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BOAS-TYPE THEOREMS FOR THE MODIFIED WHITTAKER TRANSFORM

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Abstract

In this paper, we give necessary and sufficient conditions in terms of $\mathcal{F}_W(f)$, the modified Whittaker transform of f, to ensure that f belongs either to one of the generalized Lipschitz classes H^p_δ and h^p_δ for $\delta > 0$.

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1 Introduction

Harmonic analysis is a fundamental branch of mathematics that deals with the representation of functions or signals as sums or integrals of simpler, oscillatory components, such as sines and cosines. At its core, harmonic analysis aims to break down complex functions into elementary waveforms, making it an indispensable tool across many areas of pure and applied mathematics.
The significance of harmonic analysis lies in its ability to transform complicated problems into more manageable ones. By decomposing a function into its harmonic components, it allows for a deeper understanding of its structure and properties. This process is particularly useful in understanding periodic phenomena, such as sound waves, light waves, and vibrations. It also underpins critical tools like the Fourier transform, which has applications across various fields including signal processing, quantum mechanics, and data analysis.

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In mathematics, harmonic analysis plays a key role in understanding the properties of functions, spaces, and operators. It bridges the gap between different areas of analysis, from pure theory to practical applications. Its techniques are used in differential equations, functional analysis, and group theory, helping to solve problems that arise in many other branches of mathematics, including number theory and geometry.
Beyond mathematics, harmonic analysis has profound applications in physics, engineering, and computer science. In signal processing, for example, it enables the compression, filtering, and transmission of data by decomposing signals into frequencies, allowing for efficient processing and storage. In physics, it aids in the study of wave phenomena, quantum states, and the behavior of physical systems over time.
In summary, harmonic analysis is a cornerstone of modern mathematics, offering both a theoretical framework and practical tools to understand and manipulate functions, signals, and systems. Its importance extends well beyond mathematics itself, impacting numerous scientific and engineering disciplines.
Integral transforms and their inverses (e. g. the Bessel transform) are widely used to solve various problems in calculus, mechanics, mathematical physics and computational mathematics (see [11], [12], [15], [16], [21], [22]).
Boas results are one of the classical topics in harmonic analysis and approximation theory which consists in finding necessary and sufficient conditions on the Fourier coefficients of a function to belong to a generalised Lipschitz class.
In 1967, Boas proved the first characterisation of this type, see [2]. Since then, this theory has been widely studied by several authors.
In [2, 7], the author has studied the continuity and smoothness properties of a function f with absolutely convergent Fourier series. He gave the best possible sufficient conditions in terms of the Fourier coefficients of f which ensure the belonging of f either to one of the Lipschitz classes $Lip(\alpha)$ and $lip(\alpha)$ for some $0 < \alpha \leq 1$.
Tikhonov, in [14, 13], considered the cases of cosine and sine series separately. Recently,Volosivets have published several papers, see [17, 18, 19], in which he generalized all previous results. Our aim is to generalize these results for the Bessel transform which is found as a very useful mathematical tool in many fields of physics, geophysics, signal processing and other fields [1, 4, 20].
In this paper, we will see an analogue of this theory for the modified Whittaker transform.
We will try to complete the work done in our research laboratory on this theory, see [6, 5, 8]. To do this, we will start by recalling the necessary results of this transformation that we need to study this theory.

2 Preliminaries

In this section, we will see the main results which will help us find the desired result.

We consider the operator L defined by

$$L = -\frac{1}{4}\left[x^2 + \frac{d^2}{dx^2} + \left(x^{-1} + (3-4\alpha)x\right)\frac{d}{dx}\right].$$

The operator L has the form of Sturm-Liouville operator

$$L = -\frac{1}{4}\left[x^2 + \frac{d^2}{dx^2} + \frac{[x^2 m(x)]'}{m(x)}\frac{d}{dx}\right],$$

being m the function

$$m(x) = x^{1-4\alpha}e^{-\frac{1}{2x^2}}. \tag{1}$$

We define

$$\mathcal{W}_{\alpha,v}(x) = 2^\alpha x^{2\alpha} e^{\frac{1}{2x^2}} W_{\alpha,v}\left(\frac{1}{2x^2}\right) = (2x^2)^{-\frac{1}{2}+\alpha+v}\Psi\left(\frac{1}{2}-\alpha-v, 1-2v, \frac{1}{2x^2}\right), \tag{2}$$

where $W_{\alpha,v}(z)$ is the Whittaker function of the second kind, $\alpha < \frac{1}{2}$ and $v \in \mathbb{C}$ are parameters. Unless stated otherwise, the parameter $\alpha < \frac{1}{2}$ is held fixed throughout the discussion.
By transformation of the Whittaker differential equation, the function $\mathcal{W}_{\alpha,v}(x)$ is a solution of the differential equation

$$Lf = \left((\frac{1}{2}-\alpha)^2 - v^2\right)f.$$

In , it gives a representation of $\mathcal{W}_{\alpha,v}(x)$ the function as follows:

Theorem 2.1. *The confluent hypergeometric-type function (2) admits the integral representation*

$$\mathcal{W}_{\alpha,v}(x) = \int_0^{+\infty} \cosh(vs)\eta_x(s)ds,$$

with $\alpha, v \in \mathbb{C}$, $x > 0$ and

$$\eta_x(s) = (2\pi)^{-\frac{1}{2}}x^{-1+2\alpha}exp\left(\frac{1}{2x^2} - \frac{1}{4x^2}\cosh^2\left(\frac{s}{2}\right)\right)D_{2\alpha}\left(\frac{1}{x}\cosh\left(\frac{s}{2}\right)\right),$$

being $D_\mu(z)$ the parabolic cylinder function, given by

$$D_\mu(z) = \frac{z^\mu e^{-\frac{z^2}{4}}}{\Gamma\left(\frac{1}{2}(1-\mu)\right)}\int_0^{+\infty} e^{-s}s^{-\frac{1}{2}(1+\mu)}\left(1+\frac{2s}{z^2}\right)^{\frac{\mu}{2}}ds,$$

with $\Re z > 0$ and $\Re\mu < 1$.
$\Re z$ the real part of z.

Now, we will see certain properties of the function $\mathcal{W}_{\alpha,v}(x)$ which are proven in [9].

Lemma 2.2. *The following inequalities are fulfilled:*

1. *$|\mathcal{W}_{\alpha,v}(x)| \leq 1$, for $x \geq 0$, $|\Re v| \leq \frac{1}{2} - \alpha$.*
2. *$1 - \mathcal{W}_{\alpha,v}(x) \leq 2\Big((\frac{1}{2} - \alpha)^2 - v^2\Big)x^2$, for $x \geq 0$, $v \in [0, \frac{1}{2} - \alpha] \cap i\mathbb{R}$.*
3. *$|1 - \mathcal{W}_{\alpha,v}(x)| \geq c$, for $x \geq 1$, $|\Re v| < \frac{1}{2} - \alpha$, where $c > 0$ is a constant which depends only on α.*

Let $L^p(\mathbb{R}^+, d\mu)$ the space of measurable function on $[0; +\infty[$ satisfying:

$$\|f\|_p = \Big(\int_0^{+\infty} |f(x)|^p d\mu(x)\Big)^{\frac{1}{p}} < +\infty,$$

if $1 \leq p < +\infty$, and

$$\|f\|_\infty = esssup_{x\in\mathbb{R}^+}|f(x)|,$$

if $p = +\infty$ and μ is a positive measure.
$Loc(\mathbb{R}^+)$ the space of locally integrable functions.
In this paper, α is a real number such that $\alpha < \frac{1}{2}$. We consider the modified Whittaler transform:

$$\mathcal{F}_W(f)(\lambda) = \int_0^{+\infty} f(x)\Phi_\lambda^\alpha(x)m(x)dx$$

for every $\lambda \geq 0$.
Where $m(x)$ is defined by formula (1) ,and

$$\Phi_\alpha^\lambda(x) = \mathcal{W}_{\alpha,v_\lambda}(x).$$

With

$$v_\lambda = \sqrt{(\frac{1}{2} - \alpha)^2 - \lambda},$$

and $\mathcal{W}_{\alpha,v}(x)$ is defined by formula (2).
We have that:

$$\Phi_\alpha^\lambda(0) = 1.$$

We denote by $C_c^2(0,\infty)$, the space of functions with class C^2 on $(0,\infty)$ with compact support.
The following inversion theorem for the modified Whittaker transform is proved in[10]:

Theorem 2.3. *Let $f \in C_c^2(0,\infty)$. For $\alpha < \frac{1}{2}$, we have*

$$f(x) = \int_0^{+\infty} \mathcal{F}_W(f)(\lambda)\Phi_\lambda^\alpha(x)\rho(\lambda)d\lambda.$$

Where the right-hand side integral converges absolutely for each $x > 0$, and

$$\rho(\lambda) = 2^{1-2\alpha}\pi^{-2}\sinh(-2\pi i v_\lambda)|\Gamma(\frac{1}{2} - \alpha + v_\lambda)|^2\mathcal{U}_\Lambda(\lambda).$$

Such that $\mathcal{U}_\Lambda$ is the characteristic function of the interval $\Lambda = \Big((\frac{1}{2} - \alpha)^2, +\infty\Big)$.

From [9], we have the following lemma:

Lemma 2.4. *The function Φ_α^λ, $\lambda > (\frac{1}{2}-\alpha)^2$ possesses the following properties:*

1. $-1 \leq \Phi_\alpha^\lambda(x) \leq 1$, *for* $x \geq 0$.

2. $1-\Phi_\alpha^\lambda(x) \leq 2\lambda x^2$, *for* $x \geq 0$.

3. $1-\Phi_\alpha^\lambda(x) \geq c$, *for* $x \geq 1$, *where* $c>0$ *is a constant which depends only on* α.

By $L^p(A;\omega(x)dx)$ with $A \subset \mathbb{R}$ and $1 \leq p \leq \infty$ we denote the weighted L^p-space with the norm

$$\|f\|_{L^p(A;\omega(x)dx)} = \left(\int_A |f(x)|^p \omega(x)dx\right)^{\frac{1}{p}},$$

if $1 \leq p < \infty$, and

$$\|f\|_{L^\infty(A;\omega(x)dx)} = ess \quad \sup_{x\in A} |f(x)|,$$

if $p=\infty$.

We now define the generalized translation operator with whom we will work for $\alpha < \frac{1}{2}$.

For every bounded continuous function on $(0,+\infty)$, the linear operator

$$T_h f(x) = \int_0^{+\infty} f(s) q(x,h,s) m(s) ds,$$

for every $x, y > 0$ will be called the Whittaker translation. Where

$$q(x,y,\xi)=\frac{(xy\xi)^{-1+2\alpha}}{(2\pi)^{\frac{1}{2}}} exp\left(\frac{1}{2x^2}+\frac{1}{2y^2}+\frac{1}{2\xi^2}-\left(\frac{x^2+y^2+\xi^2}{4xy\xi}\right)^2\right) D_{2\alpha}\left(\frac{x^2+y^2+\xi^2}{2xy\xi}\right).$$

With:

$$D_\mu(z) = \frac{z^\mu e^{-\frac{z^2}{4}}}{\Gamma\big((\frac{1}{2}(1-\mu)\big)} \int_0^{+\infty} e^{-s} s^{-\frac{1}{2}(1+\mu)} \left(1+\frac{2s}{z^2}\right)^{\frac{\mu}{2}} ds,$$

for $Re z > 0$ and $Re\mu < 1$. In particular, for $x, y, \xi > 0$ we have

$$q(x,y,\xi) = q(y,x,\xi) = q(x,\xi,y) = q(\xi,y,x),$$

and

$$\int_0^{+\infty} q(x,y,\xi) m(\xi) d\xi = 1.$$

In addition, if $\alpha < \frac{1}{2}$, we have the positivity condition

$$q(x,y,\xi) > 0,$$

for every $x, y, \xi > 0$.

The Whittaker translation operator is connected with the modified Whittaker transform $\mathcal{F}_W$ via the following formula proven in [9]:

Lemma 2.5. *If $f \in L^2(\mathbb{R}^+; m(x)dx)$ and $h > 0$, we have:*

$$\mathcal{F}_W(T_h f)(\lambda) = \Phi_\alpha^\lambda(h)\mathcal{F}_W(f)(\lambda),$$

for every $\lambda \geq 0$.

3 Main results

We define differences of the order $p \in \mathbb{N}$ by

$$\Delta_h^p f(x) = (2I - T_h)^p f(x).$$

We have that:

$$\mathcal{F}_W(\Delta_h^p f)(\lambda) = \big(2 - \Phi_\alpha^\lambda(h)\big)^p \mathcal{F}_W(f)(\lambda).$$

If $f \in C_c^2(\mathbb{R}^+)$, by using de formula of inversion of modified Whittaker transform we get:

$$\Delta_h^p f(x) = \int_0^{+\infty} \big(2 - \Phi_\alpha^\lambda(h)\big)^p \mathcal{F}_w(f)(\lambda)\rho(\lambda)d\lambda.$$

Now, we define the generalized Lipschitz H_α^p and h_α^p.
A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said belong to H_δ^p for $\delta > 0$ if:

$$\Delta_h^p f(x) = O(h^\delta),$$

as $h \to 0$. and is said belong to h_δ^p for $\delta > 0$ if:

$$\Delta_h^p f(x) = o(h^\delta),$$

as $h \to 0$.
Next, we will prove a result that will be essential for establishing the first result in this paper.

Lemma 3.1. *Let g be a non-negative, measurable function defined on $\mathbb{R}^+$, and $p \in \mathbb{N}$. Then:*

1. *if $0 < \delta \leq p$ is given number, $x^p g(x) \in L^1(\mathbb{R}^+, d\mu) \cap L_{loc}(\mathbb{R}^+)$ such that μ is a positive measure and*

$$\int_0^y x^p g(x)d\mu(x) = O(y^{p-\delta}), \tag{3}$$

   for every $y > 0$. Then $g \in L^1(]y; +\infty[, d\mu)$ and

$$\int_y^{+\infty} g(x)d\mu(x) = O(y^{-\delta}), \tag{4}$$

   for every $y > 0$.

2. *Conversely, if* $0 \leq \delta < p$ *and* (4) *holds, then* (3) *also holds.*

Proof. Proof analogous to the proof of Lemma 3.1 of [6]. ■
Throughout the rest, we use the following notation:

$$dv_\alpha(\lambda) = \rho(\lambda)\mathcal{U}_\Lambda(\lambda)d\lambda.$$

So, we get:

$$\Delta_h^p f(x) = \int_0^{+\infty} \big(2 - \Phi_\alpha^\lambda(h)\big)^p \mathcal{F}_w(f)(\lambda)dv_\alpha(\lambda).$$

We will see the first main result of this paper.

Theorem 3.2. *Let* $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$, $p \in \mathbb{N}$ *and* $f \in L^1(\mathbb{R}^+, m(x)dx)$.

1. *Suppose that* $f \in L^1(\mathbb{R}^+, m(x)dx) \cap C_c^2(\mathbb{R}^+)$. *If*

$$\int_0^y |\lambda|^{2p} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) = O\big(y^{2p-\delta}\big), \tag{5}$$

for every $y > 0$. *Then* $\mathcal{F}_W(f) \in L^1(]y; +\infty[, dv_\alpha(\lambda))$ *and* $f \in H_\delta^p$.

2. *Conversely, suppose that* $f \in L^1(\mathbb{R}^+, m(x)dx)$, $\mathcal{F}_W(f) \in L^1(]y; +\infty[, dv_\alpha(\lambda))$. *If* $f \in H_p^\delta$ *and* $\mathcal{F}_W(f)$ *is non-negative or nonpositive then* (5) *holds.*

Remark 3.3. *From Lemma 3.1, it follows that the statement* (5) *implies*

$$\int_y^{+\infty} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) = O\big(y^{-\delta}\big).$$

Proof.

1. If $C_c^2(\mathbb{R}^+)$, then f has has a compact support, let $supp(f) = [a, b] \subset (0, +\infty)$. Since $f \in C^2(\mathbb{R}^+)$ then f admits a primitive F. So

$$\begin{aligned}\int_0^{+\infty} f(x)dx &= \int_a^b f(x)dx \\ &= F(b) - F(a) < +\infty.\end{aligned}$$

Which shows that $f \in L_{loc}(\mathbb{R}^+)$.
By Remark 3.3 we have

$$\int_y^{+\infty} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) = O\big(y^{-\delta}\big),$$

and

$$\mathcal{F}_W(f) \in L^1(]y; +\infty[, dv_\alpha(\lambda)).$$

By Lemma 2.4 we get:

$$1 \leq 2 - \Phi_\alpha^\lambda(x) \leq 3,$$

for each $x \in \mathbb{R}^+$.
We have that:

$$\begin{aligned}
|\Delta_h^p f(x)| &= |\int_0^{+\infty} \big(2-\Phi_\alpha^\lambda(h)\big)^p \Phi_\alpha^\lambda(x)\mathcal{F}_W(f)(\lambda)dv_\alpha(\lambda)| \\
&\le \int_0^{+\infty} \big|2-\Phi_\alpha^\lambda(h)\big|^p |\Phi_\alpha^\lambda(x)||\mathcal{F}_W(f)(\lambda)|dv_\alpha(\lambda) \\
&\le 3^p \int_0^{+\infty} |\mathcal{F}_W(f)(\lambda)|dv_\alpha(\lambda) \\
&= 3^p(I_1+I_2),
\end{aligned}$$

such that:

$$I_1 = \int_0^{\frac{1}{h}} |\mathcal{F}_W(f)(\lambda)|dv_\alpha(\lambda),$$

and

$$I_2 = \int_{\frac{1}{h}}^{+\infty} |\mathcal{F}_W(f)(\lambda)|dv_\alpha(\lambda).$$

By hypothesis, we have

$$\int_0^{\frac{1}{h}} |\lambda|^{2p}|\mathcal{F}_W(f)(\lambda)|dv_\alpha(\lambda) = O\big(\frac{1}{h^{2p-\delta}}\big).$$

By Lemma 3.1 we get:

$$\int_{\frac{1}{h}}^{+\infty} |\mathcal{F}_W(f)(\lambda)|dv_\alpha(\lambda) = O(h^\delta).$$

To estimate I_1, set

$$\phi(x) = \int_x^{\infty} |\mathcal{F}_W(f)(\lambda)|dv_\alpha(\lambda).$$

We have that

$$\int_0^x \lambda\phi'(\lambda)d\lambda \le x\int_0^x \phi'(\lambda)d\lambda.$$

So:

$$\int_0^x \lambda\phi'(\lambda)d\lambda \le x\Bigg(\phi(x)-\phi(\Big(\frac{1}{2}-\alpha\Big)^2)\Bigg).$$

It follows that:

$$\int_0^x \lambda\phi'(\lambda)d\lambda \le -x\int_{\left(\frac{1}{2}-\alpha\right)^2}^x |\mathcal{F}_W(f)(\lambda)|dv_\alpha(\lambda).$$

Then:

$$\int_0^x |\mathcal{F}_W(f)(\lambda)|dv_\alpha(\lambda) \le \frac{-1}{x}\int_0^x \lambda\phi'(\lambda)d\lambda.$$

Upon an integration by parts, we obtain:

$$\int_0^x \lambda\phi'(\lambda)d\lambda = x\phi(x) - \int_{\left(\frac{1}{2}-\alpha\right)^2}^x \phi(\lambda)d\lambda.$$

So:

$$\frac{-1}{x}\int_0^x \lambda\phi'(\lambda)d\lambda = \frac{1}{x}\int_0^x \phi(\lambda)d\lambda - \phi(x).$$

Then

$$\frac{-1}{x}\int_0^x \lambda\phi'(\lambda)d\lambda \leq \frac{1}{x}\int_0^x \phi(\lambda)d\lambda.$$

So, if $x \to +\infty$

$$\frac{-1}{x}\int_0^x \lambda\phi'(\lambda)d\lambda \leq \frac{1}{x}\int_0^x O(\lambda^{-\alpha})d\lambda = O(x^{-\delta}).$$

Then

$$\int_0^x |\mathcal{F}_W(f)(\lambda)|dv_\alpha(\lambda) = O(x^{-\delta}).$$

So

$$\int_0^{\frac{1}{h}} |\mathcal{F}_W(f)(\lambda)|dv_\alpha(\lambda) = O(h^\delta).$$

Which shows that

$$I_1 = O(h^\delta).$$

Then

$$\Delta_h^p f(x) = 3^p\Big(O(h^\delta) + O(h^\delta)\Big).$$

It follows that

$$\Delta_h^p f(x) = O(h^\delta).$$

Which shows that

$$f \in H_\delta^p.$$

2. We have that:

$$\int_0^{\frac{\beta}{h}} |\lambda|^{2p}|\mathcal{F}_W(f)(\lambda)|dv_\alpha(\lambda) \leq \left(\frac{\beta}{h}\right)^{2p}\int_0^{\frac{\beta}{h}} |\mathcal{F}_W(f)(\lambda)|dv_\alpha(\lambda).$$

So

$$\int_0^{\frac{\beta}{h}} |\lambda|^{2p}|\mathcal{F}_W(f)(\lambda)|dv_\alpha(\lambda) \leq \left(\frac{\beta}{h}\right)^{2p}\int_0^{+\infty} |\mathcal{F}_W(f)(\lambda)|dv_\alpha(\lambda).$$

Since

$$1 \leq \big(2 - \Phi_\alpha^\lambda(h)\big)^p.$$

Then

$$\int_0^{\frac{\beta}{h}} |\lambda|^{2p}|\mathcal{F}_W(f)(\lambda)|dv_\alpha(\lambda) \leq \left(\frac{\beta}{h}\right)^{2p}\int_0^{+\infty} \big(2 - \Phi_\alpha^\lambda(h)\big)^p|\mathcal{F}_W(f)(\lambda)|dv_\alpha(\lambda).$$

First case: If $\mathcal{F}_W(f)(\lambda) \geq 0$ for every $\lambda > 0$, Then

$$|\Delta_h^p f(0)| = \int_0^{+\infty} \big(2 - \Phi_\alpha^\lambda(h)\big)^p \mathcal{F}_W(f)(\lambda) dv_\alpha(\lambda).$$

Since $f \in H_\delta^p$, so:

$$\begin{aligned}\int_0^{\frac{\beta}{h}} |\lambda|^{2p} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) &\leq \left(\frac{\beta}{h}\right)^{2p} \int_0^{+\infty} \big(2 - \Phi_\alpha^\lambda(h)\big)^p \mathcal{F}_W(f)(\lambda) dv_\alpha(\lambda) \\ &= |\Delta_h^p f(0)| \left(\frac{\beta}{h}\right)^{2p} \\ &= O(h^\delta)\left(\frac{\beta}{h}\right)^{2p}.\end{aligned}$$

Hence, for $y = \frac{\beta}{h}$, we get:

$$\int_0^{y} |\lambda|^{2p} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) = O\left(\left(\frac{\beta}{y}\right)^{\delta - 2p}\right).$$

So:

$$\int_0^{y} |\lambda|^{2p} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) = O\big(y^{2p-\delta}\big).$$

Thus, (5) is holds.

Second case: If $\mathcal{F}_W(f)(\lambda) \leq 0$ for every $\lambda > 0$, Then

$$|\Delta_h^p f(0)| = -\int_0^{+\infty} \big(2 - \Phi_\alpha^\lambda(h)\big)^p \mathcal{F}_W(f)(\lambda) dv_\alpha(\lambda).$$

Since $f \in H_\delta^p$ so:

$$\begin{aligned}\int_0^{\frac{\beta}{h}} |\lambda|^{2p} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) &\leq \left(\frac{\beta}{h}\right)^{2p} \left(-\int_0^{+\infty} \big(2 - \Phi_\alpha^\lambda(h)\big)^p \mathcal{F}_W(f)(\lambda) dv_\alpha(\lambda)\right) \\ &= |\Delta_h^p f(0)| \left(\frac{\beta}{h}\right)^{2p} \\ &= O(h^\delta)\left(\frac{\beta}{h}\right)^{2p}.\end{aligned}$$

Hence, for $y = \frac{\beta}{h}$, we get:

$$\int_0^{y} |\lambda|^{2p} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) = O\left(\left(\frac{\beta}{y}\right)^{\delta - 2p}\right).$$

So:

$$\int_0^y |\lambda|^{2p}|\mathcal{F}_W(f)(\lambda)|dv_\alpha(\lambda) = O\big(y^{2p-\delta}\big).$$

Thus, (5) is holds.

■

Now, to present the second main result of this paper, we will state the following Lemma:

Lemma 3.4. *Let g be a non-negative, measurable function defined on $\mathbb{R}^+$, and $p \in \mathbb{N}$. Then:*

1. *if $0 < \alpha < p$ is given number, $x^p g(x) \in L^1(\mathbb{R}^+, d\mu) \cap L_{loc}(\mathbb{R}^+)$ such that μ is a positive measure and*

$$\int_0^y x^p g(x)d\mu(x) = o(y^{p-\delta}), \tag{6}$$

as $y \to +\infty$. Then $g \in L^1(]y; +\infty[, d\mu)$ for y large and

$$\int_y^{+\infty} g(x)d\mu(x) = o(y^{-\delta}), \tag{7}$$

as $y \to +\infty$.

2. *Conversely, if $0 < \delta \leq p$ and (4),(7) holds, then (6) also holds.*

Proof. Proof analogous to the proof of Lemma 3.3 of [6]. ■

Now, we state and prove the second main result of this paper:

Theorem 3.5. *Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$, $p \in \mathbb{N}$ and $f \in L^1(\mathbb{R}^+, m(x)dx)$.*

1. *Suppose that $f \in L^1(\mathbb{R}^+, m(x)dx) \cap C_c^2(\mathbb{R}^+)$. If*

$$\int_0^y |\lambda|^{2p}|\mathcal{F}_W(f)(\lambda)|dv_\alpha(\lambda) = o\big(y^{2p-\delta}\big), \tag{8}$$

for every $y > 0$. Then $\mathcal{F}_W(f) \in L^1(]y; +\infty[, dv_\alpha(\lambda))$ and $f \in h_\delta^p$

2. *Conversely, suppose that $f \in L^1(\mathbb{R}^+, m(x)dx)$, $\mathcal{F}_W(f) \in L^1(]y; +\infty[, dv_\alpha(\lambda))$. If $f \in h_p^\delta$ and $\mathcal{F}_W(f)$ is non-negative or nonpositive then (8) holds.*

Remark 3.6. *From Lemma 3.4, it follows that:*
The statement (8) implies

$$\int_y^{+\infty} |\mathcal{F}_W(f)(\lambda)|dv_\alpha(\lambda) = o\big(y^{-\delta}\big).$$

Proof.

1. By Remark 3.6 we have

$$\int_{y}^{+\infty} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) = o\big(y^{-\delta}\big),$$

and

$$\mathcal{F}_W(f) \in L^1(]y;+\infty[, dv_\alpha(\lambda)).$$

We have that:

$$\begin{aligned} |\Delta_h^p f(x)| &= |\int_0^{+\infty} \big(2-\Phi_\alpha^\lambda(h)\big)^p \Phi_\alpha^\lambda(x) \mathcal{F}_W(f)(\lambda) dv_\alpha(\lambda)| \\ &\leq \int_0^{+\infty} \big|2-\Phi_\alpha^\lambda(h)\big|^p |\Phi_\alpha^\lambda(x)| |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) \\ &\leq 3^p \int_0^{+\infty} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) \\ &= 3^p(I_1+I_2), \end{aligned}$$

such that:

$$I_1 = \int_0^{\frac{1}{h}} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda),$$

and

$$I_2 = \int_{\frac{1}{h}}^{+\infty} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda).$$

By hypothesis we have

$$\int_0^{\frac{1}{h}} |\lambda|^{2m} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) = o\big(\frac{1}{h^{2p-\delta}}\big).$$

By Lemma 3.1 we get:

$$\int_{\frac{1}{h}}^{+\infty} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) = o(h^\delta).$$

Using the same method that we used in the proof of the Theorem 3.2 we will obtain:

$$I_1 = o(h^\delta).$$

Then

$$\Delta_h^p f(x) = 3^p \Big(o(h^\delta) + o(h^\delta)\Big).$$

It follows that

$$\Delta_h^p f(x) = o(h^\delta).$$

Which show that

$$f \in h_\delta^p.$$

2. We have that:

$$\int_0^{\frac{\beta}{h}} |\lambda|^{2p} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) \leq \left(\frac{\beta}{h}\right)^{2p} \int_0^{\frac{\beta}{h}} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda).$$

So

$$\int_0^{\frac{\beta}{h}} |\lambda|^{2p} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) \leq \left(\frac{\beta}{h}\right)^{2p} \int_0^{+\infty} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda).$$

Since

$$1 \leq \big(2 - \Phi_\alpha^\lambda(h)\big)^p.$$

Then

$$\int_0^{\frac{\beta}{h}} |\lambda|^{2p} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) \leq \left(\frac{\beta}{h}\right)^{2p} \int_0^{+\infty} \big(2 - \Phi_\alpha^\lambda(h)\big)^p |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda).$$

First case: If $\mathcal{F}_W(f)(\lambda) \geq 0$ for every $\lambda > 0$, then

$$|\Delta_h^p f(0)| = \int_0^{+\infty} \big(2 - \Phi_\alpha^\lambda(h)\big)^p \mathcal{F}_W(f)(\lambda) dv_\alpha(\lambda).$$

Since $f \in h_\delta^p$, so:

$$\begin{aligned}\int_0^{\frac{\beta}{h}} |\lambda|^{2p} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) &\leq \left(\frac{\beta}{h}\right)^{2p} \int_0^{+\infty} \big(2 - \Phi_\alpha^\lambda(h)\big)^p \mathcal{F}_W(f)(\lambda) dv_\alpha(\lambda) \\ &= |\Delta_h^p f(0)| \left(\frac{\beta}{h}\right)^{2p} \\ &= o(h^\delta) \left(\frac{\beta}{h}\right)^{2p}.\end{aligned}$$

Hence, for $y = \frac{\beta}{h}$, we get:

$$\int_0^y |\lambda|^{2p} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) = o\left(\left(\frac{\beta}{y}\right)^{\delta - 2p}\right).$$

So:

$$\int_0^y |\lambda|^{2p} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) = o\big(y^{2p-\delta}\big).$$

Thus, (8) is holds.
Second case: If $\mathcal{F}_W(f)(\lambda) \leq 0$ for every $\lambda > 0$, then

$$|\Delta_h^p f(0)| = -\int_0^{+\infty} \big(2 - \Phi_\alpha^\lambda(h)\big)^p \mathcal{F}_W(f)(\lambda) dv_\alpha(\lambda).$$

Since $f \in h^p_\delta$ so:

$$\begin{aligned}\int_0^{\frac{\beta}{h}} |\lambda|^{2p} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) &\leq \left(\frac{\beta}{h}\right)^{2p} \left(- \int_0^{+\infty} \left(2 - \Phi_\alpha^\lambda(h)\right)^p \mathcal{F}_W(f)(\lambda) dv_\alpha(\lambda) \right) \\ &= |\Delta_h^p f(0)| \left(\frac{\beta}{h}\right)^{2p} \\ &= o(h^\delta) \left(\frac{\beta}{h}\right)^{2p}.\end{aligned}$$

Hence, for $y = \frac{\beta}{h}$, we get:

$$\int_0^y |\lambda|^{2p} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) = o\left(\left(\frac{\beta}{y}\right)^{\delta - 2p} \right).$$

So:

$$\int_0^y |\lambda|^{2p} |\mathcal{F}_W(f)(\lambda)| dv_\alpha(\lambda) = o\big(y^{2p-\delta}\big).$$

Thus, (8) is holds.

■

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