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DIFFERENTIAL TRANSFORMATION METHOD FOR CIRCULAR MEMBRANE VIBRATIONS

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Abstract

The purpose of this research is to present the steps of one-dimensional differential transformation method (DTM) to find the series solutions for the vibrations of a circular membrane under the specified initial and boundary conditions. The problems will be studied in the both cases of vibrations depending only on radius and of the vibrations depending on both radius and angle. We illustrate four examples of problems which the exact solutions can be solve analytically and compare them to the DTM results, to show that the DTM is reliable and of high accuracy. This work shows that the DTM is easier to use than the analytical method from the point of view of programming.

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Key words: analytical solution, approximate series solution, differential transform method, vibrations of a circular membrane.

1 Introduction

Circular membranes are important parts of drums, pumps, microphones, and other devices. This accounts for their great importance in engineering. We consider the case when the circular membrane is plane and its material is elastic, but offers no resistance to bending (this excludes thin metallic membranes). Then the vibrations of the circular membrane is given in the form of two-dimensional wave equation in polar coordinates,

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$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right), \tag{1}$$
$$u(R,\theta,t) = 0 \text{ for all } t \geq 0, u(r,\theta,0) = f(r,\theta), u_t(r,\theta,0) = g(r,\theta).$$

where $0 \leq r \leq R, 0 \leq \theta \leq 2\pi, c^2 = T/\rho$ in term of the membranes tension T and density ρ, R is a radius of a membrane, a membrane is fixed along the boundary circle radius $R, f(r,\theta)$ is the initial shape at time $t = 0$ and $g(r,\theta)$ is the initial velocity (see [4]).

The differential transformation method (DTM) is an alternative procedure for obtaining an approximate Taylor series solution or the semi-analytical solution of differential equations. The main advantage of this method is that it can be applied directly to nonlinear differential equations without the requiring linearization and discretization. The concept of the DTM was introduced by Zhou [10], who solved linear and nonlinear problems in electrical circuits and many other problems related to differential equations (see also [3], [5], [6], [7], [8] and [9])

In the present paper, we will show how to extend the method of differential transformation to the problem of vibrations of a circular membrane. The computation consists of three steps. The first step is using the method of separation of variables to obtain ODEs from the wave equation in Eq.(1). The next step is applying the DTM to ODEs from the previous step to obtain recursive relations. The last step is to find the coefficients of the series solutions for ODEs using the recursive relations.

The present paper has been organized as follows. In the section 2, the one-dimensional differential transformation method is introduced, and the Fourier Bessel series are described. In the section 3, the analysis of the method for the vibrations of a circular membrane both the vibrations depending on only radius and the vibrations depending on both radius and angle are described, as shown in subsection 3.1 and subsection 3.2, respectively. Four examples of vibrations of a circular membrane with different conditions corresponding to the three steps in the section 3, have been presented in the section 4. The conclusion is given at the end of the paper in the section 5.

2 Preliminaries

The basic definitions and fundamental operations of the differential transform are introduced as follows.

2.1 The one-dimensional differential transformation

Definition 1. *The one-dimensional differential transform of the function $x(t)$ is defined as*

$$X(k) = \frac{1}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0}, \quad k \in \mathbb{I}^+ \cup \{0\}. \tag{2}$$

In Eq.(2), $x(t)$ is called the original function and $X(k)$ is called the transformed function.

Definition 2. *The inverse one-dimensional differential transforms of $X(k)$ is defined as*

$$x(t) = \sum_{k=0}^{\infty} X(k)t^k, \tag{3}$$

that is,

$$x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left[\frac{d^k x(t)}{dt^k} \right]_{t=0}. \tag{4}$$

Equation (4) implies that the concept of differential transformation method is derived from Taylor series expansion. Actually, in concrete applications, the function $x(t)$ is expressed by a truncated series and Eq.(3) becomes

$$x(t) = \sum_{k=0}^{N} X(k)t^k. \tag{5}$$

The fundamental operations of one-dimensional the DTM are shown in Table 1.

Table 1: The fundamental operations of one-dimensional DTM.

Original function $x(t)$	Transformed function $X(k)$
$x(t) \pm y(t)$	$X(k) \pm Y(k)$
$\lambda x(t)$	$\lambda X(k)$
$x(t)y(t)$	$\sum_{r=0}^{k} X(r)Y(k-r)$
$x(t)y(t)z(t)$	$\sum_{r=0}^{k} \sum_{l=0}^{r} X(l)Y(r-l)Z(k-r)$
$\frac{d^r}{dt^r} x(t)$	$\frac{(k+r)!}{k!} X(k+r)$

2.2 Fourier-Bessel series

The series solutions of the presented problems consist of the coefficients of the Fourier-Bessel series corresponding to the Bessel functions of the first kind (see also [1]). The following theorem explain the meaning of the Fourier-Bessel series based on the orthogonality relations.

Theorem 1 (Orthogonality of the Bessel Functions [3]). *For each fixed nonnegative integer n the sequence of the Bessel functions of the first kind $J_n(h_{n1}r)$, $J_n(h_{n2}r), \ldots$ with $h_{nm} = \alpha_n m/R$ where $\alpha_n m$ is the mth positive zero of $J_n, (m = 1, 2, 3, \ldots)$, forms an orthogonal set on the interval $0 \leq r \leq R$ with respect to the weight function r, that is*

$$\int_0^R r J_n(h_{nm}r) J_n(h_{nj}r) dr = 0 \quad (j \neq m, n \text{ fixed}). \tag{6}$$

The Fourier-Bessel series for vibrations of a circular membrane independent of angle in subsection 3.1 corresponding to J_n (n fixed) is $f(r) = \sum_{m=1}^{\infty} A_{nm} J_n(h_{nm} r)$, (with $h_{nm} = \alpha_{nm}/R$). Here the coefficients are

$$A_{nm} = \frac{2}{R^2 J_{n+1}^2(\alpha_{nm})} \int_0^R r f(r) J_n(h_{nm} r) dr, \tag{7}$$

where $J_n(h_{nm} r) = \sum_{l=0}^{\infty} \dfrac{(-1)^l (h_{nm} r)^{2l+n}}{2^{2l+n} l!(n+l)!}$ is the Bessel function of order n of the first kind and α_{nm} is the mth positive zero of $J_n, (m = 1, 2, 3, ...)$,

The Fourier-Bessel series for vibrations of a circular membrane depending on both radius and angle in subsection 3.2 corresponding to J_n is

$$f(r, \theta) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} J_n(h_{nm} r) \Big(A_{nm} \cos(n\theta) + B_{nm} \sin(n\theta) \Big).$$

Here the coefficients are

$$A_{nm} = \frac{2}{\pi R^2 J_{n+1}^2(\alpha_{nm})} \int_0^{2\pi} \int_0^R f(r, \theta) J_n(h_{nm} r) \cos(n\theta) r dr d\theta, \tag{8}$$

$$B_{nm} = \frac{2}{\pi R^2 J_{n+1}^2(\alpha_{nm})} \int_0^{2\pi} \int_0^R f(r, \theta) J_n(h_{nm} r) \sin(n\theta) r dr d\theta, \tag{9}$$

where $J_n(h_{nm} r) = \sum_{l=0}^{\infty} \dfrac{(-1)^l (h_{nm} r)^{2l+n}}{2^{2l+n} l!(n+l)!}$ is the Bessel function of order n of the first kind and α_{nm} is the mth positive zero of $J_n, (n = 0, 1, 2, ..., m = 1, 2, 3, ...)$.

3 Analysis of method

In this section, we will show how to use the DTM to the problems of vibrations of a circular membrane. The presented problems include the vibrations of a circular membrane independent of angle studied in subsection 3.1 and the vibrations of a circular membrane depending on both radius and angle studied in subsection 3.2.

3.1 Vibrations of a circular membrane independent of angle θ.

In this section, we consider a circular membrane of radius R which is fixed along the boundary circle, the initial shape $f(r)$, and assume that the initial velocity $g(r)$ is equal to zero. The model of the problem is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right), \tag{10}$$

$$u(R, t) = 0 \text{ for all } t \geq 0, u(r, 0) = f(r), u_t(r, 0) = 0. \tag{11}$$

The calculation consists of the following three steps.

Step 1 By the method of separation of variables, we obtain two linear ODEs form the wave equation in Eq.(10).
The method of separation of variables uses the substitution

$$u(r,t) = w(r)g(t). \tag{12}$$

Differentiating Eq.(12), we obtain

$$\frac{\partial^2 u}{\partial t^2} = w\frac{d^2 g}{dt^2}, \frac{\partial u}{\partial r} = \frac{dw}{dr}g, \text{ and } \frac{\partial^2 u}{\partial r^2} = \frac{d^2 w}{dr^2}g. \tag{13}$$

By substituting Eq.(13) into Eq.(10), we obtain

$$w\frac{d^2 g}{dt^2} = c^2\left(\frac{d^2 w}{dr^2}g + \frac{1}{r}\frac{dw}{dr}g\right). \tag{14}$$

Dividing the result by $c^2 wg$, we have

$$\frac{1}{c^2 g}\frac{d^2 g}{dt^2} = \frac{1}{w}\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right). \tag{15}$$

The variables are now separated. Hence, both sides are independent and this can only be so if they are equal to a constant. This constant must be negative in order to obtain solutions that satisfy the boundary condition without being identically zero. Thus,

$$\frac{1}{c^2 g}\frac{d^2 g}{dt^2} = \frac{1}{w}\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) = -h_{nm}^2. \tag{16}$$

Eq.(16) gives the two linear ODEs,

$$\frac{d^2 g}{dt^2} + \lambda_{nm}^2 g = 0, \quad \lambda_{nm} = ch_{nm}, \tag{17}$$

by defining $s = h_{nm}r$ we reduce $\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + h_{nm}^2 w = 0$ to the Bessel equation, that is

$$s^2\frac{d^2 w}{ds^2} + s\frac{dw}{ds} + s^2 w = 0. \tag{18}$$

Step 2 We apply the differential transform method to ODEs in Eqs.(17) and (18) to obtain the recursive formulas.
Applying the fundamental operations of DTM in Table 1 to Eqs.(17) and (18), respectively, we obtain

$$G(k+2) = -\frac{\lambda_{nm}^2 G(k)}{(k+1)(k+2)} \tag{19}$$

$$\begin{aligned}&\sum_{l=0}^{k}\delta(l-2)(k-l+1)(k-l+2)W(k-l+2)+\\&\sum_{l=0}^{k}\delta(l-1)(k-l+1)W(k-l+1)+\sum_{l=0}^{k}\delta(l-2)W(k-l)=0,\end{aligned} \tag{20}$$

with the initial values $G(0) = A_{nm}, G(1) = 0, W(0) = 1$ and $W(1) = 0$, where A_{nm} are the coefficients of the Fourier-Bessel series in Eq.(7).

Step 3 Using the recursive formulas in Eqs.(19) and (20) we find the coefficients of the series solutions of the Eqs. (17) and (18), respectively.
To find the series solution of Eq.(17), we substitute into Eq.(19), then we obtain

$$\begin{aligned}
G(2) &= -\frac{-\lambda_{nm}^2 G(0)}{2} = -\frac{\lambda_{nm}^2 A_{nm}}{2}, & G(3) &= -\frac{-\lambda_{nm}^2 G(1)}{6} = 0,\\
G(4) &= -\frac{-\lambda_{nm}^2 G(2)}{12} = \frac{\lambda_{nm}^2 A_{nm}}{24}, & G(5) &= -\frac{-\lambda_{nm}^2 G(3)}{20} = 0,\\
G(6) &= -\frac{-\lambda_{nm}^2 G(4)}{30} = -\frac{\lambda_{nm}^2 A_{nm}}{720}, & G(7) &= -\frac{-\lambda_{nm}^2 G(5)}{42} = 0,\\
&\vdots & &\vdots
\end{aligned}$$

where, $G(0), G(1), G(2), ...$ are the coefficients of the series solution.
Hence, we obtain the solution corresponding to Eq.(17), by substituting $\lambda_{nm} = c\dfrac{\alpha_{nm}}{R}$,

$$g_{nm}(t) = A_{nm}\left(1 - \frac{2c^2\alpha_{nm}^2 t^2}{R^2} + \frac{2c^4\alpha_{nm}^4 t^4}{3R^4} - \frac{4c^6\alpha_{nm}^6 t^6}{45R^6} + \frac{2c^8\alpha_{nm}^8 t^8}{315R^8} - \cdots\right). \tag{21}$$

To find the series solution of Eq.(18), we substitute $k = 1, 2, ...$ into Eq.(20) and we obtain

$$\begin{aligned}
k=1;\quad & \delta(0-2)(1-0+1)(1-0+2)W(1-0+2)+\\
& \delta(0-1)(1-0+1)W(1-0+1)+\\
& \delta(0-2)W(1-0)+\delta(1-2)(1-1+1)(1-1+2)W(1-1+2)+\\
& \delta(1-1)(1-1+1)W(1-1+1)+\delta(1-2)W(1-1)=0\\
& W(1)=0
\end{aligned}$$

$$\begin{aligned}
k=2;\quad & W(2) = \frac{-W(0)}{4} = -\frac{1}{4}, & k=3;\quad & W(3) = \frac{-W(1)}{9} = 0\\
k=4;\quad & W(4) = \frac{-W(2)}{16} = \frac{1}{64}, & k=5;\quad & W(5) = \frac{-W(3)}{25} = 0\\
& \vdots & & \vdots
\end{aligned}$$

Since, $W(k) = \dfrac{-W(k-2)}{k^2}$, $k = 2, 3, 4, ...$ or $W(k+2) = \dfrac{-W(k)}{(k+2)^2}$, $k = 0, 1, 2, 3, ...$ and $W(0), W(1), W(2), ...$ are the coefficients of the series solution. Hence, we obtain the series solution corresponding to Eq.(18), by substituting $s = h_{nm}r = \dfrac{\alpha_{nm}}{R}r$,

$$w_{nm}(r) = 1 - \frac{\alpha_{nm}^2 r^2}{4R^2} + \frac{\alpha_{nm}^4 r^4}{64R^4} - \frac{\alpha_{nm}^6 r^6}{2304R^6} + \frac{\alpha_{nm}^8 r^8}{147456R^8} - \frac{\alpha_{nm}^{10} r^{10}}{14745600R^{10}} + \cdots . \quad (22)$$

Therefore, the series solution of vibrations of a circular membrane is

$$u(r,t) = \sum_{m=1}^{\infty} w_{nm}(r) g_{nm}(t) = \sum_{m=1}^{\infty} A_{nm}\Big(1 - \frac{2c^2\alpha_{nm}^2 t^2}{R^2} + \cdots\Big)\Big(1 - \frac{\alpha_{nm}^2 r^2}{4R^2} + \cdots\Big), \quad (23)$$

where A_{nm} are the coefficients of the Fourier-Bessel series corresponding to J_n which can be calculated by Eq.(7) and α_{nm} is the mth positive zero of J_n.
Next, let us consider the general case, when the solution can also depend on angle θ.

3.2 Vibrations of a circular membrane depending on both radius and angle.

We now consider a circular membrane of radius R which is fixed along the boundary circle, with the initial shape $f(r,\theta)$, and the initial velocity $g(r,\theta)$ equal zero. The model of the problem is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right), \quad (24)$$

$$u(R,\theta,t) = 0 \text{ for all } t \geq 0, u(r,\theta,0) = f(r,\theta), u_t(r,\theta,0) = 0. \quad (25)$$

The calculation consists of the following three steps.

Step 1 Three ODEs form the wave equation in Eq.(24) using the method of separation variables.
We define a solution in the method of separation of variables,

$$u(r,\theta,t) = z(r,\theta)g(t). \quad (26)$$

Differentiating Eq.(26), we obtain

$$\frac{\partial^2 u}{\partial t^2} = z\frac{d^2 g}{dt^2}, \quad \frac{\partial u}{\partial r} = \frac{\partial z}{\partial r}g, \quad \frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 z}{\partial r^2}g \text{ and } \frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 z}{\partial \theta^2}g. \quad (27)$$

Substituting Eq.(27) into Eq.(24) gives

$$z\frac{d^2 g}{dt^2} = c^2\left(\frac{\partial^2 z}{\partial r^2}g + \frac{1}{r^2}\frac{\partial z}{\partial r}g + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}g\right). \quad (28)$$

Dividing both sides by $c^2 zg$ yields

$$\frac{1}{c^2 g}\frac{d^2 g}{dt^2} = \frac{1}{z}\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right). \quad (29)$$

The variables are now separated. Hence, both sides must equal a constant, that is

$$\frac{1}{c^2 g}\frac{d^2 g}{dt^2} = \frac{1}{z}\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right) = -h_{nm}^2, \tag{30}$$

Eq.(30) gives an ODE and a PDE, as follows:

$$\frac{d^2 g}{dt^2} + \lambda_{nm}^2 g = 0, \quad \lambda_{nm} = ch_{nm} \tag{31}$$

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + h_{nm}^2 z = 0. \tag{32}$$

The PDE as Eq.(32) can be separated by substituting $z = w(r)q(\theta)$ and its derivatives into Eq.(32), we obtain

$$\frac{d^2 w}{dr^2}q + \frac{1}{r}\frac{dw}{dr}q + \frac{1}{r^2}w\frac{d^2 q}{d\theta^2} + h_{nm}^2 wq = 0. \tag{33}$$

On the both sides, multiplying by r^2/wq and then rearranging the equation, we obtain

$$\frac{1}{q}\frac{d^2 q}{d\theta^2} = -\frac{1}{w}\left(r^2\frac{d^2 w}{dr^2} + r\frac{dw}{dr}\right) - h_{nm}^2 r^2. \tag{34}$$

The variables are now separated. The expressions on both sides must equal a constant, that is

$$\frac{1}{q}\frac{d^2 q}{d\theta^2} = -\frac{1}{w}\left(r^2\frac{d^2 w}{dr^2} + r\frac{dw}{dr}\right) - h_{nm}^2 r^2 = -n^2, \tag{35}$$

Eq.(35) gives two ODEs, as follows:

$$\frac{d^2 q}{d\theta^2} + n^2 q = 0, \tag{36}$$

$$r^2\frac{d^2 w}{dr^2} + r\frac{dw}{dr} + (h_{nm}^2 r^2 - n^2)w = 0, \tag{37}$$

Eq.(37) is known as the Bessel equation of order n where $n = 1, 2, 3, ...$ The nonnegative integer n in Eqs.(36) and (37) depends on the initial shape as shown in Table 2.

Table 2: The values of nonnegative integer n corresponding to given initial shape $f(r,\theta)$, and the initial values $G(0), G(1), Q(0)$ and $Q(1)$.

Initial shape $f(r,\theta)$	Value of n	G(0)	G(1)	Q(0)	Q(1)
$w(r)$	0	A_{0m}	0	1	0
$w(r)\sin(N\theta), N = 0, 1, ...$	N	B_{Nm}	0	0	N
$w(r)\cos(N\theta), N = 0, 1, ...$	N	A_{Nm}	0	1	0

Step 2 We apply the differential transform method to ODEs in Eqs.(31), (36) and (37) to obtain recursive formulas.
Taking the DTM of Eqs. (31), (36) and (37),respectively, we obtain

$$G(k+2) = -\frac{-\lambda_{nm}^2 G(k)}{(k+1)(k+2)}, \tag{38}$$

$$Q(k+2) = -\frac{n^2 Q(k)}{(k+1)(k+2)}, \tag{39}$$

$$\begin{aligned} n^2 W(k) = &\sum_{l=0}^{k} \delta(l-2)(k-l+1)(k-l+2)W(k-l+2)+ \\ &\sum_{l=0}^{k} \delta(l-1)(k-l+1)W(k-l+1) + \frac{\alpha_{nm}^2}{R^2}\sum_{l=0}^{k} \delta(l-2)W(k-l). \end{aligned} \tag{40}$$

Step 3 Using the recursive formulas in Eqs.(38), (39) and (40) to find the coefficients of the series solutions of ODEs.
Substituting $k = 0, 1, 2, ...$ into Eq.(38), we obtain

$$\begin{aligned} G(2) &= -\frac{-\lambda_{nm}^2 G(0)}{2}, & G(3) &= -\frac{-\lambda_{nm}^2 G(1)}{6} = 0, \\ G(4) &= -\frac{-\lambda_{nm}^2 G(2)}{12} = \frac{\lambda_{nm}^2 G(0)}{24}, & G(5) &= -\frac{-\lambda_{nm}^2 G(3)}{20} = 0, \\ G(6) &= -\frac{-\lambda_{nm}^2 G(4)}{30} = -\frac{\lambda_{nm}^2 G(0)}{720}, & G(7) &= -\frac{-\lambda_{nm}^2 G(5)}{42} = 0, \\ &\vdots & &\vdots \end{aligned}$$

where, $G(0), G(1), G(2), ...$ are the coefficients of the series solution. Hence, we obtain the solution corresponding to Eq.(31),

$$g_{nm}(t) = G(0)\left(1 - \frac{c^2\alpha_{nm}^2 t^2}{2R^2} + \frac{c^4\alpha_{nm}^4 t^4}{24R^4} - \frac{c^6\alpha_{nm}^6 t^6}{720R^6} + \frac{c^8\alpha_{nm}^8 t^8}{40320R^8} - \cdots\right). \tag{41}$$

where the values of $G(0)$ depend on the initial shape, as shown in Table 2.

Substituting $k = 0, 1, 2, ...$ into Eq.(36), we obtain

$$\begin{aligned} Q(2) &= -\frac{n^2 Q(0)}{2}, & Q(3) &= -\frac{n^2 Q(1)}{6}, \\ Q(4) &= -\frac{n^2 G(2)}{12} = \frac{n^4 Q(0)}{24}, & Q(5) &= -\frac{n^2 Q(3)}{20} = \frac{n^4 Q(1)}{120}, \\ Q(6) &= -\frac{n^2 Q(4)}{30} = -\frac{n^2 Q(0)}{720}, & Q(7) &= -\frac{n^2 Q(5)}{42} = -\frac{n^6 Q(1)}{5040}, \\ &\vdots & &\vdots \end{aligned}$$

where, $Q(0), Q(1), Q(2), ...$ are the coefficients of the series solution. Hence, we obtain the solution corresponding to Eq.(39),

$$q_n(\theta) = Q(0)+Q(1)\theta-\frac{n^2Q(0)}{2}\theta^2-\frac{n^2Q(1)}{6}\theta^3+\frac{n^4Q(0)}{24}\theta^4+\frac{n^4Q(0)}{120}\theta^5-\cdots. \quad (42)$$

Substituting $k = 1, 2, 3, ...$ into Eq.(37), we obtain

$$k = 1;\ \ n^2W(1) = W(1), \qquad k = 2;\ \ W(2) = \frac{\alpha_{nm}^2 W(0)}{n^2-4},$$
$$k = 3;\ \ W(3) = \frac{\alpha_{nm}^2 W(1)}{n^2-9}, \qquad k = 4;\ \ W(4) = \frac{\alpha_{nm}^2 W(2)}{n^2-16},$$
$$k = 5;\ \ W(3) = \frac{\alpha_{nm}^2 W(3)}{n^2-25}, \qquad k = 6;\ \ W(6) = \frac{\alpha_{nm}^2 W(4)}{n^2-36},$$
$$\vdots \qquad\qquad \vdots$$

Since, $W(k) = \dfrac{\alpha_{nm}^2 W(k-2)}{n^2-k^2}, k = 2, 3, 4, ...,$ or $W(k+2) = \dfrac{\alpha_{nm}^2 W(k)}{n^2-(k+2)^2}, k = 0, 1, 2, ...$ Hence, we obtain the solution corresponding to Eq.(40),

$$w_{nm}(r) = W(0) + W(1)r + W(2)r^2 + W(3)r^3 + \cdots \quad (43)$$

Therefore, the solutions of vibrations of a circular membrane is

$$\begin{aligned} u(r,\theta,t) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} w_{nm}(r)q_{nm}(t)q_n(\theta) \\ &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} G(0)\Big(W(0)+W(1)r+W(2)r^2+\cdots\Big) \\ &\quad \left(1-\frac{c^2\alpha_{nm}^2t^2}{2R^2}+\frac{c^4\alpha_{nm}^4t^4}{24R^4}-\cdots\right) \\ &\quad \left(Q(0)+Q(1)\theta-\frac{n^2Q(0)}{2}\theta^2-\frac{n^2Q(1)}{6}\theta^3+\cdots\right), \end{aligned} \quad (44)$$

where the values of $G(0)$ depend on the initial shape function, it can be A_{nm} or B_{nm} (see Table 2), here A_{nm} and B_{nm} are calculated by Eqs.(8) and (9), respectively, and α_{nm} is the mth positive zero of J_n.

The following shows the calculations of A_{nm} and B_{nm} corresponding to the value of nonnegative integer n. Let us recall Eqs.(8) and (9), we have

$$A_{nm} = \frac{2}{\pi R^2 J_{n+1}^2(\alpha_{nm})}\int_0^{2\pi}\int_0^R f(r,\theta)J_n(h_{nm}r)\cos(n\theta)r\,dr\,d\theta, \text{ and}$$
$$B_{nm} = \frac{2}{\pi R^2 J_{n+1}^2(\alpha_{nm})}\int_0^{2\pi}\int_0^R f(r,\theta)J_n(h_{nm}r)\sin(n\theta)r\,dr\,d\theta,$$

As we can see Eqs.(8) and (9) consist of the integral forms of the initial shape function $f(r,\theta) = w(r)q(r)$. Here, we illustrate the three cases of $q(\theta)$ i.e., (i) $q(\theta) = 1$ (ii) $q(\theta) = \cos(N\theta)$ and (iii) $q(\theta) = \sin(N\theta)$, here the calculations of A_{nm} and B_{nm} are as follows:

1. If $q(\theta) = 1$, then $\int_0^{2\pi} \cos(n\theta)d\theta = 0$ for $N \leq 1$ except $n = 0$. Thus A_{nm} are available for $n = 0$. That is A_{0m} are obtainable when $q(\theta) = 1$.

2. If $q(\theta) = \cos(N\theta), N = 0, 1, 2, ...$, then $\int_0^{2\pi} \cos(N\theta)\cos(n\theta)d\theta = 0$ for $n \neq N$. Thus $A_{nm} = 0$ when $n \neq N$. Besides $\int_0^{2\pi} \cos(N\theta)\sin(n\theta)d\theta = 0$ for all n. Thus $B_{nm} = 0$, for all n. Therefore A_{nm} are available for $n = N$. That is A_{Nm} are obtainable when $q(\theta) = \cos(N\theta), N = 0, 1, 2, ...$

3. If $q(\theta) = \sin(N\theta), N = 0, 1, 2, ...$, then $\int_0^{2\pi} \sin(N\theta)\cos(n\theta)d\theta = 0$ for all n. Thus $A_{nm} = 0$, for all n. Besides $\int_0^{2\pi} \sin(N\theta)\sin(n\theta)d\theta = 0$ for $n \neq N$. Thus $B_{nm} = 0$ when $n \neq N$. Therefore B_{nm} are available for $n = N$. That is B_{Nm} are obtainable when $q(\theta) = \sin(N\theta), N = 0, 1, 2, ...$

As summarized in the Table 2, if we know the value of n then we obtain the initial values $G(0), G(1), Q(0), Q(1)$, and the similar for the initial values of the recursive relation in Eq.(40). Observe that $W(k), k = 0, 1, 2, ...$ also depend on the value of n as shown in Table 3.

Table 3: The initial values $W(k), k = 0, 1, 2, ...$ depending on n of Bessel equation in Eq.(37).

Value of n	$W(0)$	$W(1)$	$W(2)$	$W(3)$	$\cdots$	$W(n-1)$	$W(n)$
0	1	0					
1	0	$\frac{\alpha_{1m}}{2}$					
2		0	$\frac{\alpha_{2m}}{8}$				
3			0	$\frac{\alpha_{3m}}{48}$			
$\vdots$							
n						0	$\frac{\alpha_{nm}}{2^n n!}$

Table 4: The values of α_{nm} where the mth is positive zero of Bessel function J_n.

n \ m	1	2	3	4	5	6	$\cdots$
0	2.40483	5.52008	8.65373	11.79153	14.93092	18.07106	$\cdots$
1	03.83171	7.01559	10.1735	13.3237	16.4706	19.6159	$\cdots$
2	5.13562	8.41724	11.6198	14.796	17.9598	21.117	$\cdots$
3	6.38016	9.76102	13.0152	16.2235	19.4094	22.5827	$\cdots$
4	7.58834	11.0647	14.3725	17.616	20.8269	24.019	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

4 Applications

In this section, four examples of the problem are illustrated corresponding to the method in the previous section. Example 1 shows the vibrations of a circular membrane independent on angle corresponding to the problem in the subsection 3.1. Examples 2, 3 and 4 show the vibrations of a circular membrane depending on both and angle corresponding to the problem in the subsection 3.2. The accuracy of the method is assessed by data value comparisons with the analytical solutions.

Example 1. *Consider the problem of vibrations of a circular membrane depending on radius in Eqs.*(10) *and* (11) *with radius 1,* $c = 2$*, the initial shape* $f(r) = 1 - r^2$ *and the initial velocity equal to zero. Hence, the series solutions in Eqs.*(21) *and* (22) *are as follows:*

$$g_{0m}(t) = A_{0m}\left(1 - 2\alpha_{0m}^2 t^2 + \frac{2\alpha_{0m}^4 t^4}{3} - \frac{4\alpha_{0m}^6 t^6}{45} + \frac{2\alpha_{0m}^8 t^8}{315} - \frac{4\alpha_{0m}^{10} t^{10}}{14175} + \cdots\right),$$

$$w_{0m}(r) = 1 - \frac{\alpha_{0m}^2 r^2}{4} + \frac{\alpha_{0m}^4 r^4}{64} - \frac{\alpha_{0m}^6 r^6}{2304} + \frac{\alpha_{0m}^8 r^8}{147456} - \frac{\alpha_{0m}^{10} r^{10}}{147456000} + \cdots,$$

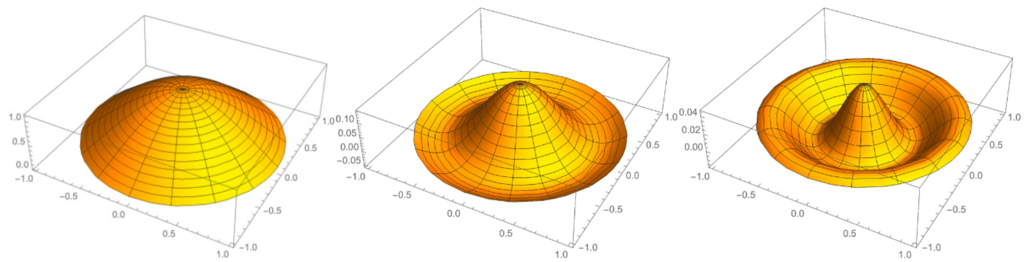
where A_{0m} *are calculated by Eq.*(7) *and* α_{0m} *is the* mth *positive zero of* J_0 *as shown in Table 4. Therefore, the series solution of vibrations of a circular membrane is*

$$\begin{aligned} u(r,t) &= \sum_{m=1}^{\infty} w_{0m}(r) g_{0m}(t) \\ &= \sum_{m=1}^{\infty} A_{0m}\left(1 - \frac{\alpha_{0m}^2 r^2}{4} + \frac{\alpha_{0m}^4 r^4}{64} - \cdots\right)\left(1 - 2\alpha_{0m}^2 t^2 + \frac{2\alpha_{0m}^4 t^4}{3} - \cdots\right) \\ &= (1 - 1.44558r^2 + 0.552586r^4 - \cdots)(1.10802 - 12.8158t^2 + 24.7055t^4 - \cdots) - \\ &\quad (1 - 7.61782)r^2 + 14.5078r^4 - \cdots)(0.139777 - 8.51839t^2 + 86.5221t^4 - \cdots) + \\ &\quad (1 - 18.722r^2 + 87.6281r^4 - \cdots)(0.045476 - 6.81119t^2 + 170.025t^4 - \cdots) - \cdots . \end{aligned}$$

The analytical solution of a circular membrane in this example is

$$\begin{aligned} u(r,t) &= \sum_{m=1}^{\infty} A_m J_0(\alpha_m r)\cos(2\alpha_m t) = 1.10801 J_0(2.40483r)\cos(4.80966t) - \\ &\quad 0.13978 J_0(5.52008r)\cos(11.04016t) + 0.04548 J_0(8.65373r)\cos(17.30746t) - \cdots . \end{aligned}$$

Figure 1 shows the motion of the series solution for the first term ($m = 1, \alpha_{01} = 2.40483$)*, the second term* ($m = 2, \alpha_{02} = 5.52008$) *and the third term* ($m = 3, \alpha_{03} = 8.65373$) *at the initial time.*



(a) $\alpha_{01} = 2.40483$ (b) $\alpha_{02} = 5.52008$ (c) $\alpha_{03} = 8.65373$

Figure 1: Normal modes of the vibrations of a circular membrane independent of the angle for Example 1.

Example 2. *Consider the problem of vibrations of a circular membrane depending on both r and θ in Eqs.*(24) *and* (25) *with radius 1, $c = 1$, the initial shape $f(r,\theta) = 1 - r^4$ and initial velocity equal to zero. From the initial shape, the nonnegative integer n can only be zero (see Table 2) because $A_{nm} = 0$ when $n \geq 1$ (i.e., $A_{0m} \neq 0$) and B_{nm} in Eq.*(9) *is always zero. The initial values are $G(0) = A_{0m}, G(1) = 0, Q(0) = 1, Q(1) = 0, W(0) = 1$ and $W(1) = 0$.*
Hence, we obtain the series solutions corresponding to Eq.(41), (42) *and* (43), *as follows:*

$$g_{0m} = A_{0m}\left(1 - \frac{\alpha_{0m}^2 t^2}{2} + \frac{\alpha_{0m}^4 t^4}{24} - \frac{\alpha_{0m}^6 t^6}{720} + \frac{\alpha_{0m}^8 t^8}{40320} - \frac{\alpha_{0m}^{10} t^{10}}{3628800} + \cdots\right)$$

$$q_0(\theta) = 1$$

$$w_{0m}(r) = 1 - \frac{\alpha_{0m}^2 r^2}{4} + \frac{\alpha_{0m}^4 r^4}{64} - \frac{\alpha_{0m}^6 r^6}{2304} + \frac{\alpha_{0m}^8 r^8}{147456} - \frac{\alpha_{0m}^{10} r^{10}}{147456000} + \cdots,$$

where A_{0m} are calculated by Eq.(8) *and α_{0m} is the mth positive zero of J_0 as shown in Table 4. Therefore, the series solution of vibrations of a circular membrane is*

$$\begin{aligned} u(r,\theta,t) &= \sum_{n=0}^{\infty} q_n(\theta) \sum_{m=1}^{\infty} w_{nm}(r) g_{nm}(t) = q_0(\theta) \sum_{m=1}^{\infty} w_{0m}(r) g_{0m}(t) \\ &= (1 - 1.4458r^2 + 0.522586r^4 - \cdots)(2.73318 - 7.90328t^2 + 3.80886t^4 - \cdots) - \\ &\quad (1 - 7.61782r^2 + 14.5078r^4 - \cdots)(0.971432 - 14.8004t^2 + 37.5822t^4 - \cdots) + \\ &\quad (1 - 18.7218r^2 + 87.6261r^4 - \cdots)(0.344381 - 12.8948t^2 + 80.4712t^4 - \cdots) - \cdots. \end{aligned}$$

The analytical solution of a circular membrane in this example is

$$\begin{aligned} u(r,\theta,t) &= \sum_{m=1}^{\infty} \frac{16(\alpha_{0m} J_2(\alpha_{0m}) - 2J_3(\alpha_{0m}))}{\alpha_{0m}^3 J_1^2(\alpha_{0m})} J_0(\alpha_{0m} r)\cos(\alpha_{0m} t) \\ &= 2.73318 J_0(2.40483r)\cos(2.40483t) - 0.971432 J_0(5.52008r)\cos(5.52008t) + \\ &\quad 0.344381 J_0(8.65373r)\cos(8.65373t) - \cdots. \end{aligned}$$

Figure 2 shows the motion of the series solution for the first term ($m = 1, \alpha_{01} = 2.40483$)*, the second term* ($m = 2, \alpha_{02} = 5.52008$) *and the third term* ($m = 3, \alpha_{03} = 8.65373$) *at the initial time.*

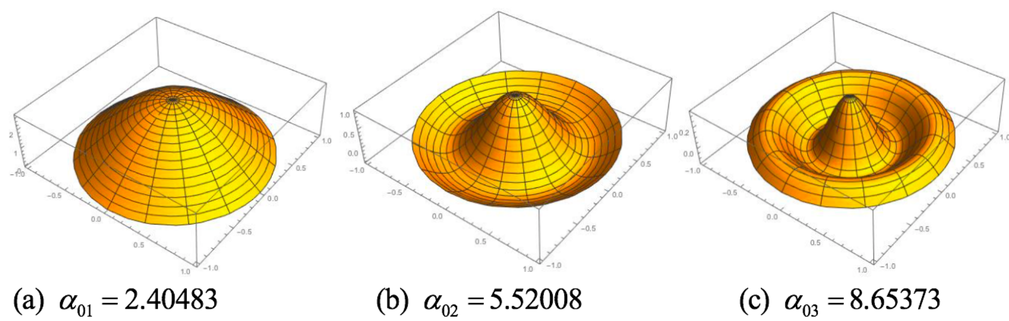


Figure 2: Normal modes of the vibrations of a circular membrane depending on both r and θ for Example 2, when $n = 0$.

Example 3. *Let us consider the Example 2 with the initial shape defined by* $f(r,\theta) = r(1-r^4)\cos(\theta)$. *From the initial shape, we obtain only* $n = 1$ *(see Table 2) because* $A_{nm} = 0$*, when* $n = 0$ *and* $n > 1$*, and* B_{nm} *in Eq.*(9) *is always zero. The initial values* $G(0) = A_{1m}, G(1) = 0, Q(0) = 1, Q(1) = 0, W(0) = 0$ *and* $W(1) = \frac{\alpha_{1m}}{2}$. *Then, we obtain the series solutions in Eqs.*(41), (42) *and* (43) *as follows:*

$$g_{1m} = A_{1m}\left(1 - \frac{\alpha_{1m}^2 t^2}{2} + \frac{\alpha_{1m}^4 t^4}{24} - \frac{\alpha_{1m}^6 t^6}{720} + \frac{\alpha_{1m}^8 t^8}{40320} - \frac{\alpha_{1m}^{10} t^{10}}{3628800} + \cdots\right)$$
$$q_1(\theta) = 1 - \frac{\theta^2}{2} - \frac{\theta^4}{24} - \frac{\theta^6}{720} + \frac{\theta^8}{40320} - \cdots$$
$$w_{1m}(r) = \frac{\alpha_{1m} r}{2} - \frac{\alpha_{1m}^3 r^3}{16} + \frac{\alpha_{1m}^5 r^5}{384} - \frac{\alpha_{1m}^7 r^7}{18432} + \frac{\alpha_{1m}^9 r^9}{1474560} - \frac{\alpha_{1m}^{11} r^{11}}{176947200} + \cdots,$$

where A_{1m} *are calculated by Eq.*(8) *and* α_{1m} *is the* m*th positive zero of* J_1 *as shown in Table 4. Therefore, the series solution of vibrations of a circular membrane is*

$$u(r,\theta,t) = \sum_{n=0}^{\infty} q_n(\theta) \sum_{m=1}^{\infty} w_{nm}(r) g_{nm}(t) = q_1(\theta) \sum_{m=1}^{\infty} w_{1m}(r) g_{1m}(t)$$
$$= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \cdots\right)\Big[(1.91586r - 3.51607r^3 + \cdot)(0.964141 - 7.07776t^2 + \cdots) -$$
$$(3.5078r - 21.5811r^3 + \cdots)(0.387905 - 9.54606t^2 + \cdots) +$$
$$(5.8099r - 98.0564r^3 + \cdots)(1.3701 \times 10^8 - 9.24954 \times 10^9 t^2 + \cdots) - \cdots\Big].$$

The analytical solution of a circular membrane in this example is

$$u(r,\theta,t) = \cos\theta \sum_{m=1}^{\infty} \frac{8(\alpha_{1m}J_3(\alpha_{1m}) - 2J_4(\alpha_{1m}))}{\alpha_{1m}^3 J_2^2(\alpha_{1m})} J_1(\alpha_{1m}r)\cos(\alpha_{1m}t)$$

$$= \cos\theta \Big[0.96414 J_1(3.83171r)\cos(3.83171t) - 0.387905 J_1(7.01559r)\cos(7.01559t) + 1.3701\times 10^8 J_1(11.6198r)\cos(11.6198t) - \cdots \Big].$$

Figure 3 shows the motion of the series solution for the first term ($m = 1, \alpha_{11} = 3.83171$)*, the second term* ($m = 2, \alpha_{12} = 7.01559$) *and the third term* ($m = 3, \alpha_{13} = 10.1735$) *at the initial time.*

As show in Table 5, the approximate series solutions of the vibrations of a circular membrane obtained by using the DTM are compared with the analytical solution. The results given in Table 5 showed that, the series solutions obtained by the DTM are equal to the analytical solutions.

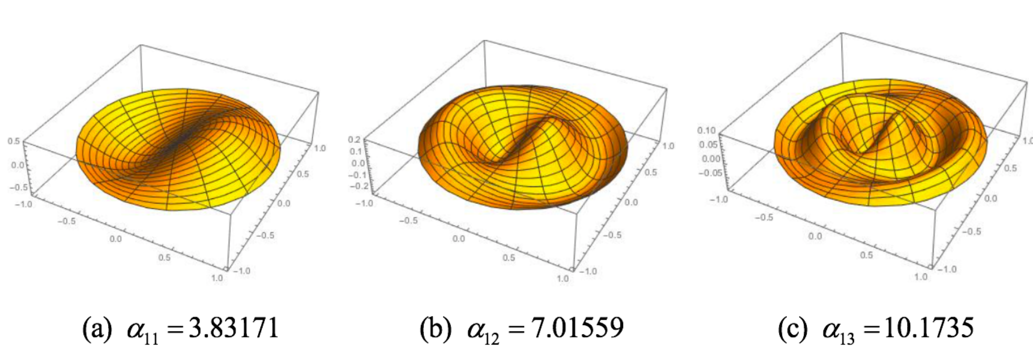


Figure 3: Normal modes of the vibrations of a circular membrane depending on both r and θ for Example 3, when $n = 1$.

Table 5: The comparison results for Example 3 between DTM solutions and analytical solutions at the initial time.

α_{nm}	r \ θ	0		π		2π	
		DTM	Analytical	DTM	Analytical	DTM	Analytical
$\alpha_{11} = 3.83171$	0	0	0	0	0	0	0
	0.01	0.01847	0.01847	-0.01847	-0.01847	0.01847	0.01847
	0.02	0.03692	0.03692	-0.03692	-0.03692	0.03692	0.03692
$\alpha_{12} = 7.01559$	0	0	0	0	0	0	0
	0.01	0.01036	0.01036	-0.01036	-0.01036	0.01036	0.01036
	0.02	0.02715	0.02715	-0.02715	-0.02715	0.02715	0.02715
$\alpha_{13} = 10.1735$	0	0	0	0	0	0	0
	0.01	0.00856	0.00856	-0.00856	-0.00856	0.00856	0.00856
	0.02	0.00170	0.00170	-0.00170	-0.00170	0.00170	0.00170

Example 4. *Let us consider the Example 2 with the initial shape defined by* $f(r,\theta) = r(1-r^4)\cos(2\theta)$. *From the initial shape, we obtain only* $n = 2$ *(see Table 2) because* $A_{nm} = 0$, *when* $n = 0, 1$ *and* $n > 2$, *and* B_{nm} *in Eq.*(9) *is always zero. The initial values are* $G(0) = A_{2m}, G(1) = 0, Q(1) = 0, Q(1) = 0, W(1) = 0$ *and* $W(2) = \frac{\alpha_{2m}}{8}$. *Therefore, the series solution of vibrations of a circular membrane is*

$$u(r,\theta,t) = \sum_{n=0}^{\infty} q_n(\theta) \sum_{m=1}^{\infty} w_{nm}(r) g_{nm}(t) = q_2(\theta) \sum_{m=1}^{\infty} w_{2m}(r) g_{2m}(t)$$

$$= \left(1 - 2\theta^2 + \frac{2\theta^4}{3} - \cdots\right)\Big[(0.641953r^2 - 1.41094r^4 + \cdot)(1.1876 - 15.6612 + \cdots) -$$

$$(1.05216r^2 - 6.21209r^4 + \cdots)(0.145193 - 5.14345t^2 + \cdots) +$$

$$(1.45248r^2 - 16.3427r^4 + \cdots)(0.197975 - 13.3653t^2 + \cdots) - \cdots\Big].$$

where A_{2m} *are calculated by Eq.*(8) *and* α_{2m} *is the mth positive zero of* J_2 *as shown in Table 4.*

5 Conclusions

The importance of the differential transformation method (DTM) lies in the initial values of the recursive formulas derived from the conversion of the ODEs problem, and the recursive formulas used to find the coefficients of the series solution of the problem. In this work, solving of the vibration problem of a circular membrane, the initial values of the recursive formulas can be calculated from the coefficients of the Fourier-Bessel series which is in the integral form of the initial shape of the membrane. The comparison of the DTM series solutions and the analytical solutions show that these are the same, hence it follows that the DTM method is suitable for this type of problems. Moreover, we consider that DTM is easier to use from the programming point of view. Since in the case of the nonlinear vibrating circular membrane problems are difficult to be unsolvable analytically, we intend to modify and apply the steps of the DTM method presented in this work to these problems, in a future research.

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