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UNIVALENCE CRITERIA FOR CERTAIN INTEGRAL OPERATOR

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Abstract

Dedicated to the memory of Professor Petrică Dicu

In this paper we introduce a new general integral operator for analytic functions in the open unit disk $\mathbb{U}$ in $\mathbb{C}$ as an extension of the integral operator defined by P. Dicu, M. Dicu and A.F. Albişoru. We focus our attention on some sufficient conditions for the univalence of the integral operator. We end this paper with a remark on a modification of the studied integral operator.

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1 Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc $\mathbb{U} = \left\{z \in \mathbb{C} : |z| < 1\right\}$ in $\mathbb{C}$ and satisfy the usual normalization $f(0) = 0$ and $f'(0) = 1$. Also, let us denote by S the subclass of $\mathcal{A}$ that contains the functions $f \in \mathcal{A}$ that are also univalent in the unit disc $\mathbb{U}$ (for details, one may consult [5], [7], [8], [9] or [11]).

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Recall that Bucur, Andrei and Breaz (see [3]) introduced the integral operator $J_\alpha : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ defined by

$$J_\alpha(f,g)(z) = \int_0^z \left[\frac{e^{f(t)}}{g'(t)}\right]^\alpha dt, \quad z \in \mathbb{U}, \tag{1}$$

where $\alpha \in \mathbb{C}$ has the property that $\mathfrak{Re}\alpha \geq 1$. For the operator J_α the authors obtained univalence conditions, starlikeness and convexity properties.

A first generalization of the integral operator J_α was introduced by Dicu, Dicu and Albişoru in [4]. The authors studied the integral operators

$$F_{n,\beta}(z) = \left[\beta \int_0^z t^{\beta-1} \prod_{j=1}^n \frac{e^{f_j(t)}}{g_j'(t)} dt\right]^{\frac{1}{\beta}}, \tag{2}$$

for $\beta \in \mathbb{C}$ with $\beta \neq 0$ and $f_j, g_j \in \mathcal{A}$, for $j \in \{1, ..., n\}$, respectively

$$G_{\beta,\gamma}(f,g)(z) = \left\{\beta \int_0^z t^{\beta-1} \left[\frac{e^{f(t)}}{g'(t)}\right]^\gamma dt\right\}^{\frac{1}{\beta}}, \tag{3}$$

where $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $f, g \in \mathcal{A}$. It is easy to observe that $G_{\beta,1} = F_{1,\beta}$.

In our paper we try to extend the ideas presented above, combining the integral operators $F_{n,\beta}$ and $G_{\beta,\gamma}$ in the form of a new integral operator $G_{n,\beta}$ that will be defined in the next section. For the operator $G_{n,\beta}$ we prove some sufficient conditions of univalence on $\mathbb{U}$ together with other particular properties.

2 Preliminaries

Let $\beta, \gamma_1, ..., \gamma_n \in \mathbb{C}$ be such that $\beta \neq 0$. For every $f_j, g_j \in \mathcal{A}$, with $j \in \{1, ..., n\}$, we introduce the integral operator $G_{n,\beta} : \mathcal{A}^n \times \mathcal{A}^n \to \mathcal{A}$ given by

$$G_{n,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1} \prod_{j=1}^n \left[\frac{e^{f_j(t)}}{g_j'(t)}\right]^{\gamma_j} dt\right\}^{\frac{1}{\beta}}, \quad z \in \mathbb{U}. \tag{4}$$

In this paper we study this operator together with some natural modifications of it (see Section 4). For this type of integral operators we present sufficient conditions of univalence on $\mathbb{U}$.

Remark 1. *It is clear that $G_{1,\beta} = G_{\beta,\gamma}$ defined by (3), for $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$. Moreover, if $\beta = 1$, then $G_{n,1}$ is a generalization of the operator J_α. In particular, $G_{1,1} = J_\alpha$, for $\alpha = \gamma \in \mathbb{C}$. Other important results related to similar integral operators were obtained by Bărbatu and Breaz (see e.g. [1]), Blezu and Pascu (see e.g. [2]), Frasin (see e.g. [6]), Pescar (see e.g. [13], [15], [14] and [16]) and Stanciu (see e.g. [17]). See also [3] and [4].*

Together with the definition presented above, let us recall some useful results obtained by Pascu (see e.g. [12]; see also [2]), respectively by Pescar (see e.g. [13]) related to the univalence of a general integral operator in $\mathbb{U}$.

Lemma 1. *Let $f \in \mathcal{A}$ and $\alpha \in \mathbb{C}$ with $\mathfrak{Re}\alpha > 0$. If*

$$\frac{1-|z|^{2\mathfrak{Re}\alpha}}{\mathfrak{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \quad z \in \mathbb{U}, \tag{5}$$

then for all $\beta \in \mathbb{C}$ with $\mathfrak{Re}\beta \ge \mathfrak{Re}\alpha$, the function

$$F_\beta(z) = \left[\beta\int_0^z t^{\beta-1}f'(t)dt\right]^{\frac{1}{\beta}} \tag{6}$$

is analytic and univalent on $\mathbb{U}$.

A natural modification of the operator F_β was considered by Pescar in [13]. He obtained a sufficient condition of univalence for this new integral operator on $\mathbb{U}$.

Lemma 2. *Let $f \in \mathcal{A}$, $\beta \in \mathbb{R}$ with $\beta \ge 1$ and let $c \in \mathbb{C}$ be such that $|c| \le \frac{1}{\beta}$ and $c \ne -1$. If*

$$\left|\frac{f''(z)}{f'(z)}\right| \le 1, \quad z \in \mathbb{U},$$

then the integral operator

$$H_\beta(z) = \left\{\beta\int_0^z \left[tf'(t)\right]^{\beta-1}dt\right\}^{\frac{1}{\beta}}$$

is analytic and univalent on $\mathbb{U}$.

Another important result that will be used in the main part of the paper is the generalized Schwarz's lemma (see e.g. [5], [8]).

Lemma 3. *Let $f : \mathbb{U} \to \mathbb{C}$ be a holomorphic function on $\mathbb{U}$ such that $|f(z)| < M$, where $M > 0$ is a fixed real constant. If in the origin f has one zero with the multiplicity order greater that $m \in \mathbb{N}$, then*

$$|f(z)| \le M|z|^m, \quad z \in \mathbb{U}.$$

The equality holds in the previous relation for $z \ne 0$ if $f(z) = e^{i\theta}Mz^m$, for all $z \in \mathbb{U}$ and $\theta \in \mathbb{R}$ a given constant.

An important subclass of $\mathcal{A}$ that can be considered in this context was introduced by Liu and Yang in [10]. For $\alpha \in (0, 2]$ they defined the family $S(\alpha)$ of functions $f \in \mathcal{A}$ that satisfy the following two properties (for details, one may consult [10]; see also [4]):

- $f(z) \ne 0$, for all $z \in \mathbb{U} \setminus \{0\}$
- $\left|\left(\frac{z}{f(z)}\right)''\right| \le \alpha$, for all $z \in \mathbb{U}$.

Lemma 4. *Let $\alpha \in (0, 2]$. If $f \in S(\alpha)$, then*

$$\left|\frac{z^2f'(z)}{[f(z)]^2} - 1\right| \le \alpha|z|^2, \quad z \in \mathbb{U}. \tag{7}$$

3 Main results

Our main results give sufficient conditions for the integral operator $G_{n,\beta}$ defined by (4) to be univalent in the open disk $\mathbb{U}$. The first theorem presented in this section is a generalization of [4, Theorem 3]. Based on the univalence criteria proved in the main result we can also obtain two consequences related to the integral operator $G_{n,\beta}$. Note that the proof of Theorem 1 follows the ideas presented in the proof of Theorem 3 in [4].

Theorem 1. *Let $f_j, g_j \in \mathcal{A}$ be such that*

$$\left|\frac{z^2 f_j'(z)}{[f_j(z)]^2} - 1\right| < 1, \quad |f_j(z)| < M_j \quad \text{and} \quad \left|\frac{g_j''(z)}{g_j'(z)}\right| \le N_j, \quad z \in \mathbb{U},$$

where $M_j, N_j \in \mathbb{R}$ are strictly positive constants, for every $j \in \{1, ..., n\}$. Also, let $\beta, \gamma_1, ..., \gamma_n \in \mathbb{C}$ be such that $\mathfrak{Re}\beta = a > 0$ and

$$c \sum_{j=1}^{n} |\gamma_j| (2M_j^2 + N_j) \le 1,$$

where $c = \frac{2}{1+2a}\left(\frac{1}{1+2a}\right)^{\frac{1}{2a}}$. Then the integral operator $G_{n,\beta}$ given by (4) belongs to the class S.

Proof. Let us consider the function $h : U \to \mathbb{C}$ be given by

$$h(z) = \int_0^z \prod_{j=1}^{n} \left[\frac{e^{f_j(t)}}{g_j'(t)}\right]^{\gamma_j} dt, \quad z \in U. \tag{8}$$

It is not difficult to observe that h is holomorphic on U and

$$h'(z) = \prod_{j=1}^{n} \left[\frac{e^{f_j(z)}}{g_j'(z)}\right]^{\gamma_j}, \quad z \in U. \tag{9}$$

Moreover,

$$h''(z) = \sum_{j=1}^{n} \gamma_j \left[\frac{e^{f_j(t)}}{g_j'(t)}\right]^{\gamma_j - 1} \left[\frac{e^{f_j(z)} f_j'(z) g_j'(z) - e^{f_j(z)} g_j''(z)}{[g_j'(z)]^2}\right] \prod_{k=1, k\neq j}^{n} \frac{e^{f_k(z)}}{g_k'(z)}.$$

and then

$$\frac{zh''(z)}{h'(z)} = \sum_{j=1}^{n} \gamma_j \left[z f_j'(z) - \frac{z g_j''(z)}{g_j'(z)}\right].$$

From the previous relation, we obtain

$$\frac{1 - |z|^{2\mathfrak{Re}\beta}}{\mathfrak{Re}\beta}\left|\frac{zh''(z)}{h'(z)}\right| = \frac{1 - |z|^{2\mathfrak{Re}\beta}}{\mathfrak{Re}\beta}\left|\sum_{j=1}^{n} \gamma_j \left[z f_j'(z) - \frac{z g_j''(z)}{g_j'(z)}\right]\right|$$

$$\le \frac{1 - |z|^{2a}}{a} |z| \sum_{j=1}^{n} |\gamma_j| \left[|f_j'(z)| + \left|\frac{g_j''(z)}{g_j'(z)}\right|\right].$$

According to the Schwarz's lemma for the functions f_j, where $j \in \{1, ..., n\}$, we deduce that

$$|f_j(z)| \leq M_j|z|, \quad z \in \mathbb{U}$$

and then

$$\begin{aligned}\frac{1-|z|^{2\mathfrak{Re}\beta}}{\mathfrak{Re}\beta}\left|\frac{zh''(z)}{h'(z)}\right| &\leq \frac{1-|z|^{2a}}{a}|z|\sum_{j=1}^{n}|\gamma_j|\left[|f_j'(z)|+\left|\frac{g_j''(z)}{g_j'(z)}\right|\right]\\ &= \frac{1-|z|^{2a}}{a}|z|\sum_{j=1}^{n}|\gamma_j|\left[\left|\frac{z^2f_j'(z)}{[f_j(z)]^2}\right|\left|\frac{f_j(z)}{z}\right|^2+\left|\frac{g_j''(z)}{g_j'(z)}\right|\right]\end{aligned}$$

Next, taking into account that

$$\frac{1-|z|^{2\mathfrak{Re}\beta}}{\mathfrak{Re}\beta}|z| = \frac{1-|z|^{2a}}{a}|z| \leq \frac{2}{1+2a}\left(\frac{1}{2a+1}\right)^{\frac{1}{2a}} = c,$$

for all $z \in U$ and $\beta \in \mathbb{C}$ with $\mathfrak{Re}\beta = a > 0$, we obtain that

$$\begin{aligned}\frac{1-|z|^{2\mathfrak{Re}\beta}}{\mathfrak{Re}\beta}\left|\frac{zh''(z)}{h'(z)}\right| &\leq c\sum_{j=1}^{n}|\gamma_j|\left[\left|\frac{z^2f_j'(z)}{[f_j(z)]^2}-1+1\right|M_j^2+N_j\right]\\ &< c\sum_{j=1}^{n}|\gamma_j|\big(2M_j^2+N_j\big)\\ &\leq 1,\end{aligned}$$

according to the hypothesis. Finally, we have that

$$\frac{1-|z|^{2\mathfrak{Re}\beta}}{\mathfrak{Re}\beta}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1. \tag{10}$$

and based on Lemma 1 we conclude that the operator $G_{n,\beta}$ given by (4) belongs to the class S. □

If we consider that $M_j = N_j = M \in \mathbb{R}$ is strictly positive, for every $j \in \{1, ..., n\}$, then we obtain the following consequence of Theorem 1:

Corollary 1. *Let $f_j, g_j \in \mathcal{A}$ be such that*

$$\left|\frac{z^2f_j'(z)}{[f_j(z)]^2}-1\right| < 1, \quad |f_j(z)| < M \quad \textit{and} \quad \left|\frac{g_j''(z)}{g_j'(z)}\right| \leq M,$$

for every $j \in \{1, ..., n\}$ and $z \in \mathbb{U}$, where $M \in \mathbb{R}$ is a strictly positive constant. Also, let $\beta, \gamma_1, ..., \gamma_n \in \mathbb{C}$ be such that $\mathfrak{Re}\beta = a > 0$ and

$$cM(2M+1)\sum_{j=1}^{n}|\gamma_j| \leq 1,$$

where $c = \frac{2}{1+2a}\left(\frac{1}{1+2a}\right)^{\frac{1}{2a}}$. Then the integral operator $G_{n,\beta}$ given by (4) belongs to the class S.

Moreover, if $M = 1$ in Corollary 1, then we deduce the following simplified result:

Corollary 2. *Let $f_j, g_j \in \mathcal{A}$ be such that*

$$\left|\frac{z^2 f_j'(z)}{[f_j(z)]^2} - 1\right| < 1, \quad |f_j(z)| < 1 \quad \textit{and} \quad \left|\frac{g_j''(z)}{g_j'(z)}\right| \leq 1,$$

for every $j \in \{1, ..., n\}$ and $z \in \mathbb{U}$. Also, let $\beta, \gamma_1, ..., \gamma_n \in \mathbb{C}$ be such that $\mathfrak{Re}\beta = a > 0$ and

$$\sum_{j=1}^{n} |\gamma_j| \leq \frac{1}{3c},$$

where $c = \frac{2}{1+2a}\left(\frac{1}{1+2a}\right)^{\frac{1}{2a}}$. Then the integral operator $G_{n,\beta}$ given by (4) belongs to the class S.

Remark 2. *For $\beta = 1$ and $\gamma_j = \alpha_j$, where $j \in \{1, ..., n\}$, we obtain a new proof of the univalence of the operator J_α defined by (1).*

Several particular integral operators can be obtained from the one introduced above. For example, if we consider $\beta = \frac{1}{2}$ in Corollary 2, then the operator $G_{n,1/2}$ is given by

$$G_{n,1/2}(z) = \left\{\frac{1}{2}\int_0^z \frac{1}{\sqrt{t}} \prod_{j=1}^{n} \left[\frac{e^{f_j(t)}}{g_j'(t)}\right]^{\gamma_j} dt\right\}^2, \quad z \in \mathbb{U} \tag{11}$$

and we can prove the following result (for $a = c = \frac{1}{2}$):

Corollary 3. *Let $f_j, g_j \in \mathcal{A}$ be such that*

$$\left|\frac{z^2 f_j'(z)}{[f_j(z)]^2} - 1\right| < 1, \quad |f_j(z)| < 1 \quad \textit{and} \quad \left|\frac{g_j''(z)}{g_j'(z)}\right| \leq 1,$$

for every $j \in \{1, ..., n\}$ and $z \in \mathbb{U}$. Also, let $\gamma_1, ..., \gamma_n \in \mathbb{C}$ be such that

$$\sum_{j=1}^{n} |\gamma_j| \leq \frac{2}{3}.$$

Then the integral operator $G_{n,1/2}$ given by (11) belongs to the class S.

It is easy to observe that all the results presented above can also be obtained in the particular case $n = 1$. Indeed, Dicu, Dicu and Albişoru proved several important properties related to the univalence of the integral operator $G_{\beta,\gamma}$ in [4] (see also [3]). Moreover, if $n = 1$ or $\gamma = 1$, then Theorem 1 reduces to [4, Theorem 3].

Theorem 2. *Let $\alpha_j \in (0,2]$ and let $f_j \in S(\alpha_j)$, for $j \in \{1,...,n\}$. Let also $g_j \in \mathcal{A}$ be such that*

$$|f_j(z)| < M_j \quad \text{and} \quad \left|\frac{g_j''(z)}{g_j'(z)}\right| \le N_j, \quad z \in \mathbb{U},$$

where $M_j, N_j \in \mathbb{R}$ are strictly positive constants, for every $j \in \{1,...,n\}$. Also, let $\beta, \gamma_1, ..., \gamma_n \in \mathbb{C}$ be such that $\mathfrak{Re}\beta = a > 0$ and

$$c\sum_{j=1}^{n} |\gamma_j|\big[(\alpha_j + 1)M_j^2 + N_j\big] \le 1,$$

where $c = \frac{2}{1+2a}\left(\frac{1}{1+2a}\right)^{\frac{1}{2a}}$. Then the integral operator $G_{n,\beta}$ given by (4) belongs to the class S.

Proof. Let $f_j \in S(\alpha_j)$, where $\alpha_j \in (0,2]$, for every $j \in \{1,...,n\}$. In view of Lemma 4 we know that

$$\left|\frac{z^2 f_j'(z)}{[f_j(z)]^2} - 1\right| \le \alpha|z|^2, \quad z \in \mathbb{U}. \tag{12}$$

According to the proof of Theorem 1, we have that

$$\frac{zh''(z)}{h'(z)} = \sum_{j=1}^{n} \gamma_j\left[zf_j'(z) - \frac{zg_j''(z)}{g_j'(z)}\right],$$

where

$$h(z) = \int_0^z \prod_{j=1}^{n}\left[\frac{e^{f_j(t)}}{g_j'(t)}\right]^{\gamma_j} dt,$$

for all $z \in \mathbb{U}$. Then

$$\begin{aligned}
\frac{1-|z|^{2\mathfrak{Re}\beta}}{\mathfrak{Re}\beta}\left|\frac{zh''(z)}{h'(z)}\right| &\le \frac{1-|z|^{2a}}{a}|z|\sum_{j=1}^{n}|\gamma_j|\left[|f_j'(z)| + \left|\frac{g_j''(z)}{g_j'(z)}\right|\right] \\
&= \frac{1-|z|^{2a}}{a}|z|\sum_{j=1}^{n}|\gamma_j|\left[\left|\frac{z^2 f_j'(z)}{[f_j(z)]^2}\right|\left|\frac{f_j(z)}{z}\right|^2 + \left|\frac{g_j''(z)}{g_j'(z)}\right|\right] \\
&= c\sum_{j=1}^{n}|\gamma_j|\left[\left|\frac{z^2 f_j'(z)}{[f_j(z)]^2} - 1 + 1\right|M_j^2 + N_j\right] \\
&\le c\sum_{j=1}^{n}|\gamma_j|\big[(\alpha_j|z|^2 + 1)M_j^2 + N_j\big] \\
&\le c\sum_{j=1}^{n}|\gamma_j|\big[(\alpha_j + 1)M_j^2 + N_j\big] \\
&\le 1,
\end{aligned}$$

based on similar arguments as in the proof of Theorem 1. Hence, in view of Lemma 1 we obtain that the integral operator $G_{n,\beta}$ given by (4) belongs to the class S. □

Remark 3. *Note that from Theorem 2 we can obtain two consequences for $M_j = N_j = M \in \mathbb{R}$ with $M > 0$, respectively for $M_j = N_j = 1$, where $j \in \{1, ..., n\}$. This particular results are the natural generalizations of Corollaries 1 and 2. On the other hand, if $\gamma_1 = ... = \gamma_n = 1$, then Theorem 2 reduces to [4, Theorem 4].*

4 Modification of the operator $G_{n,\beta}$

Following the ideas presented above and the result proved by Pescar in Lemma 2 (see also [13]), we can introduce the integral operator $\mathcal{G}_{n,\beta}^{\beta-1} : \mathcal{A}^n \times \mathcal{A}^n \to \mathcal{A}$, given by

$$\mathcal{G}_{n,\beta}^{\beta-1}(z) = \left\{\beta \int_0^z t^{\beta-1} \left[\prod_{j=1}^n \left(\frac{e^{f_j(t)}}{g_j'(t)}\right)^{\gamma_j}\right]^{\beta-1} dt\right\}^{\frac{1}{\beta}}, \quad z \in \mathbb{U}, \tag{13}$$

where $\beta, \gamma_1, ..., \gamma_n \in \mathbb{C}$ with $\beta \neq 0$ and $f_j, g_j \in \mathcal{A}$, for $j \in \{1, ..., n\}$. For this operator we can obtain a sufficient univalent condition on $\mathbb{U}$ (similar to the one presented in Theorem 1):

Theorem 3. *Let $f_j, g_j \in \mathcal{A}$ be such that*

$$\left|\frac{z^2 f_j'(z)}{[f_j(z)]^2} - 1\right| < 1, \quad |f_j(z)| < M_j \quad \textit{and} \quad \left|\frac{g_j''(z)}{g_j'(z)}\right| \leq N_j, \quad z \in \mathbb{U},$$

where $M_j, N_j \in \mathbb{R}$ are strictly positive constants, for every $j \in \{1, ..., n\}$. Also, let $\beta \in \mathbb{R}$ be such that $\beta \geq 1$, $c \in \mathbb{C}$ with $|c| \leq \frac{1}{\beta}$ and $c \neq -1$, respectively $\gamma_1, ..., \gamma_n \in \mathbb{C}$ such that

$$\sum_{j=1}^n |\gamma_j|(2M_j^2 + N_j) \leq 1.$$

Then the integral operator $\mathcal{G}_{n,\beta}^{\beta-1}$ given by (13) belongs to the class S.

Proof. According to the ideas presented in the proof of Theorem 1, it is not difficult to observe that we can consider the holomorphic function $h : \mathbb{U} \to \mathbb{C}$, given by

$$h(z) = \int_0^z \prod_{j=1}^n \left[\frac{e^{f_j(t)}}{g_j'(t)}\right]^{\gamma_j} dt, \quad z \in \mathbb{U}.$$

Then

$$\left|\frac{h''(z)}{h'(z)}\right| \leq \sum_{j=1}^n |\gamma_j| \left[|f_j'(z)| + \left|\frac{g_j''(z)}{g_j'(z)}\right|\right].$$

Based on the assumptions imposed in Theorem 3, respectively on Lemma 3 we deduce that

$$\left|\frac{h''(z)}{h'(z)}\right| \leq \sum_{j=1}^n |\gamma_j|(2M_j^2 + N_j) \leq 1, \quad z \in \mathbb{U}.$$

Hence, in view of Lemma 2 it follows that the operator

$$\begin{aligned} \mathcal{G}_{n,\beta}^{\beta-1}(z) &= \left\{ \beta \int_0^z \left[th'(t) \right]^{\beta-1} dt \right\}^{\frac{1}{\beta}} \\ &= \left\{ \beta \int_0^z t^{\beta-1} \left[\prod_{j=1}^n \left(\frac{e^{f_j(t)}}{g_j'(t)} \right)^{\gamma_j} \right]^{\beta-1} dt \right\}^{\frac{1}{\beta}}, \quad z \in \mathbb{U} \end{aligned}$$

belongs to the class S. □

Remark 4. *Taking into account the definition of the operator* $\mathcal{G}_{n,\beta}^{\beta-1}$*, we can obtain the following particular case (for* $n = 1$*):*

$$\mathcal{G}_{1,\beta}^{\beta-1}(z) = \left\{ \beta \int_0^z t^{\beta-1} \left[\frac{e^{f(t)}}{g'(t)} \right]^{\gamma(\beta-1)} dt \right\}^{\frac{1}{\beta}}, \quad z \in \mathbb{U}.$$

Note that the operator $\mathcal{G}_{1,\beta}^{\beta-1}$ *can be seen as a generalization of the operator* $G_{\beta,\gamma}$ *defined by (3), since* $\mathcal{G}_{1,\beta}^{\beta-1} = G_{\beta,\gamma^*}$ *for* $\gamma^* = \gamma(\beta - 1)$*. Such operators, as well as their properties, will be studied in future works.*

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