{v} \in H^1(\Omega)^d \mid \mathbf{v} = 0 \text{ on } \Gamma_1 \right\}.$$

For the temperature we consider the  $\Omega$ , denote the closed subspace of  $H^1(\Omega)$  given by

$$\Omega = \left\{ \gamma \in H^1(\Omega) \mid \gamma = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \right\},$$

We consider the inner product and the corresponding norm, defined as follows

$$(\mathbf{v}, \mathbf{w})_V = (\varepsilon(\mathbf{v}), \varepsilon(\mathbf{w}))_{\mathcal{H}}, \quad \|\mathbf{w}\|_V = \|\varepsilon(\mathbf{w})\|_{\mathcal{H}}. \quad (12)$$

$$(\theta, \gamma)_{\Omega} = (\nabla \theta, \nabla \gamma)_H, \quad \|\gamma\|_{\Omega} = (\gamma, \gamma)_{\Omega}^{\frac{1}{2}}. \quad (13)$$

Given that  $\text{meas}(\Gamma_1) > 0$ , Korn's inequality is applicable, and there exists a constant  $C_0 > 0$  depending solely on  $\Omega$  and  $\Gamma_1$  such that

$$\|\varepsilon(\mathbf{v})\|_{\mathcal{H}} \geq C_0 \|\mathbf{v}\|_{H^1(\Omega)^d}, \quad \forall \mathbf{v} \in V.$$

A proof of Korn's inequality may be found in ([13], p.79). The Friedrichs-Poincaré inequality holds

$$\|\nabla \gamma\|_H \geq c_F \|\gamma\|_{\Omega}, \quad \forall \gamma \in \Omega, \quad (14)$$

where  $c_F > 0$  is a constant which depends only on  $\Omega$ ,  $\Gamma_1$  and  $\Gamma_2$ .

Moreover, by the Sobolev trace theorem, there exist two positive constants  $c_0$  and  $\tilde{c}_0$  such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leq c_0 \|\mathbf{v}\|_V, \quad \forall \mathbf{v} \in V, \quad \|\gamma\|_{L^2(\Gamma_3)} \leq \tilde{c}_0 \|\gamma\|_{\Omega}, \quad \forall \gamma \in \Omega. \quad (15)$$

We define  $E$  as the space of fourth-order tensor fields, expressed as:

$$E = \left\{ \mathcal{E} = (\mathcal{E}_{ijkl}) \mid \mathcal{E}_{ijkl} = \mathcal{E}_{klij} = \mathcal{E}_{jikl} \in L^\infty(\Omega), \quad 1 \leq i, j, k, l \leq d \right\}.$$

This space is a real Banach space with the following norm:

$$\|\mathcal{E}\|_E = \max_{1 \leq i, j, k, l \leq d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}. \quad (16)$$

A simple calculation reveals that:

$$\|\mathcal{E}\xi\|_{\mathcal{H}} \leq d \|\mathcal{E}\|_E \|\xi\|_{\mathcal{H}}, \quad \forall \mathcal{E} \in E, \quad \xi \in \mathcal{H}. \quad (17)$$

For any real Hilbert space  $X$ , we make use of the standard notation for the spaces  $L^p(0, T; X)$  and  $W^{k,p}(0, T; X)$ , where  $1 \leq p \leq \infty$  and  $k \geq 1$ . For  $T > 0$ , we denote  $C(0, T; X)$  and  $C^1(0, T; X)$  as the spaces of continuous and continuously differentiable functions from  $[0, T]$  to  $X$ , respectively, with the following norms

$$\|\mathbf{h}\|_{C(0, T; X)} = \max_{t \in [0, T]} \|\mathbf{h}(t)\|_X.$$

$$\|\mathbf{h}\|_{C^1(0,T;X)} = \max_{t \in [0,T]} \|\mathbf{h}(t)\|_X + \max_{t \in [0,T]} \|\dot{\mathbf{h}}(t)\|_X.$$

The following results, which will be referenced in Section 5, begin with a recall of the convergence criterion for a sequence  $(x_k)_k$  converging to an element  $x$  in  $C(\mathbb{R}^+; X)$ . This criterion states that:

$$\begin{aligned} x_k \rightarrow x \text{ in } C(\mathbb{R}^+; X) \text{ as } k \rightarrow \infty \text{ if and only if} \\ \max_{r \in [0,n]} \|x_k(r) - x(r)\|_X \rightarrow 0 \text{ as } k \rightarrow \infty \text{ for every } n \in \mathbb{N}. \end{aligned} \quad (18)$$

Additionally, the convergence of a sequence  $(x_k)_k$  to an element  $x \in C^1(\mathbb{R}^+; X)$  is characterized by:

$$\begin{aligned} x_k \rightarrow x \text{ in } C^1(\mathbb{R}^+; X) \text{ as } k \rightarrow \infty \text{ if and only if} \\ x_k \rightarrow x \text{ in } C(\mathbb{R}^+; X) \text{ and } \dot{x}_k \rightarrow \dot{x} \text{ in } C(\mathbb{R}^+; X) \text{ as } k \rightarrow \infty, \end{aligned} \quad (19)$$

where  $\dot{x}$  represents the time derivative of  $x$  for all  $x \in C(\mathbb{R}^+; X)$ .

In the analysis of problem  $P$ , we consider the following assumptions. The viscosity operator  $\mathcal{A} : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d$  satisfies

$$\left\{ \begin{array}{l} (a) \text{ There exists } L_{\mathcal{A}} > 0 \text{ such that} \\ \|\mathcal{A}(\mathbf{x}, \mathbf{v}_1) - \mathcal{A}(\mathbf{x}, \mathbf{v}_2)\| \leq L_{\mathcal{A}} \|\mathbf{v}_1 - \mathbf{v}_2\|, \\ \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{S}^d, \text{ a.e } \mathbf{x} \in \Omega. \\ (b) \text{ There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ (\mathcal{A}(\mathbf{x}, \mathbf{v}_1) - \mathcal{A}(\mathbf{x}, \mathbf{v}_2)) \cdot (\mathbf{v}_1 - \mathbf{v}_2) \geq m_{\mathcal{A}} \|\mathbf{v}_1 - \mathbf{v}_2\|^2, \\ \text{for all } \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{S}^d, \text{ a.e } \mathbf{x} \in \Omega. \\ (c) \text{ The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \mathbf{v}) \text{ is Lebesgue measurable on } \Omega, \\ \text{for any } \mathbf{v} \in \mathbb{S}^d. \\ (d) \text{ The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \mathbf{0}) \in \mathcal{H}. \end{array} \right. \quad (20)$$

The elastic function  $\mathcal{B} : \Omega \times \mathbb{S}^d \times \mathbb{R} \rightarrow \mathbb{S}^d$  satisfies

$$\left\{ \begin{array}{l} (a) \text{ There exists } L_{\mathcal{B}} > 0 \text{ such that} \\ \|\mathcal{B}(\mathbf{x}, \mathbf{v}_1) - \mathcal{B}(\mathbf{x}, \mathbf{v}_2)\| \leq L_{\mathcal{B}} \|\mathbf{v}_1 - \mathbf{v}_2\|, \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{S}^d, \text{ a.e } \mathbf{x} \in \Omega. \\ (b) \text{ The mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \mathbf{v}) \text{ is Lebesgue measurable on } \Omega, \\ \forall \mathbf{v} \in \mathbb{S}^d. \\ (c) \text{ The mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, 0) \in \mathcal{H}. \end{array} \right. \quad (21)$$

The thermal expansion operator  $C_e : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  satisfies

$$\left\{ \begin{array}{l} \text{(a) There exists } L_{C_e} > 0 \text{ such that} \\ \|C_e(\mathbf{x}, \varsigma_1) - C_e(\mathbf{x}, \varsigma_2)\| \leq L_{C_e} \|\varsigma_1 - \varsigma_2\| \text{ for all } \varsigma_1, \varsigma_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Omega. \\ \text{(b) } C_e = (c_{ij}), c_{ij} = c_{ji} \in L^\infty(\Omega). \\ \text{(c) The mapping } \mathbf{x} \mapsto C_e(\mathbf{x}, \varsigma) \text{ is Lebesgue measurable on } \Omega, \\ \text{for any } \varsigma \in \mathbb{R}. \\ \text{(d) The mapping } \mathbf{x} \mapsto C_e(\mathbf{x}, 0) \in \mathcal{H}. \end{array} \right. \quad (22)$$

The thermal conductance  $k_c$  satisfy the following hypothesis

$$\left\{ \begin{array}{l} k_c : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}^+ \\ \text{(a) There exists } M_{k_c} > 0 \text{ such that } \|k_c(\mathbf{x}, \delta)\| \leq M_{k_c}, \\ \quad \forall \delta \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3 \\ \text{(b) } \mathbf{x} \mapsto k_c(\mathbf{x}, \delta) \text{ is measurable on } \Gamma_3, \forall \delta \in \mathbb{R}, \\ \text{(c) } \mathbf{x} \mapsto k_c(\mathbf{x}, \delta) = 0, \quad \forall \delta \leq 0. \\ \text{(d) There exists } L_{k_c} > 0 \text{ such that} \\ \|k_c(\cdot, \delta_1) - k_c(\cdot, \delta_2)\| \leq L_{k_c} \|\delta_1 - \delta_2\|, \forall \delta_1, \delta_2 \in \mathbb{R} \end{array} \right. \quad (23)$$

The normal compliance function  $p_e : \Gamma_3 \times \mathbb{R} \rightarrow \mathbb{R}_+$ , ( $e = \nu, \tau$ ) satisfies

$$\left\{ \begin{array}{l} \text{(a) There exists } L_e > 0 \text{ such that} \\ \|p_e(\mathbf{x}, \delta_1) - p_e(\mathbf{x}, \delta_2)\| \leq L_e \|\delta_1 - \delta_2\| \\ \forall \delta_1, \delta_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3 \\ \text{(d) For any } \delta \in \mathbb{R}, \mathbf{x} \mapsto p_e(\mathbf{x}, \delta) \text{ is measurable on } \Gamma_3 \\ \text{(e) } \mathbf{x} \mapsto p_e(\mathbf{x}, \delta) = 0, \text{ for all } \delta \leq 0. \end{array} \right. \quad (24)$$

$$\mathcal{M} \in C(\mathbb{R}^+, E) \quad (25)$$

We assume that the initial data  $\mathbf{u}_0$  and  $\theta_0$  the volume of forces  $f_0$ , the traction  $f_2$  and the thermal flux  $q_0$  satisfy

$$\mathbf{u}_0 \in V, \quad (26)$$

$$\theta_0 \in H^1(\Omega), \quad (27)$$

$$f_0 \in C(0, T; L^2(\Omega)^d), \quad f_2 \in C(0, T; L^2(\Gamma_2)^d), \quad (28)$$

$$q_0 \in L^2(0, T; L^2(\Omega)), \quad \theta_F \in L^2(0, T; L^2(\Gamma_3)). \quad (29)$$

Next. We define four mappings  $j : V \times V \rightarrow \mathbb{R}$ ,  $q_{th} : [0, T] \rightarrow \mathbb{Q}$ ,  $f : [0, T] \rightarrow V$

and  $\chi : V \times \mathcal{Q} \times \mathcal{Q} \rightarrow \mathbb{R}$ , as follows

$$j(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} p_\nu(u_\nu) v_\nu da + \int_{\Gamma_3} p_\tau(u_\nu) \|\mathbf{v}_\tau\| da, \quad (30)$$

$$(q_{th}(t), \zeta)_\mathcal{Q} = \int_\Omega q_0(t) \zeta dx, \quad \forall \zeta \in \mathcal{Q}. \quad (31)$$

$$(\mathbf{f}(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_0(t) \cdot \mathbf{v} dx + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot \mathbf{v} da, \quad (32)$$

$$\chi(u(t), \theta(t), \zeta) = \int_{\Gamma_3} k_c(u_\nu(t)) \phi_l(\theta(t) - \theta_F) \zeta da, \quad (33)$$

and the following operators

$$\mathcal{T} : \mathcal{Q} \times \mathcal{Q} \rightarrow \mathbb{R}, \quad \mathcal{T}(\theta, \zeta) = (\mathcal{K} \nabla \theta, \nabla \zeta)_H, \quad (34)$$

$$\mathcal{F} : V \times \mathcal{Q} \rightarrow \mathbb{R}, \quad \mathcal{F}(\mathbf{u}, \zeta) = (\mathcal{R} \varepsilon(\mathbf{u}), \zeta)_{L^2(\Omega)}. \quad (35)$$

The operators  $\mathcal{T}$  and  $\mathcal{F}$  satisfy the usual property of symmetry

$$\mathcal{K}_{ij} = \mathcal{K}_{ji} \in L^\infty(\Omega), \quad (36)$$

$$\mathcal{R}_{ij} = \mathcal{R}_{ji} \in L^\infty(\Omega), \quad (37)$$

There exist positive constant and such that

$$\mathcal{T}(\zeta, \zeta) \geq m_{\mathcal{T}} \|\zeta\|_{\mathcal{Q}}^2. \quad (38)$$

The operators  $\mathcal{T}$  and  $\mathcal{F}$  satisfy the usual property of boundedness

$$|\mathcal{T}(\theta, \zeta)| \leq M_{\mathcal{T}} \|\theta\|_{\mathcal{Q}} \|\zeta\|_{\mathcal{Q}}, \quad (39)$$

$$|\mathcal{F}(\mathbf{u}, \zeta)| \leq M_{\mathcal{F}} \|\mathbf{u}\|_V \|\zeta\|_{\mathcal{Q}}. \quad (40)$$

Using a standard procedure that employs Green's formula, we can obtain the following variational formulation of the contact problem (1)-(11).

### Problem *PV*

Find a displacement field  $\mathbf{u} : (0, T) \rightarrow V$ , a stress field  $\boldsymbol{\sigma} : (0, T) \rightarrow \mathcal{H}$ , and a temperature  $\theta : (0, T) \rightarrow H^1(\Omega)$  such that

$$\boldsymbol{\sigma}(t) = \mathcal{A} \varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{B}(\varepsilon(\mathbf{u}(t))) + \int_0^t \mathcal{M}(t-s) \varepsilon(\mathbf{u}(s)) ds - C_e \theta(t), \quad (41)$$

$$(\boldsymbol{\sigma}(t), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}(t)))_{\mathcal{H}} + j(\dot{\mathbf{u}}(t), \mathbf{v}) - j(\dot{\mathbf{u}}(t), \dot{\mathbf{u}}(t)) \geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V, \quad (42)$$

$$(\dot{\theta}(t), \vartheta) + \mathcal{T}(\theta(t), \vartheta) - \mathcal{F}(\mathbf{u}(t), \vartheta) + \chi(\mathbf{u}(t), \theta(t), \vartheta) = (q_{th}(t), \vartheta)_\mathcal{Q}, \quad (43)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \theta(0) = \theta_0. \quad (44)$$

Our primary existence and uniqueness result for Problem *PV* is presented in the following section.

## 4 Existence and uniqueness

**Theorem 1.** *Assume that (20)-(29) and (36)-(40) hold, Then there exists a unique solution  $(\mathbf{u}, \boldsymbol{\sigma}, \theta)$  to problem PV. Moreover, the solution has the regularity*

$$\mathbf{u} \in C^1(0, T; V), \quad (45)$$

$$\boldsymbol{\sigma} \in C(0, T; \mathcal{H}), \quad (46)$$

$$\theta \in C(0, T; L^2(\Omega)) \cap L^2(0, T; \mathcal{Q}). \quad (47)$$

The functions  $\mathbf{u}$ ,  $\boldsymbol{\sigma}$  and  $\theta$  which satisfy (41)-(44) are called a weak solution of the contact problem  $P$ .

The proof relies on formulating three intermediate problems for the displacement field, the temperature field, and the stress field, respectively. We establish the unique solvability of these intermediate problems and subsequently construct a contraction mapping. The unique fixed point of this mapping corresponds to the solution of the original problem.

Throughout this proof, we use the symbol  $C$  to represent a constant whose value may vary from one line to the next, as long as it does not cause confusion.

Given  $\eta \in C(0, T; \mathcal{H})$ , we examine the following variational problem.

### Problem $\mathcal{P}_\eta^1$

Find a displacement field  $\mathbf{u}_\eta : [0, T] \rightarrow V$  such that for all  $t \in [0, T]$

$$\begin{aligned} & (\mathcal{A}\varepsilon(\dot{\mathbf{u}}_\eta(t)), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}_\eta(t)))_{\mathcal{H}} + (\mathcal{B}\varepsilon(\mathbf{u}_\eta(t)), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}_\eta(t)))_{\mathcal{H}} \\ & + (\boldsymbol{\eta}(t), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}_\eta(t)))_{\mathcal{H}} + j(\dot{\mathbf{u}}_\eta(t), \mathbf{v}) - j(\dot{\mathbf{u}}_\eta(t), \dot{\mathbf{u}}_\eta(t)) \geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t))_V, \\ & \forall \mathbf{v} \in V, \text{ a.e. } t \in (0, T), \end{aligned} \quad (48)$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0. \quad (49)$$

The following result pertains to problem  $\mathcal{P}_\eta^1$ .

**Lemma 1.** *There exists a unique solution  $\mathbf{u}_\eta \in C^1(0, T; V)$  for the problem defined by (48) and (49).*

*Proof.* We introduce the operators  $A : V \rightarrow V$  and  $B : V \rightarrow V$  as follows:

$$(A\mathbf{u}, \mathbf{v})_V = (\mathcal{A}\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad \forall \mathbf{u}, \mathbf{v} \in V, \quad (50)$$

$$(B\mathbf{u}, \mathbf{v})_V = (\mathcal{B}\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad \forall \mathbf{u}, \mathbf{v} \in V. \quad (51)$$

Thus, we can rewrite (48) as:

$$\begin{aligned} & (A\dot{\mathbf{u}}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V + (B\mathbf{u}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V + j(\dot{\mathbf{u}}_\eta(t), \mathbf{v}) \\ & - j(\dot{\mathbf{u}}_\eta(t), \dot{\mathbf{u}}_\eta(t)) \geq (\mathbf{f}_\eta(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V, \end{aligned} \quad (52)$$

where

$$\mathbf{f}_\eta(t) = \mathbf{f}(t) - \boldsymbol{\eta}(t), \quad \text{a.e. } t \in [0, T].$$

Using equations (50)-(51) and (20)-(21), we conclude that  $A$  and  $B$  are Lipschitz continuous operators. Furthermore, from (50) and (20), we deduce that  $A$  is a strongly monotone operator on  $V$ :

$$\begin{aligned} (A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_V &= (\mathcal{A}(\varepsilon(\mathbf{u}_1)) - \mathcal{A}(\varepsilon(\mathbf{u}_2)), \varepsilon(\mathbf{u}_1) - \varepsilon(\mathbf{u}_2))_{\mathcal{H}} \\ &\geq m_{\mathcal{A}} \|\varepsilon(\mathbf{u}_1) - \varepsilon(\mathbf{u}_2)\|_{\mathcal{H}}^2 \geq C \|\mathbf{u}_1 - \mathbf{u}_2\|_V^2. \end{aligned}$$

From (24), we know that the functional  $j$  defined in (30) is continuous, and therefore, it is a convex lower semicontinuous function on  $V$ . Finally, given that  $\mathbf{f}_\eta \in C([0, T]; V)$  and  $\mathbf{u}_0 \in V$ , and by employing classical functional analysis techniques concerning evolutionary quasivariational inequalities [17], we conclude that there exists a unique solution  $\mathbf{u}_\eta \in C^1(0, T; V)$  for problem  $\mathcal{P}_\eta^1$ .  $\square$

For  $\lambda \in L^2(0, T; L^2(\Omega))$ , we consider the following variational problem.

### Problem $\mathcal{P}_\lambda$

Find the temperature field  $\theta_\lambda : (0, T) \rightarrow L^2(\Omega)$

$$\left( \dot{\theta}_\lambda(t), \vartheta \right) + \mathcal{J}(\theta_\lambda(t), \vartheta) + (\lambda(t), \vartheta) = (q_{th}(t), \vartheta)_\Omega, \quad (53)$$

$$\theta_\lambda(0) = \theta_0. \quad (54)$$

**Lemma 2.** *There exists a unique solution  $\theta_\lambda$  to the auxiliary problem  $\mathcal{P}_\lambda$  satisfying (47).*

*Proof.* Using Riesz's representation theorem, there exists an operator  $q_\lambda$  defined by

$$(q_\lambda(t), \vartheta)_\Omega = (q_{th}(t), \vartheta)_\Omega - (\lambda(t), \vartheta)_\Omega. \quad (55)$$

to get

$$\left( \dot{\theta}_\lambda(t), \vartheta \right) + \mathcal{J}(\theta_\lambda(t), \vartheta) = (q_\lambda(t), \vartheta), \quad (56)$$

$$\theta_\lambda(0) = \theta_0. \quad (57)$$

By applying (36)-(40), we conclude that the operator  $\mathcal{J}$  is both hemicontinuous and monotone. Additionally, from (55) and the regularity of  $q_{th}$ , it follows that  $q_\lambda \in L^2(0, T; \Omega)$ .

Thus, the theorem presented [16] affirms that, there exists a unique function  $\theta_\lambda$  which satisfies (47).  $\square$

Ultimately, as an outcome of these findings and leveraging the properties of the operator  $\mathcal{M}$ , the operator  $C_e$ ,  $\mathcal{F}$ , and the functions  $\chi$  for  $t \in (0, T)$ , we examine the element

$$\Lambda(\boldsymbol{\eta}, \lambda)(t) = (\Lambda^1(\boldsymbol{\eta}, \lambda)(t), \Lambda^2(\boldsymbol{\eta}, \lambda)(t)) \in \mathcal{H} \times \Omega, \quad (58)$$

defined by for all  $\mathbf{v} \in V$  and  $\vartheta \in \mathcal{Q}$

$$(\Lambda^1(\boldsymbol{\eta}, \lambda)(t), \mathbf{v})_{\mathcal{H} \times V} = (C_e \theta_\lambda(t), \varepsilon(\mathbf{v}))_{\mathcal{H}} + \left( \int_0^t \mathcal{M}(t-s) \varepsilon(\mathbf{u}(s)) ds, \varepsilon(\mathbf{v}) \right)_{\mathcal{H}}, \quad (59)$$

$$(\Lambda_2(\boldsymbol{\eta}, \lambda)(t), \vartheta) = -\mathcal{F}(\mathbf{u}_\eta(t), \vartheta) + \chi(\mathbf{u}_\eta(t), \theta_\lambda(t), \vartheta) \quad (60)$$

Here, for every  $(\boldsymbol{\eta}, \lambda) \in C(0, T; \mathcal{H} \times \mathcal{Q})$ .  $\mathbf{u}_\eta$  and  $\theta_\lambda$ , represent the displacement field, the temperature field, obtained in Lemmas 1 and 2 respectively. We have the following result.

**Lemma 3.** *The mapping  $\Lambda$  has a fixed point  $(\boldsymbol{\eta}, \lambda) \in C(0, T; \mathcal{H} \times \mathcal{Q})$ , such that  $\Lambda(\boldsymbol{\eta}^*, \lambda^*) = (\boldsymbol{\eta}^*, \lambda^*)$ .*

*Proof.* Let  $t \in (0, T)$  and  $(\boldsymbol{\eta}_1, \lambda_1), (\boldsymbol{\eta}_2, \lambda_2) \in C(0, T; \mathcal{H} \times \mathcal{Q})$ . We use the notation that  $\mathbf{u}_{\eta_i} = \mathbf{u}_i$ ,  $\theta_{\lambda_i} = \theta_i$ , and  $\boldsymbol{\sigma}_{\eta_i, \theta_i} = \boldsymbol{\sigma}_i$  for  $i = 1, 2$ .

Let us start by using (22), we have

$$\begin{aligned} \|\Lambda^1(\boldsymbol{\eta}_1, \lambda_1)(t) - \Lambda^1(\boldsymbol{\eta}_2, \lambda_2)(t)\|_{\mathcal{H}}^2 &\leq C \left( \|\theta_1(t) - \theta_2(t)\|_{\mathcal{Q}}^2 \right. \\ &\quad \left. + \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 ds \right). \end{aligned} \quad (61)$$

By similar arguments, from (23), (40) and (15) we obtain

$$\|\Lambda^2(\boldsymbol{\eta}_1, \lambda_1)(t) - \Lambda^2(\boldsymbol{\eta}_2, \lambda_2)(t)\|_{\mathcal{H}}^2 \leq C \left( \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + \|\theta_1(t) - \theta_2(t)\|_{\mathcal{Q}}^2 \right). \quad (62)$$

It follows now from (58), (61) and (62) that

$$\begin{aligned} \|\Lambda(\boldsymbol{\eta}_1, \lambda_1)(t) - \Lambda(\boldsymbol{\eta}_2, \lambda_2)(t)\|_{\mathcal{H}}^2 &\leq C \left( \|\theta_1(t) - \theta_2(t)\|_{\mathcal{Q}}^2 + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 \right. \\ &\quad \left. + \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 ds \right). \end{aligned} \quad (63)$$

Using inequality (48) for  $\boldsymbol{\eta} = \boldsymbol{\eta}_1$ , we find

$$\begin{aligned} &(\mathcal{A}\varepsilon(\dot{\mathbf{u}}_1(t)), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}_1(t)))_{\mathcal{H}} + (\mathcal{B}\varepsilon(\mathbf{u}_1(t)), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}_1(t)))_{\mathcal{H}} \\ &+ (\boldsymbol{\eta}_1(t), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}_1(t)))_{\mathcal{H}} + j(\dot{\mathbf{u}}_1(t), \mathbf{v}) - j(\dot{\mathbf{u}}_1(t), \dot{\mathbf{u}}_1(t)) \\ &\geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}_1(t))_V, \forall \mathbf{v} \in V, \text{ a.e. } t \in (0, T), \end{aligned} \quad (64)$$

for  $\boldsymbol{\eta} = \boldsymbol{\eta}_2$ , we find

$$\begin{aligned} &(\mathcal{A}\varepsilon(\dot{\mathbf{u}}_2(t)), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}_2(t)))_{\mathcal{H}} + (\mathcal{B}\varepsilon(\mathbf{u}_2(t)), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}_2(t)))_{\mathcal{H}} \\ &+ (\boldsymbol{\eta}_2(t), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}_2(t)))_{\mathcal{H}} + j(\dot{\mathbf{u}}_2(t), \mathbf{v}) - j(\dot{\mathbf{u}}_2(t), \dot{\mathbf{u}}_2(t)) \\ &\geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}_2(t))_V, \forall \mathbf{v} \in V, \text{ a.e. } t \in (0, T), \end{aligned} \quad (65)$$

we take  $\mathbf{v} = \dot{\mathbf{u}}_2(t)$  in (64) and  $\mathbf{v} = \dot{\mathbf{u}}_1(t)$  in (65), add the two inequalities to obtain

$$\begin{aligned} & (\mathcal{A}\varepsilon(\dot{\mathbf{u}}_1(t)) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}_2(t)), \varepsilon(\dot{\mathbf{u}}_1(t)) - \varepsilon(\dot{\mathbf{u}}_2(t)))_{\mathcal{H}} \\ & \leq (\mathcal{B}\varepsilon(\mathbf{u}_1(t)) - \mathcal{B}\varepsilon(\mathbf{u}_2(t)), \varepsilon(\dot{\mathbf{u}}_2(t)) - \varepsilon(\dot{\mathbf{u}}_1(t)))_{\mathcal{H}} \\ & \quad + (\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t), \varepsilon(\dot{\mathbf{u}}_2(t)) - \varepsilon(\dot{\mathbf{u}}_1(t))) + j(\dot{\mathbf{u}}_1(t), \dot{\mathbf{u}}_2(t)) \\ & \quad - j(\dot{\mathbf{u}}_1(t), \dot{\mathbf{u}}_1(t)) + j(\dot{\mathbf{u}}_2(t), \dot{\mathbf{u}}_1(t)) - j(\dot{\mathbf{u}}_2(t), \dot{\mathbf{u}}_2(t)), \end{aligned}$$

then we use assumptions (20), (21) and (24) to find

$$\begin{aligned} m_{\mathcal{A}} \|\dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2\|_V^2 & \leq L_{\mathcal{B}} \|\mathbf{u}_1 - \mathbf{u}_2\|_V \|\dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2\|_V + \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{\mathcal{H}} \|\dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2\|_V \\ & \quad + c_0^2(L_{\nu} + L_{\tau}) \|\dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2\|_V^2 \end{aligned} \quad (66)$$

It follows that

$$\|\dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2\|_V \leq C (\|\mathbf{u}_1 - \mathbf{u}_2\|_V + \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{\mathcal{H}}). \quad (67)$$

Since  $\mathbf{u}_i(t) = \int_0^t \dot{\mathbf{u}}_i(s) ds + \mathbf{u}_0, \forall t \in [0, T]$ , we have

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V \leq \int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V ds. \quad (68)$$

Then,

$$\|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|_V \leq c \left( \|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_Q + \int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V ds \right) \quad (69)$$

This inequality implies that

$$\begin{aligned} \int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V ds & \leq c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_Q ds \\ & \quad + c \int_0^t \int_0^s \|\dot{\mathbf{u}}_1(r) - \dot{\mathbf{u}}_2(r)\|_V dr ds \end{aligned}$$

It follows now from the Gronwall inequality that

$$\int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V ds \leq c \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_Q ds \quad (70)$$

From (53) we deduce that

$$\left( \dot{\theta}_1 - \dot{\theta}_2, \theta_1 - \theta_2 \right) + \mathcal{T}(\theta_1 - \theta_2, \theta_1 - \theta_2) + (\lambda_1 - \lambda_2, \theta_1 - \theta_2) = 0.$$

Integrating this equality with respect to time, employing the initial conditions  $\theta_1(0) = \theta_2(0) = \theta_0$ , and using the inequality  $\mathcal{T}(\theta_1 - \theta_2, \theta_1 - \theta_2) \geq 0$ , we find

$$\frac{1}{2} \|\theta_1(t) - \theta_2(t)\|_{\Omega}^2 \leq \int_0^t (\lambda_1(s) - \lambda_2(s), \theta_1(s) - \theta_2(s))_{\Omega} ds,$$

which implies that

$$\|\theta_1(t) - \theta_2(t)\|_{\mathcal{Q}}^2 \leq \int_0^t \|\lambda_1(s) - \lambda_2(s)\|_{\mathcal{Q}}^2 ds + \int_0^t \|\theta_1(s) - \theta_2(s)\|_{\mathcal{Q}}^2 ds.$$

Combining this inequality with Gronwall's inequality leads to

$$\|\theta_1(t) - \theta_2(t)\|_{\mathcal{Q}}^2 \leq C \int_0^t \|\lambda_1(s) - \lambda_2(s)\|_{\mathcal{Q}}^2 ds. \quad (71)$$

Form the estimates (63), (70) and (71) it follows now that

$$\begin{aligned} & \|\Lambda(\boldsymbol{\eta}_1, \lambda_1)(t) - \Lambda(\boldsymbol{\eta}_2, \lambda_2)(t)\|_{\mathcal{H} \times \mathcal{Q}}^2 \\ & \leq C \int_0^T \|(\boldsymbol{\eta}_1, \lambda_1)(s) - (\boldsymbol{\eta}_2, \lambda_2)(s)\|_{\mathcal{H} \times \mathcal{Q}}^2 ds. \end{aligned} \quad (72)$$

Repeating this inequality  $m$  times, we obtain

$$\begin{aligned} & \|\Lambda^m(\boldsymbol{\eta}_1, \lambda_1) - \Lambda^m(\boldsymbol{\eta}_2, \lambda_2)\|_{C(0, T; \mathcal{H} \times \mathcal{Q})}^2 \\ & \leq \frac{C^m T^m}{m!} \|(\boldsymbol{\eta}_1, \lambda_1) - (\boldsymbol{\eta}_2, \lambda_2)\|_{C(0, T; \mathcal{H} \times \mathcal{Q})}^2. \end{aligned}$$

Indeed, for sufficiently large  $m$ , the operator  $\Lambda^m$  acts as a contraction on the Banach space  $C(0, T; \mathcal{H} \times \mathcal{Q})$ . Consequently, the operator  $\Lambda$  has a unique fixed point.  $\square$

## Existence

Consider  $(\boldsymbol{\eta}^*, \lambda^*) \in C(0, T; \mathcal{H} \times \mathcal{Q})$  as the fixed point of the operator  $\Lambda$ , defined by (58)-(60). Let  $(\mathbf{u}_\eta, \boldsymbol{\sigma}_\eta)$  be the solution to problem  $\mathcal{P}_\eta^1$ , and let  $\theta_{\lambda^*}$  be the solution to problem  $\mathcal{P}_\lambda$  for  $\lambda = \lambda^*$ . The equalities  $\Lambda^1(\boldsymbol{\eta}^*, \lambda^*) = \boldsymbol{\eta}^*$ ,  $\Lambda^2(\boldsymbol{\eta}^*, \lambda^*) = \lambda^*$ , along with (59)-(60), demonstrate that conditions (41)-(43) are satisfied. Consequently, (44) and the regularity conditions (45)-(47) follow from Lemmas 2 and 3, thereby concluding the existence proof.

## Uniqueness

The uniqueness of the solution arises from the uniqueness of the fixed point of the operator  $\Lambda$ , combined with the unique solvability of problems  $\mathcal{P}_\eta^1$ ,  $\mathcal{P}_\lambda$  which completes the proof.

## 5 A Continuous Dependence Result

In this section we study the dependence of the solution of Problem  $PV$  with respect to the contact conditions. We assume that the hypotheses (20)-(29) and (34)-(40) are verified and let  $(\mathbf{u}, \boldsymbol{\sigma}, \theta)$  be the solution of Problem  $PV$  obtained by Theorem 1. Also, for all  $n > 0$  we denote by  $p_\nu^n$  a perturbation of  $p_\nu$  which satisfies (24) such that  $L_\nu$  is replaced by  $L_\nu^n$ . We introduce the functional  $j^n$  defined by (30), replacing  $p_\nu$  by  $p_\nu^n$  and we consider the following variational problem

**Problem  $P_V^n$** 

Find a displacement field  $\mathbf{u}^n : (0, T) \rightarrow V$ , a stress field  $\boldsymbol{\sigma}^n : (0, T) \rightarrow \mathcal{H}$ , and a temperature  $\theta^n : (0, T) \rightarrow H^1(\Omega)$  such that

$$\begin{aligned} \boldsymbol{\sigma}^n(t) = & \mathcal{A}\varepsilon(\dot{\mathbf{u}}^n(t)) + \mathcal{B}\varepsilon(\mathbf{u}^n(t)) \\ & + \int_0^t \mathcal{M}(t-s)\varepsilon(\mathbf{u}^n(s)) ds - C_e\theta^n(t), \end{aligned} \quad (73)$$

$$(\boldsymbol{\sigma}^n(t), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}^n(t)))_{\mathcal{H}} + j^n(\dot{\mathbf{u}}^n(t), \mathbf{v}) - j^n(\dot{\mathbf{u}}^n(t), \dot{\mathbf{u}}^n(t)) \geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}^n(t))_V, \quad (74)$$

$$(\dot{\theta}^n(t), \vartheta) + \mathcal{T}(\theta^n(t), \vartheta) - \mathcal{F}(\mathbf{u}^n(t), \vartheta) + \chi(\mathbf{u}^n(t), \theta^n(t), \vartheta) = (q_{th}(t), \vartheta)_{\Omega}, \quad (75)$$

$$\mathbf{u}^n(0) = \mathbf{u}_0, \quad \theta^n(0) = \theta_0. \quad (76)$$

It follows from Theorem 1 that for all  $n > 0$ , the Problem  $P_V^n$  admits a unique solution  $\mathbf{u}^n, \boldsymbol{\sigma}^n, \theta^n$  with the regularity (45)-(47). Let us now consider the following hypotheses on the operators  $\mathcal{M}^n$  and  $\mathcal{M}$  as well as on the normal compliance functions  $p_e^n$  and  $p_e$  ( $e = \tau, \nu$ ).

$$\left\{ \begin{array}{l} \text{There exists } \omega_\nu \in \mathbb{R} \text{ and } G_\nu : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ such that:} \\ (i) |p_e^n(x, r) - p_e(x, r)| \leq G_e(n), \quad \forall r \in \mathbb{R}^N, \text{ a.e. } x \in \Gamma_3, \\ (ii) \lim_{n \rightarrow 0} G_e(n) = 0 \end{array} \right. \quad (77)$$

$$\mathcal{M}^n \rightarrow \mathcal{M} \quad \text{in } C(\mathbb{R}_+, E), \quad \text{as } n \rightarrow 0. \quad (78)$$

With these considerations, we have the following convergence result

**Theorem 2.** *The solution  $(\mathbf{u}^n, \boldsymbol{\sigma}^n, \theta^n)$  of Problem  $P_V^n$  converges to the solution  $(\mathbf{u}, \boldsymbol{\sigma}, \theta)$  of Problem  $PV$ , i.e.*

$$\mathbf{u}^n \rightarrow \mathbf{u} \quad \text{in } C^1(0, T; V), \quad (79)$$

$$\boldsymbol{\sigma}^n \rightarrow \boldsymbol{\sigma} \quad \text{in } C(0, T; \mathcal{H}), \quad (80)$$

$$\theta^n \rightarrow \theta \quad \text{in } C(0, T; \mathcal{Q}) \text{ as } n \rightarrow 0. \quad (81)$$

Besides its significance in the context of asymptotic analysis, this convergence result holds mechanical importance as it suggests that minor alterations in the contact conditions correspond to minor perturbations in the weak solution of problem  $P$ .

For all subsequent discussions, let  $n > 0$ . In the expressions that follow, the symbol  $C$  always represents a positive constant. This constant may rely on the problem's data and the solution  $(\mathbf{u}, \boldsymbol{\sigma}, \theta)$  but remains independent of both  $n$  and the time variable. Its specific value can vary from one context to another.

*Proof.* Let  $n > 0$  and  $t \in [0, T]$ , we take  $v = \dot{\mathbf{u}}(t)$  in (74) and  $v = \dot{\mathbf{u}}^n(t)$  in (42)

and we add the resulting inequalities to obtain

$$\begin{aligned}
& (\mathcal{A}\varepsilon(\dot{\mathbf{u}}^n(t)) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)), \varepsilon(\dot{\mathbf{u}}^n(t)) - \varepsilon(\dot{\mathbf{u}}(t)))_{\mathcal{H}} \leq \\
& (\mathcal{B}\varepsilon(\mathbf{u}^n(t)) - \mathcal{B}\varepsilon(\mathbf{u}(t)), \varepsilon(\dot{\mathbf{u}}^n(t)) - \varepsilon(\dot{\mathbf{u}}(t)))_{\mathcal{H}} \\
& + \left( \int_0^t \mathcal{M}^n(t-s)\varepsilon(\mathbf{u}^n(s)) ds - \int_0^t \mathcal{M}(t-s)\varepsilon(\mathbf{u}(s)) ds, \varepsilon(\dot{\mathbf{u}}^n(t)) - \varepsilon(\dot{\mathbf{u}}(t)) \right)_{\mathcal{H}} \\
& + (C_e\theta^n(t) - C_e\theta(t), \varepsilon(\dot{\mathbf{u}}^n(t)) - \varepsilon(\dot{\mathbf{u}}(t))) \\
& + j^n(\dot{\mathbf{u}}^n(t), \dot{\mathbf{u}}(t)) - j^n(\mathbf{u}^n(t), \dot{\mathbf{u}}^n(t)) \\
& + j(\mathbf{u}(t), \dot{\mathbf{u}}^n(t)) - j(\mathbf{u}(t), \dot{\mathbf{u}}(t)).
\end{aligned} \tag{82}$$

We use (20), (21) and (15), to obtain

$$(\mathcal{A}\varepsilon(\dot{\mathbf{u}}^n(t)) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)), \varepsilon(\dot{\mathbf{u}}^n(t)) - \varepsilon(\dot{\mathbf{u}}(t)))_{\mathcal{H}} \geq m_{\mathcal{A}} \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V^2, \tag{83}$$

and

$$(\mathcal{B}\varepsilon(\mathbf{u}^n(t)) - \mathcal{B}\varepsilon(\mathbf{u}(t)), \varepsilon(\dot{\mathbf{u}}^n(t)) - \varepsilon(\dot{\mathbf{u}}(t)))_{\mathcal{H}} \leq C \|\mathbf{u}^n(t) - \mathbf{u}(t)\|_V \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V, \tag{84}$$

and

$$\begin{aligned}
& \left( \int_0^t \mathcal{M}^n(t-s)\varepsilon(\mathbf{u}^n(s)) ds - \int_0^t \mathcal{M}(t-s)\varepsilon(\mathbf{u}(s)) ds, \varepsilon(\dot{\mathbf{u}}^n(t)) - \varepsilon(\dot{\mathbf{u}}(t)) \right)_{\mathcal{H}} \leq \\
& \left\| \int_0^t \mathcal{M}^n(t-s)\varepsilon(\mathbf{u}^n(s)) ds - \int_0^t \mathcal{M}(t-s)\varepsilon(\mathbf{u}(s)) ds \right\|_{\mathcal{H}} \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V.
\end{aligned} \tag{85}$$

We write

$$\begin{aligned}
\mathcal{M}^n(t-s)\varepsilon(\mathbf{u}^n(s)) - \mathcal{M}(t-s)\varepsilon(\mathbf{u}(s)) &= \mathcal{M}^n(t-s)\varepsilon((\mathbf{u}^n(s)) - \varepsilon(\mathbf{u}(s))) \\
&+ (\mathcal{M}^n(t-s) - \mathcal{M}(t-s))\varepsilon(\mathbf{u}(s)),
\end{aligned}$$

then, we obtain

$$\begin{aligned}
& \left( \int_0^t \mathcal{M}^n(t-s)\varepsilon(\mathbf{u}^n(s)) ds - \int_0^t \mathcal{M}(t-s)\varepsilon(\mathbf{u}(s)) ds, \varepsilon(\dot{\mathbf{u}}(t)) - \varepsilon(\dot{\mathbf{u}}^n(t)) \right)_{\mathcal{H}} \leq \\
& \left( \max_{r \in [0, m]} \|\mathcal{M}^n(r)\|_E \int_0^t \|\mathbf{u}^n(s) - \mathbf{u}(s)\|_V ds \right. \\
& \quad \left. + \max_{r \in [0, m]} \|\mathcal{M}^n(r) - \mathcal{M}(r)\|_E \int_0^t \|\mathbf{u}(s)\|_V ds \right) \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V \\
& \leq d_n \int_0^t \|\mathbf{u}^n(s) - \mathbf{u}(s)\|_V ds \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V \\
& \quad + \xi_m^n \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V,
\end{aligned} \tag{86}$$

where

$$\begin{aligned}\xi_m^n &= \max_{r \in [0, m]} \|\mathcal{M}^n(r) - \mathcal{M}(r)\|_E \int_0^t \|\mathbf{u}(s)\|_V ds, \\ d_n &= \max_{r \in [0, m]} \|\mathcal{M}^n(r)\|_E, \quad m \in \mathbb{N}, \quad t \in [0, m].\end{aligned}$$

Next, we replace  $v$  by  $\theta - \theta^n$  in (43) and in (75) we obtain

$$\begin{aligned}& \left( \dot{\theta}(t) - \dot{\theta}^n(t), \theta(t) - \theta^n(t) \right) + \mathcal{T}(\theta(t) - \theta^n(t), \theta(t) - \theta^n(t)) \\ & - \mathcal{F}(\mathbf{u}(t) - \mathbf{u}^n(t), \theta(t) - \theta^n(t)) + \chi(\mathbf{u}(t), \theta(t), \theta(t) - \theta^n(t)) \\ & - \chi(\mathbf{u}^n(t), \theta^n(t), \theta(t) - \theta^n(t)) = 0.\end{aligned}\tag{87}$$

It follows now from (38) and (40) that

$$\begin{aligned}m_{\mathcal{T}} \|\theta(t) - \theta^n(t)\|_{\mathbb{Q}}^2 + \frac{1}{2} \frac{d}{dt} \|\theta(t) - \theta^n(t)\|_{\mathbb{Q}}^2 &\leq \\ M_{\mathcal{F}} \|\mathbf{u}(t) - \mathbf{u}^n(t)\|_V \|\theta(t) - \theta^n(t)\|_{\mathbb{Q}} + |R_{\chi}|,\end{aligned}\tag{88}$$

use (15), (23) and (29) to obtain

$$|R_{\chi}| \leq M_{n_c} \cdot L \cdot \tilde{c}_0^2 \|\theta(t) - \theta^n(t)\|_{\mathbb{Q}}^2.\tag{89}$$

We deduce that

$$\|\theta(t) - \theta^n(t)\|_{\mathbb{Q}}^2 \leq C \int_0^t \|\mathbf{u}(s) - \mathbf{u}^n(s)\|_V^2 ds.\tag{90}$$

We use the definition of the functional  $j$  and  $j^n$ , to obtain

$$\begin{aligned}& j^n(\mathbf{u}^n(t), \dot{\mathbf{u}}(t)) - j^n(\mathbf{u}^n(t), \dot{\mathbf{u}}^n(t)) + j(\mathbf{u}(t), \dot{\mathbf{u}}^n(t)) - j(\mathbf{u}(t), \dot{\mathbf{u}}(t)) \\ & \leq \int_{\Gamma_3} (p_v^n(u_v) - p_v(u_v)) (\dot{u}_v - \dot{u}_v^n) da \\ & + \int_{\Gamma_3} (p_{\tau}^n(u_v) - p_{\tau}(u_v)) (|\dot{\mathbf{u}}_{\tau}| - |\dot{\mathbf{u}}_{\tau}^n|) da \\ & \leq \int_{\Gamma_3} |p_v^n(u_v) - p_v(u_v)| |\dot{u}_v - \dot{u}_v^n| da \\ & + \int_{\Gamma_3} |p_{\tau}^n(u_v) - p_{\tau}(u_v)| \left| |\dot{\mathbf{u}}_{\tau}| - |\dot{\mathbf{u}}_{\tau}^n| \right| da.\end{aligned}\tag{91}$$

Then, using the inequality (15) and the hypothesis (77) and after some calculations we find that

$$\begin{aligned}& j^n(\mathbf{u}^n(t), \dot{\mathbf{u}}^n(t)) - j^n(\mathbf{u}^n(t), \mathbf{u}(t)) + j(\mathbf{u}(t), \dot{\mathbf{u}}(t)) - j(\mathbf{u}(t), \mathbf{u}^n(t)) \\ & \leq \text{mes}(\Gamma_3)^{1/2} (\Gamma_3) c_0 (G_{\nu}(n) + G_{\tau}(n)) \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V.\end{aligned}\tag{92}$$

We note by  $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  the function given by

$$G(n) = \text{meas}(\Gamma_3)^{1/2} c_0 (G_{\nu}(n) + G_{\tau}(n)).\tag{93}$$

We use (82)–(93) to obtain

$$\begin{aligned}
 m_{\mathcal{A}} \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V^2 &\leq C \|\mathbf{u}^n(t) - \mathbf{u}(t)\|_V \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V \\
 &+ d_n \int_0^t \|\mathbf{u}^n(s) - \mathbf{u}(s)\|_V ds \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V \\
 &+ \xi_m^n \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V \\
 &+ C \|\theta^n(t) - \theta(t)\|_V \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V \\
 &+ \text{meas}(\Gamma_3)^{1/2} c_0 (G_\nu(n) + G_\tau(n)) \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V.
 \end{aligned} \tag{94}$$

We use to find

$$\begin{aligned}
 m_{\mathcal{A}} \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V^2 &\leq (C \|\mathbf{u}^n(t) - \mathbf{u}(t)\|_V \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V \\
 &+ d_n \int_0^t \|\mathbf{u}^n(s) - \mathbf{u}(s)\|_V ds \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V \\
 &+ \xi_m^n \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V \\
 &+ C \int_0^t \|\mathbf{u}(s) - \mathbf{u}^n(s)\|_V ds \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V \\
 &+ \text{meas}(\Gamma_3)^{1/2} c_0 (G_\nu(n) + G_\tau(n)) \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V).
 \end{aligned} \tag{95}$$

This means

$$\begin{aligned}
 m_{\mathcal{A}} \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V^2 &\leq C \left( \|\mathbf{u}^n(t) - \mathbf{u}(t)\|_V + d_n \int_0^t \|\mathbf{u}^n(s) - \mathbf{u}(s)\|_V ds \right. \\
 &\left. + \xi_m^n + \int_0^t \|\mathbf{u}(s) - \mathbf{u}^n(s)\|_V ds + (G_\nu(n) + G_\tau(n)) \right) \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V.
 \end{aligned} \tag{96}$$

By inequality

$$ab \leq \frac{1}{2m_{\mathcal{A}}} a^2 + \frac{m_{\mathcal{A}}}{2} b^2,$$

and after calculation, we find

$$\begin{aligned}
 \frac{m_{\mathcal{A}}}{2} \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V^2 &\leq C \left( (G_\nu(n) + G_\tau(n))^2 + \|\mathbf{u}^n(t) - \mathbf{u}(t)\|_V^2 \right. \\
 &\left. + \xi_m^{n2} + (1 + d_n)^2 \int_0^t \|\mathbf{u}^n(s) - \mathbf{u}(s)\|_V^2 ds \right).
 \end{aligned} \tag{97}$$

By using (68), we find

$$\begin{aligned}
 \frac{m_{\mathcal{A}}}{2} \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V^2 &\leq C \left( (G_\nu(n) + G_\tau(n))^2 + \int_0^t \|\dot{\mathbf{u}}^n(s) - \dot{\mathbf{u}}(s)\|_V^2 ds \right. \\
 &\left. + \xi_m^{n2} + (1 + d_n)^2 \int_0^t \|\mathbf{u}^n(s) - \mathbf{u}(s)\|_V^2 ds \right).
 \end{aligned} \tag{98}$$

With Gronwall inequality we obtain

$$\int_0^t \|\dot{\mathbf{u}}^n(s) - \dot{\mathbf{u}}(s)\|_V^2 ds \leq C \left( (G_\nu(n) + G_\tau(n))^2 + \xi_m^{n2} + (1 + d_n)^2 \int_0^t \|\mathbf{u}^n(s) - \mathbf{u}(s)\|_V^2 ds \right), \quad (99)$$

which turns into

$$\|\mathbf{u}^n(t) - \mathbf{u}(t)\|_V^2 \leq C \left( (G_\nu(n) + G_\tau(n))^2 + \xi_m^{n2} + (1 + d_n)^2 \int_0^t \|\mathbf{u}^n(s) - \mathbf{u}(s)\|_V^2 ds \right). \quad (100)$$

Again, by Gronwall inequality we have

$$\|\mathbf{u}^n(t) - \mathbf{u}(t)\|_V^2 \leq C \left( (G_\nu(n) + G_\tau(n))^2 + \xi_m^{n2} \right) e^{(1+d_n)^2 t} \quad (101)$$

and therefore,

$$\max_{t \in [0, m]} \|\mathbf{u}^n(t) - \mathbf{u}(t)\|_V^2 \leq C \left( (G_\nu(n) + G_\tau(n))^2 + \xi_m^{n2} \right) e^{(1+d_n)^2 m} \quad (102)$$

Then, we apply the convergence definition (18) along with assumptions (77)-(78) to conclude that

$$(G_\nu(n) + G_\tau(n))^2 \rightarrow 0, \quad \xi_m^{n2} \rightarrow 0 \quad \text{as } n \rightarrow 0. \quad (103)$$

Based on (102)–(103), we conclude that

$$\max_{t \in [0, m]} \|\mathbf{u}^n(t) - \mathbf{u}(t)\|_V^2 \rightarrow 0 \quad \text{as } n \rightarrow 0. \quad (104)$$

We use (97), and the convergence (103), to conclude that

$$\max_{t \in [0, m]} \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V^2 \rightarrow 0 \quad \text{as } n \rightarrow 0. \quad (105)$$

We use (73), (20), (21), (22) and (86)

$$\begin{aligned} \|\boldsymbol{\sigma}^n(t) - \boldsymbol{\sigma}(t)\|_{\mathcal{H}} &\leq C \left( \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V + \|\mathbf{u}^n(t) - \mathbf{u}(t)\|_V \right. \\ &\quad \left. + d_n \int_0^t \|\mathbf{u}^n(s) - \mathbf{u}(s)\|_V ds + \xi_m^n + \|\theta^n(t) - \theta(t)\|_V \right). \end{aligned} \quad (106)$$

We use inequality (90) to obtain

$$\begin{aligned} \|\boldsymbol{\sigma}^n(t) - \boldsymbol{\sigma}(t)\|_{\mathcal{H}} &\leq C \left( \|\dot{\mathbf{u}}^n(t) - \dot{\mathbf{u}}(t)\|_V + \|\mathbf{u}^n(t) - \mathbf{u}(t)\|_V \right. \\ &\quad \left. + (1 + d_n) \int_0^t \|\mathbf{u}^n(s) - \mathbf{u}(s)\|_V ds + \xi_m^n \right). \end{aligned} \quad (107)$$

We use (103)-(105), to conclude that

$$\max_{t \in [0, m]} \|\boldsymbol{\sigma}^n(t) - \boldsymbol{\sigma}(t)\|_V^2 \rightarrow 0 \quad \text{as } n \rightarrow 0. \quad (108)$$

We utilize (90), to conclude that

$$\max_{t \in [0, m]} \|\theta^n(t) - \theta(t)\|_V^2 \rightarrow 0 \quad \text{as } n \rightarrow 0. \quad (109)$$

We apply (104)–(105), (108)–(109) and the convergence definition (18)–(19) to obtain the following convergences:

$$u^n \rightarrow \mathbf{u} \quad \text{in } C^1(0, T; V), \quad (110)$$

$$\boldsymbol{\sigma}^n \rightarrow \boldsymbol{\sigma} \quad \text{in } C(0, T; \mathcal{H}), \quad (111)$$

$$\theta^n \rightarrow \theta \quad \text{in } C(0, T; \mathcal{Q}) \text{ as } n \rightarrow 0. \quad (112)$$

□

## References

- [1] Amassad, A. and Sofonea, M., *Analysis of a quasistatic viscoplastic problem involving Tresca friction law*, Discrete Contin. Dyn. Syst. **4** (1998), 55–72.
- [2] Anderson, L.-E., *A quasistatic frictional problem with normal compliance*, Nonlinear Anal. **16** (1991), 347–370.
- [3] Awbi, B., Essoufi, El-H. and Sofonea, M., *A viscoelastic contact problem with normal damped response and friction*, Ann. Polon. Math. **75** (2000), no. 3, 233–246.
- [4] Bouallala, M. and Essoufi, E. H., *Analysis results for dynamic contact problem with friction in thermo-viscoelasticity*, Methods Funct. Anal. Topology **26** (2020), no. 4, 317–326.
- [5] Chandrasekhariah, D. S., *A generalized linear thermoelasticity theory of piezoelectric media*, Acta Mech. **71** (1988), 39–49.
- [6] Chau, O., Han, W. and Sofonea, M., *A dynamic frictional contact problem with normal damped response*, Acta Appl. Math. **71** (2002), 159–178.
- [7] Chau, O., Fernandez, J. R., Shillor, M. and Sofonea, M., *Variational and numerical analysis of a quasistatic viscoelastic contact problem with adhesion*, J. Comput. Appl. Math. **159** (2003), 431–465.
- [8] Dai, H. L. and Wang, X., *Thermo-electro-elastic transient responses in piezoelectric hollow structures*, Int. J. Solids Struct. **42** (2005), 1151–1171.

- [9] Hamidat, A. and Aissaoui, A., *A quasistatic frictional contact problem with normal damped response for thermo-electro-elastic-viscoelastic bodies*, Adv. Math. Sci. J. **10** (2021), no. 12.
- [10] Hamidat, A. and Aissaoui, A., *A quasi-static contact problem with friction in electro viscoelasticity with long-term memory body with damage and thermal effects*, Int. J. Nonlinear Anal. Appl. (2022).
- [11] Han, W. and Sofonea, M., *Evolutionary variational inequalities arising in viscoelastic contact problems*, SIAM J. Numer. Anal. **38** (2000), 556–579.
- [12] Han, W. and Sofonea, M., *Quasistatic contact problems in viscoelasticity and viscoplasticity*, Studies in Advanced Mathematics, American Mathematical Society and International Press, 2002.
- [13] Necas, J. and Hlaváček, I., *Mathematical theory of elastic and elastoplastic bodies: an introduction*, Elsevier, Amsterdam, 1981.
- [14] Rochdi, M., Shillor, M. and Sofonea, M., *Quasistatic viscoelastic contact with normal compliance and friction*, J. Elasticity **51** (1998), 105-126.
- [15] Shillor, M., Sofonea, M. and Telega, J. J., *Models and analysis of quasistatic contact*, Lecture Notes in Physics, Vol. 655, Springer, Berlin, 2004.
- [16] Sofonea, M., Han, W. and Shillor, M., *Analysis and approximations of contact problems with adhesion or damage*, Pure and Applied Mathematics, Chapman and Hall/CRC Press, Boca Raton, Florida, 2006.
- [17] Sofonea, M. and Matei, A., *Mathematical models in contact mechanics*, London Mathematical Society Lecture Note Series, Vol. 398, Cambridge University Press, 2012.