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FRACTIONAL ψ-HILFER DERIVATIVE SPACES: STUDY OF KIRCHHOFF PROBLEM WITH $P(\cdot)$-LAPLACIAN OPERATOR

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Abstract

The paper focuses on the existence and multiplicity of weak solution to the nonlinear Kirchhoff-type equations involving ψ-Hilfer derivatives with $p(\cdot)$-Laplacian operators and Dirichlet boundary conditions. Through the application of a critical point approach, along with genus theory and variational techniques, we establish the existence of infinitely many positive solutions within appropriate ψ-Hilfer fractional derivative spaces. Our novel main results contribute to the advancement of the literature on differential equations involving ψ-Hilfer fractional derivative.

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1 Introduction

Fractional derivatives serve as extensions of classical derivatives. The idea of fractional differentiation used to align with ordinary differentiation until relatively recently. However, contemporary research has shifted towards emphasizing fractional differentiation as a broader concept than ordinary differentiation. Fractional analysis, within mathematical analysis, explores various ways of defining real number powers or complex powers of the differentiation operator and integration. Fractional differential equations, representing generalized and noninteger differential equations, emerge in both time and space domains, featuring a power-law memory kernel that captures nonlocal relationships [22].

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There are various options for introducing fractional integro-differentiation operations, in particular, the Riemann-Liouville, Caputo, Grunwald-Letnikov approaches, and their various modifications. It is important to highlight that a majority of the literature on fractional differentiation predominantly focuses on Riemann–Liouville and Caputo fractional derivatives. Additionally, there are other established definitions, such as the fractional derivative of Hadamard, the fractional derivative of Erdélyi-Kober, and more. For further details on fractional calculus, interested readers are directed to [20, 33].

Currently, it is important to factor in the changing landscape of modern physics and mechanics when creating mathematical models. Based on relevant studies, we need to focus on specific areas to better grasp the theory behind the main problem in this research. To achieve this, we use the generalized ψ-Hilfer fractional derivative to examine a nonlinear Kirchhoff equation with positive parameter. This equation follows Dirichlet boundary conditions and is expressed as:

$$\begin{cases} \left(\alpha+\beta\displaystyle\int_\Omega \frac{1}{\tau(x)}\left|\ {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u\right|^{\tau(x)}\right)\mathbf{L}_{\tau(x)}^{\gamma,\kappa;\psi}(u)=\xi|u|^{r(x)-2}u-h(x)|u|^{\tau(x)-2}u, & \text{in } \Omega,\\ u=0, \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

with

$$\mathbf{L}_{\tau(x)}^{\gamma,\kappa;\psi}(u):=\ {}^{\mathrm{H}}\mathrm{D}_{T}^{\gamma,\kappa;\psi}\left(|\ {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u|^{\tau(x)-2}\ {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u\right),$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N>3)$ with smooth boundary $\partial\Omega$, $h\in C(\bar{\Omega},\mathbb{R})$, ξ is a positive parameters, $\tau,r\in C^+(\bar{\Omega})$ such that:

$$1<r^-\le r(x)\le r^+<\tau^-\le\tau(x)\le\tau^+<2\tau^-<\tau^\star(x)=\frac{N\tau(x)}{N-\gamma\tau(x)}<N \tag{2}$$

for all $x\in\Omega$ and h satisfies the following hypothesis:
$(\mathbf{H_0})$ $h:\Omega\longrightarrow[0,\infty)$ such that $h\in L^\infty(\Omega)$.
${}^{\mathrm{H}}\mathrm{D}_{T}^{\gamma,\kappa;\psi}$ and ${}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}$ are ψ-Hilfer fractional derivatives of order $\frac{1}{\tau(x)}<\gamma<1$ and type $0\le\kappa\le1$ defined later, this type of derivative are introduced by Sousa et al. [27], by means of the Gronwall inequality. They discussed some of its particular cases like the ψ-Riemann-Liouville fractional partial derivative and the ψ-Caputo fractional partial derivative. For more detail we refer to in [28]. For some applications of this type of operator we refer to [3, 5, 2, 4, 12, 13, 31]. It is essential to note that, in the case where $\kappa\longrightarrow1$ and $\psi(x)=x$, our problem (1) reduces to the integer case, for more detail see [27]. For this reason, we observe that our problem generalizes many papers of Kirchhoff type in the literature (integre of fractional case). Recently, many paper take into account the study of Kirchhoff equation by using diferent operator. In [6] the authors utilazed the variational approach with mauntaine pass theorem to show the existence and

multiplicity of solutions for the following $p(\cdot)$-Kirchhoff type equation

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)|\nabla u|^{p(x)}}dx\right)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $p \in C(\overline{\Omega})$ with $1 < p(\cdot) < N$. Chung in [8] used the mountain pass Theorem combined with the minimum principle, to obtained at least two non-negative, non-trivial weak solutions for the following $p(x)$-Kirchhoff-type equations,

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)|\nabla u|^{p(x)}}dx\right)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda f(x,u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega,\\ u\geq 0, & \text{in } \Omega \end{cases} \tag{4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $p \in C(\overline{\Omega})$ with $1 < p(\cdot) < N$ and λ is a positive real parameter and $f \in C(\Omega\times\mathbb{R},\mathbb{R})$ is a Caratheodory function that may change sign. In [19] Shapour et al., used variational methods and critical point theory, to established the existence of multiple solutions for problem (3). For more examples of Kirchhoff-type problem, we refer the reder to [1, 14, 18, 30, 32].

In the context of Kirchhoff-type problem involving tempered fractional derivatives, we refer to [26]. The authors employed variational methods to establish the existence of infinitely many solutions to the following:

$$\begin{cases} M\left(\int_{\mathbb{R}} |D_+^{\alpha,\lambda}u(t)|^2dt\right) D_-^{\alpha,\lambda}(D_+^{\alpha,\lambda}u(t)) = f(t,u(t)), \quad t\in\mathbb{R},\\ u\in W_\lambda^{\alpha,2}(\mathbb{R}), \end{cases}$$

where $D_\pm^{\alpha,\lambda}(\cdot)$ denote the left and right tempered fractional derivatives of order $\alpha \in (1/2,1]$, $\lambda>0$, $f\in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $M\in C(\mathbb{R}^+,\mathbb{R}^+)$. In [24], the authors studied the existence and multiplicity of solutions to the following Kirchhoff equation with singular nonlinearity and Riemann-Liouville fractional derivative:

$$\begin{cases} \left(a+b\int_0^T |{}_0D_t^\alpha(u(t))|^p dt\right) {}_tD_T^\alpha(\Phi({}_0D_t^\alpha(u(t)))) = \frac{\lambda g(t)}{u^\gamma(t)} + f(t,u(t)),\, t\in(0,T),\\ u(0)=u(T)=0, \end{cases}$$

where $a\geq 1$, b, $\lambda>0$, $p>1$ are constants, $\frac{1}{p}<\alpha\leq 1$, $0<\gamma<1$, $g\in C([0,1])$ and $f\in C^1([0,T]\times\mathbb{R},\mathbb{R})$. Under appropriate assumptions on the function f, they employed variational methods to show the existence and multiplicity of positive solutions of the above problem with respect to the parameter λ. All this problem discussed above are associated to the stationary version of the Kirchhoff problem.

$$\varrho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{5}$$

presented by Kirchhoff in 1883 [21], which extends D'Alembert's wave equation. One notable feature of model (5) is that it contains a nonlocal term $\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2$. The parameters L, h, E, u, P_0 in model (5) represent different physical meanings, which we will not cover here.

Our approach to proving the existence and multiplicity results for problems (1) relies on the utilization of the critical points theorem together with genus theory and variational approach.

This work is organized as follows. In Section 2, we provide a brief overview of the key features of variable exponent Lebesgue spaces and ψ-Hilfer fractional derivative spaces. Moving to Section 3, we present the existing solutions to problem (1), along with their corresponding proofs.

2 Preliminary

In this section we collect preliminary concepts of the theory of variable exponent Lebesgue space, classical and ψ-Hilfer fractional derivative space (see [10, 11, 17, 16, 15]).

2.1 Variable exponent Lebesgue space

In the following, we define

$$C^+(\bar{\Omega}) = \Big\{\tau \in C(\Omega) : \ 1 < \tau^- \leq \tau^+ < +\infty\Big\},$$

where

$$\tau^- := \inf_{x\in\bar{\Omega}} \tau(x) \quad \text{and} \quad \tau^+ := \sup_{x\in\bar{\Omega}} \tau(x).$$

Denote by $\mathbf{U}(\Omega)$ the set of all measurable real-valued functions defined in Ω. For any $\tau \in C^+(\bar{\Omega})$, we denote the variable exponent Lebesgue space by

$$L^{\tau(x)}(\Omega) = \left\{u \in \mathbf{U}(\Omega) : \int_\Omega |u(x)|^{\tau(x)} \mathrm{d}x < \infty\right\},$$

equipped with the Luxemburg norm

$$\|u\|_{L^{\tau(x)}} = \inf\left\{\lambda > 0 : \ \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{\tau(x)} \mathrm{d}x \leq 1\right\},$$

then, the variable exponent Lebesgue space $\left(L^{\tau(x)}(\Omega), \|\cdot\|_{L^{\tau(x)}}\right)$ becomes a Banach space.

We have the following generalized Hölder inequality

$$\left|\int_\Omega u(x)v(x)\mathrm{d}x\right| \leq \ 2\|u\|_{L^{\tau(x)}}\|v\|_{L^{\tau'(x)}}, \tag{6}$$

for $u \in L^{\tau(x)}(\Omega)$, $v \in L^{\tau'(x)}(\Omega)$ such that $\frac{1}{\tau(x)} + \frac{1}{\tau'(x)} = 1$.
At this point, let define the following map $\sigma_{\tau(x)} : L^{\tau(x)}(\Omega) \longrightarrow \mathbb{R}$ by

$$\sigma_{\tau(x)}(u) = \int_{\Omega} |u(x)|^{\tau(x)} dx.$$

Then, we can see the important relationship between the norm $||\cdot||_{L^{\tau(x)}}$ and the corresponding modular function $\sigma_{\tau(x)}(\cdot)$ given in the next proposition.

Proposition 1. *[17] If u and $(u_k)_{k\in\mathbb{N}} \in L^{\tau(x)}(\Omega)$, we have*

$$\|u\|_{L^{\tau(x)}} < 1 \ (=1, >1) \text{ if and only if } \sigma_{\tau(x)}(u) < 1 \ (=1, >1), \tag{7}$$

$$\text{If} \quad \|u\|_{L^{\tau(x)}} > 1, \text{ then } \|u\|^{\tau^-}_{L^{\tau(x)}} \le \sigma_{\tau(x)}(u) \le \|u\|^{\tau^+}_{L^{\tau(x)}}, \tag{8}$$

$$\text{If} \quad \|u\|_{L^{\tau(x)}} < 1, \text{ then } \|u\|^{\tau^+}_{L^{\tau(x)}} \le \sigma_{\tau(x)}(u) \le \|u\|^{\tau^-}_{L^{\tau(x)}}, \tag{9}$$

$$\lim_{k\to\infty} \|u_k - u\|_{L^{\tau(x)}} = 0, \text{ if and only if } \lim_{k\to\infty} \sigma_{\tau(x)}(u_k - u) = 0. \tag{10}$$

Remark 1. *Note that, by (8) and (9), we can derive the two subsequent inequalities:*

$$\|u\|_{L^{\tau(x)}} \le \sigma_{\tau(x)}(u) + 1, \tag{11}$$

$$\sigma_{\tau(x)}(u) \le \|u\|^{\tau^+}_{L^{\tau(x)}} + \|u\|^{\tau^-}_{L^{\tau(x)}}. \tag{12}$$

2.2 ψ-Hilfer fractional derivative space

Let $A := [c, d]$ $(-\infty \le c < d \le \infty)$, $n - 1 < \gamma < n$, $n \in \mathbb{N}$, $\mathbf{f}$, $\psi \in C^n(I, \mathbb{R})$ such that ψ is increasing and $\psi'(x) \neq 0$, for all $x \in A$.
• The left-sided fractional ψ-Hilfer integrals of a function $\mathbf{f}$ is given by

$$\mathbf{I}^{\gamma;\psi}_{c^+}\mathbf{f}(x) = \frac{1}{\Gamma(\gamma)} \int_0^x \psi'(y)(\psi(x) - \psi(y))^{\gamma-1}\mathbf{f}(y)\mathrm{d}y, \tag{13}$$

• The right-sided fractional ψ-Hilfer integrals of a function $\mathbf{f}$ is given by

$$\mathbf{I}^{\gamma;\psi}_{d^-}\mathbf{f}(x) = \frac{1}{\Gamma(\gamma)} \int_0^x \psi'(y)(\psi(y) - \psi(x))^{\gamma-1}\mathbf{f}(y)\mathrm{d}y. \tag{14}$$

• The left-sided ψ-Hilfer fractional derivatives for a function $\mathbf{f}$ of order γ and type $0 \le \kappa \le 1$ is defined by

$$^{\mathrm{H}}\mathrm{D}^{\gamma,\kappa;\psi}_{c^+}\mathbf{f}(x) = \mathbf{I}^{\kappa(n-\gamma);\psi}_{c^+}\Big(\frac{1}{\psi'(x)}\frac{d}{\mathrm{d}x}\Big)^n \mathbf{I}^{(1-\kappa)(n-\gamma);\psi}_{c^+}\mathbf{f}(x),$$

• The right-sided ψ-Hilfer fractional derivatives for a function $\mathbf{f}$ of order γ and type $0 \leq \kappa \leq 1$ is defined by

$$^{\mathrm{H}}\mathrm{D}^{\gamma,\kappa;\psi}_{c^+}\mathbf{f}(x) = \mathbf{I}^{\kappa(n-\gamma);\psi}_{d^-}\Big(-\frac{1}{\psi'(x)}\frac{d}{\mathrm{d}x}\Big)^n \mathbf{I}^{(1-\kappa)(n-\gamma);\psi}_{d^-}\mathbf{f}(x).$$

Choosing $\kappa \to 1$, we obtain ψ-Caputo fractional derivatives left-sided and right-sided, given by

$$\mathrm{D}^{\gamma;\psi}_{c^+}\mathbf{f}(x) = \mathbf{I}^{(n-\gamma);\psi}_{c^+}\Big(\frac{1}{\psi'(x)}\frac{d}{\mathrm{d}x}\Big)^n \mathbf{f}(x), \tag{15}$$

$$\mathrm{D}^{\gamma;\psi}_{d^-}\mathbf{f}(x) = \mathbf{I}^{(n-\gamma);\psi}_{d^-}\Big(-\frac{1}{\psi'(x)}\frac{d}{\mathrm{d}x}\Big)^n \mathbf{f}(x). \tag{16}$$

Remark 2. *The ψ-Hilfer fractional derivatives defined as above can be written in the following form*

$$^{\mathrm{H}}\mathrm{D}^{\gamma,\beta;\psi}_{c^+}\mathbf{f}(x) = \mathbf{I}^{\mu-\gamma;\psi}_{c^+}\mathrm{D}^{\gamma;\psi}_{c^+}\mathbf{f}(x),$$

and

$$^{\mathrm{H}}\mathrm{D}^{\gamma,\kappa;\psi}_{d^-}\mathbf{f}(x) = \mathbf{I}^{\mu-\gamma;\psi}_{d^-}\mathrm{D}^{\gamma;\psi}_{d^-}\mathbf{f}(x),$$

with $\mu = \gamma + \kappa(n-\gamma)$ and $\mathbf{I}^{\mu-\gamma;\psi}_{c^+}$, $\mathbf{I}^{\mu-\gamma;\psi}_{d^-}$, $\mathrm{D}^{\gamma;\psi}_{c^+}$ and $\mathrm{D}^{\gamma;\psi}_{d^-}$ as defined in (13), (14), (15) *and* (16)*.*

In this paper we take $\Omega = A_1 \times \cdots \times A_N = [c_1, d_1] \times \cdots \times [c_N, d_N]$ where $0 < c_i < d_i$ for all $i \in \mathbb{N}$, $0 < \gamma_1, ..., \gamma_N < 1$.

• The ψ-Riemann-Liouville fractional partial integral of order γ of N-variables $\mathbf{f} = (\mathbf{f}_1, ..., \mathbf{f}_N)$ is defined by

$$\mathbf{I}^{\gamma;\psi}_{c,x}\mathbf{f}(x) = \frac{1}{\Gamma(\gamma)}\int_{A_1}\int_{A_2}\cdots\int_{A_N}\psi'(y)(\psi(x)-\psi(y))^{\gamma-1}\mathbf{f}(y)\mathrm{d}y,$$

with $\psi'(y)(\psi(x)-\psi(y))^{\gamma-1} = \psi'(y_1)(\psi(x_1)-\psi(y_1))^{\gamma_1-1}\cdots\psi'(y_N)(\psi(x_N)-\psi(y_N))^{\gamma_N-1}$ and $\Gamma(\gamma) = \Gamma(\gamma_1)\Gamma(\gamma_2)\cdots\Gamma(\gamma_N)$, $x_i = x_1x_2\cdots x_N$ and $\mathrm{d}y_i = \mathrm{d}y_1\mathrm{d}y_2\cdots\mathrm{d}y_N$, for all $i \in \{1, 2, ..., N\}$.

• $^{\mathrm{H}}\mathrm{D}^{\gamma,\kappa;\psi}_{c,x_i}$ is defined by

$$^{\mathrm{H}}\mathrm{D}^{\gamma,\kappa;\psi}_{c,x_i}\mathbf{f}(x_i) = \mathbf{I}^{\kappa(n-\gamma);\psi}_{c,x_i}\Big(\frac{1}{\psi'(x_i)}\frac{\partial^N}{\partial x_i}\Big)\mathbf{I}^{(1-\kappa)(n-\gamma);\psi}_{c,x_i}\mathbf{f}(x_i),$$

with $\partial x_i = \partial x_1, \partial x_2, ..., \partial x_N$ and $\psi'(x_i) = \psi'(x_1)\psi'(x_2)\cdots\psi'(x_N)$ for all $i \in \{1, 2, .., N\}$. Analogously, it is defined $^{\mathrm{H}}\mathrm{D}^{\gamma,\kappa;\psi}_{d,x_i}(\cdot)$.

Now that we have all the necessary tools, we are ready to commence our study. To facilitate this, we define the ψ-Hilfer fractional derivative space $\mathcal{H}^{\gamma,\kappa,\psi}_{\tau(x)}(\Omega)$ as follow

$$\mathcal{H}^{\gamma,\kappa,\psi}_{\tau(x)}(\Omega) = \mathcal{H}_{\tau(x)}(\Omega) := \Big\{u \in L^{\tau(x)}(\Omega) : \ |\, ^{\mathrm{H}}\mathrm{D}^{\gamma,\kappa;\psi}_{0^+}u| \in L^{\tau(x)}(\Omega)\Big\},$$

enduid with the norm

$$\|u\|_{\mathcal{H}_{\tau(x)}} = ||u||_{L^{\tau(x)}} + ||^{\mathrm{H}}\mathrm{D}^{\gamma,\kappa,\psi}_{0^+}u||_{L^{\tau(x)}}.$$

Proposition 2. *[27] Let $0 < \gamma \leq 1$, $0 \leq \kappa \leq 1$ and $1 < \tau(x)$. $\mathcal{H}_{\tau(x)}(\Omega)$ is a reflexive and separable Banach space.*

Remark 3. *We can define $\mathcal{H}^{\gamma,\kappa,\psi}_{\tau(x),0}(\Omega) := \mathcal{H}_{\tau(x),0}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $\mathcal{H}^{\gamma,\kappa,\psi}_{\tau(x)}(\Omega)$ which can be renormed by the equivalent norm $\|u\| := \||{}^{\mathrm{H}}\mathrm{D}^{\gamma,\kappa,\psi}_{0^+}u|\|_{L^{\tau(x)}}$. This space is a separable and reflexive Banach space.*

Proposition 3. *[23] Let Ω a Lipschitz bounded domain in $\mathbb{R}^N$. Let $p \in C^0(\overline{\Omega})$. If $r : \overline{\Omega} \longrightarrow (1, +\infty)$ such that*

$$1 \leq r(x) < \tau^\star(x) = \begin{cases} \frac{N\tau(x)}{N-\gamma\tau(x)}, & \text{if } \ \gamma\tau(x) < N, \\ \infty, & \text{if } \gamma\tau(x) \geq N. \end{cases} \quad \text{for all} \quad x \in \overline{\Omega}.$$

Then, the embedding

$$\mathcal{H}_{\tau(x),0}(\Omega) \hookrightarrow L^{r(x)}(\Omega) \tag{17}$$

is compact and there is a constant $c_0 > 0$, such that $\|u\|_{L^{r(x)}} \leq c_0\|u\|$.

2.3 Genus Theory

We introduce fundamental concepts related to Krasnoselskii's genus (refer to [7], [25]) which will be employed in the proof of our main results. Let X be a real Banach space and

$$\mathfrak{R} := \Big\{A \subset X\backslash\{0\};\ A \text{ is compact and symmetric}\Big\}.$$

Definition 1. *Let $A \in \mathfrak{R}$ and $X = \mathbb{R}^k$. We define the genus of A as follows:*

$$\mathfrak{G}(A) := \inf\Big\{k \geq 1;\ \exists g \in C\left(A, \mathbb{R}^k\backslash\{0\}\right);\ f \ \textit{is odd}\ \Big\},$$

and $\mathfrak{G}(A) = \infty$, if does not exist such a map for any $k > 0$.

Theorem 1. *[25] Let $\Omega \subset \mathbb{R}^N$ be bounded symmetric with boundary $\partial\Omega$. Assume that $0 \in \Omega$, then $\mathfrak{G}(\partial\Omega) = N$.*

Corollary 1. *[25] The genus of unit sphere $\mathbb{S}^{N-1}$ of the space $\mathbb{R}^N$ is N, i.e. $\mathfrak{G}(\mathbb{S}^{N-1}) = N$.*

Definition 2. *Let X be a real Banach space and $\Upsilon \in C^1(X, \mathbb{R})$. We say that Υ satisfies the Palais-Smale condition ((PS) for short) if any sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $\{\Upsilon(u_n)\}_{n\in\mathbb{N}}$ is bounded and $\Upsilon'(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$, admits a convergent subsequence.*

Theorem 2. *[9] Let $\Upsilon \in C^1(X, \mathbb{R})$ and satisfies the (PS) condition. Additionally, we assume the following conditions:*
(i) Υ is bounded from below and even,
(ii) There is a compact set $N \in \mathfrak{R}$ such that $\mathfrak{G}(N) = k$ and $\sup_{x\in N} \Upsilon(x) < \Upsilon(0)$. Then Υ has at least k pairs of distinct critical points, and their corresponding critical values are less than $\Upsilon(0)$.

3 Main result

Firstly, let us outline the definition of a weak solution to the problem described by (1).

Definition 3. *We say that $u \in \mathcal{H}_{\tau(x),0}(\Omega)$ is a weak solution of* (1) *if*

$$\left(\alpha+\beta\int_\Omega \frac{1}{\tau(x)}| {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u|^{\tau(x)}dx\right)\int_\Omega | {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u|^{\tau(x)-2}\, {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u\, {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}\varphi dx$$
$$=\xi\int_\Omega |u|^{r(x)-2}u\varphi dx-\int_\Omega h(x)|u|^{\tau(x)-2}\varphi dx,$$

for all $\varphi \in \mathcal{H}_{\tau(x),0}(\Omega)$.

Let us introduce the energy functional $\mathfrak{E} : \mathcal{H}_{\tau(x),0}(\Omega) \longrightarrow \mathbb{R}$ associated to problem (1)

$$\mathfrak{E}(u)=\alpha\int_\Omega \frac{1}{\tau(x)}| {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u|^{\tau(x)}dx+\frac{\beta}{2}\left(\int_\Omega \frac{1}{\tau(x)}| {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u|^{\tau(x)}dx\right)^2$$
$$-\xi\int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx+\int_\Omega \frac{h(x)}{\tau(x)}|u|^{\tau(x)}dx,$$

for all $u \in \mathcal{H}_{\tau(x),0}(\Omega)$.
Observe that $\mathfrak{E} \in C^1\Big(\mathcal{H}_{\tau(x),0}(\Omega), \mathbb{R}\Big)$ and it is noteworthy that the critical points of $\mathfrak{E}$ correspond to weak solutions of (1) and its Gateaux derivative is

$$\begin{aligned}&\langle \mathfrak{E}'(u), v\rangle\\ &=\left(\alpha+\beta\int_\Omega \frac{1}{\tau(x)}| {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u|^{\tau(x)}dx\right)\int_\Omega | {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u|^{\tau(x)-2}\, {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u\, {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}v dx\\ &-\xi\int_\Omega |u|^{r(x)-2}uv dx+\int_\Omega h(x)|u|^{\tau(x)-2}v dx,\end{aligned}$$

for all $v \in \mathcal{H}_{\tau(x),0}(\Omega)$.
Let consider the following functional:

$$\mathcal{L}(u)=\int_\Omega \frac{1}{\tau(x)}| {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u|^{\tau(x)}dx, \quad \text{for all } u \in \mathcal{H}_{\tau(x),0}(\Omega).$$

Note that $\mathcal{L} \in C^1\left(\mathcal{H}_{\tau(x),0}(\Omega), \mathbb{R}\right)$ and the derivative operator of $\mathcal{L}$ in weak sense $\mathcal{L}' : \mathcal{H}_{\tau(x),0}(\Omega) \longrightarrow \left(\mathcal{H}_{\tau(x),0}(\Omega)\right)^*$ is such that

$$\langle \mathcal{L}'(u), v\rangle=\int_\Omega | {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u|^{\tau(x)-2}\, {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u\, {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}v dx,$$

for all $u, v \in \mathcal{H}_{\tau(x),0}(\Omega)$.

Theorem 3. *Problem* (1) *admits at least k pairs of different critical points if* (2) *hold.*

Proposition 4. *[29] The functional $\mathcal{L}$ is a convex. Moreover, the mapping $\mathcal{L}' : \mathcal{H}_{\tau(x),0}(\Omega) \longrightarrow \left(\mathcal{H}_{\tau(x),0}(\Omega)\right)^*$ is bounded homeomorphism and strictly monotone operator, and is a mapping of type (S_+), i.e., if $u_n \rightharpoonup u$ in $\mathcal{H}_{\tau(x),0}(\Omega)$ and $\overline{\lim}_{n\to\infty}\langle \mathcal{L}'(u_n) - \mathcal{L}'(u), u_n - u\rangle \leq 0$, then $u_n \longrightarrow u$ in $\mathcal{H}_{\tau(x),0}(\Omega)$.*

Lemma 1. *The functional $\mathfrak{E}$ satisfies the (PS) condition.*

Proof. Let show that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_{\tau(x),0}(\Omega)$. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_{\tau(x),0}(\Omega)$ be a (PS) sequence. Employing a proof by contradiction, we assume that, possibly after considering a subsequence, still denote by $\{u_n\}_{n\in\mathbb{N}}$, one has $\|u_n\| \longrightarrow +\infty$ as $n \to +\infty$. Let choose $0 < \omega < \left\{\frac{1}{r^+}, \frac{1}{\tau^+}, \frac{\tau^-}{2(\tau^+)^2}\right\}$. According to Proposition 3, for sufficiently large n, we have

$$
\begin{aligned}
&c + \|u_n\| \geq \mathfrak{E}(u_n) - \omega\left\langle \mathfrak{E}'\left(u_n\right), u_n\right\rangle \\
&\geq \alpha\left(\frac{1}{\tau^+} - \omega\right)\int_\Omega |\,{}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u_n|^{\tau(x)}dx + \beta\left(\frac{1}{2(\tau^+)^2} - \frac{\omega}{\tau^-}\right)\left(\int_\Omega |\,{}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u_a|^{\tau(x)}dx\right)^2 \\
&- \xi\left(\frac{1}{r^-} - \omega\right)\int_\Omega |u_n|^{r(x)}dx + \left(\frac{1}{\tau^+} - \omega\right)\int_\Omega h(x)|u_n|^{\tau(x)}dx. \\
&\geq \alpha\left(\frac{1}{\tau^+} - \omega\right)\|u_n\|^{\tau^-} + \beta\left(\frac{1}{2(\tau^+)^2} - \frac{\omega}{\tau^-}\right)\|u_n\|^{2\tau^-} - \xi c_0\left(\frac{1}{r^-} - \omega\right)\|u_n\|.
\end{aligned}
$$

Dividing the aforementioned inequality by $\|\mathrm{u}_n\|$ and taking the limit as $n \longrightarrow \infty$, we arrive at a contradiction. It is implied by (2) that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_{\tau(x),0}(\Omega)$.
Moreover, based on Proposition 3, we can assume that

$$
\begin{cases}
u_n \longrightarrow u \ \text{ strongly in } \ L^{r(x)}(\Omega), \\
u_n(x) \longrightarrow u(x) \ \text{ a.e in } \Omega, \\
u_n \rightharpoonup u \ \text{ weakly in } \ \mathcal{H}_{\tau(x),0}(\Omega),
\end{cases} \tag{18}
$$

Using Holder's inequality and (18), one has

$$
\begin{aligned}
\Big|\int_\Omega |u_n|^{\tau(x)-2}u_n(u_n - u)dx\Big| &\leq \int_\Omega |u_n|^{\tau(x)-1}|u_n - u|dx \\
&\leq \|u_n\|^{\tau^+ - 1}_{L^{\frac{\tau(x)}{\tau(x)-1}}}\|u_n - u\|_{L^{\tau(x)}} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.
\end{aligned}
$$

Therefore,

$$
\int_\Omega |u_n|^{\tau(x)-2}u_n(u_n - u)dx \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty, \tag{19}
$$

and

$$
\int_\Omega |u_n|^{r(x)-2}u_n(u_n - u)dx \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{20}
$$

Then,

$$
\langle \mathfrak{E}'(u_n), u_n - u\rangle \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.
$$

This implies that,

$$\langle \mathfrak{E}'(u_n), u_n - u\rangle = \left(\alpha + \beta \int_\Omega \frac{1}{\tau(x)} \left| {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u_n \right|^{\tau(x)} dx\right)$$
$$\times \int_\Omega \left| {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u_n \right|^{\tau(x)-2} {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u_n \left({}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u_n - {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u\right) dx$$
$$- \xi \int_0 |u_n|^{r(x)-2} u_n \left(u_n - u\right) dx + \int_\Omega h(x)|u_n|^{\tau(x)-2} u_n \left(u_n - u\right) dx \longrightarrow 0 \,\text{as}\, n \to \infty.$$

Thus, we can infer from equation (19) and (20) that

$$\left(\alpha + \beta \int_\Omega \frac{1}{\tau(x)} \left| {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u_n \right|^{\tau(x)} dx\right)$$
$$\times \int_\Omega \left| {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u_n \right|^{\tau(x)-2} {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u_n \left({}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u_n - {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u\right) dx \to 0 \text{ as } n \longrightarrow \infty.$$

Therefore, according to Proposition 4, we get $u_n \longrightarrow u$ in $\mathcal{H}_{\tau(x),0}(\Omega)$. Hence, we conclude the proof. □

Lemma 2. *$\mathfrak{E}$ is coercive and bounded from belows.*

Proof. For any $u \in \mathcal{H}_{\tau(x),0}(\Omega)$, we have

$$\mathfrak{E}(u) \geq \frac{\alpha}{\tau^+} \int_\Omega | {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u|^{\tau(x)} + \frac{\beta}{2(\tau^+)^2} \left(\int_\Omega | {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi} u|^{\tau(x)} dx\right)^2 - \frac{\xi}{r^-} \int_\Omega |u|^{r(x)} dx$$
$$+ \frac{1}{\tau^+} \int h(x)|u|^{\tau(x)} dx.$$

Using Proposition 1 and 3, we have two cases:
Case 1: If $\|u\|_{L^{\tau(x)}} > 1$, then

$$\mathfrak{E}(u) \geq \frac{\alpha}{\tau^+}\|u\|^{\tau^-} + \frac{\beta}{2(\tau^+)^2}\|u\|^{2\tau^-} - \frac{\xi c_0}{r^-}\|u\|^{r^+}.$$

According to (2), $\mathfrak{E}$ is coercive and bounded from below.
Case 2: If $\|u\|_{L^{\tau(x)}} < 1$, then

$$\mathfrak{E}(u) \geq \frac{\alpha}{\tau^+}\|u\|^{\tau^+} + \frac{\beta}{2(\tau^+)^2}\|u\|^{2\tau^+} - \frac{\xi c_0}{r^-}\|u\|^{r^-}.$$

Since $2\tau^+ > \tau^+$ and $2\tau^+ > \tau^-$, this implies that $\mathfrak{E}$ is coercive and bounded from below. □

Proof of Theorem 3

Let consider $(s_n)_{n=1}^\infty$ a schauder basis for $\mathcal{H}_{\tau^+,0}(\Omega)$ and $Y_k = \text{span}\{s_1, s_2, \dots, s_k\}$, the subspace of $\mathcal{H}_{\tau^+,0}(\Omega)$ generated by $s_1, s_2, \dots, s_k$. Clearly Y_k is subspace of $\mathcal{H}_{\tau^+,0}(\Omega)$. Then, since $\mathcal{H}_{\tau^+,0}(\Omega) \subset \mathcal{H}_{\tau(x),0}(\Omega) \subset L^{r(x)}(\Omega)$ we have $Y_k \subset L^{r(x)}(\Omega)$.

Also, since Y_k is a finite dimension space, the norms $\|\cdot\|$ and $\|\cdot\|_{L^{r(x)}}$ are equivalent on Y_k. Therefore, there exists a positive constant c_k such that

$$\|u\|_{L^{r(x)}} \geq c_k\|u\|, \text{ for all } u \in Y_k.$$

Let $u \in Y_k$ such that $\|u\| < 1$, then using ($\mathbf{H_0}$) one has

$$\begin{aligned}
\mathfrak{E}(u) &= \alpha \int_\Omega \frac{1}{\tau(x)}|\ {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u|^{\tau(x)}dx + \frac{\beta}{2}\left(\int_\Omega \frac{1}{\tau(x)}|\ {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u|^{\tau(x)}dx\right)^2 \\
&\quad - \xi \int_\Omega \frac{1}{r(x)}|u|^{r(x)}dx + \int_\Omega \frac{h(x)}{\tau(x)}|u|^{\tau(x)}dx \\
&\leq \frac{\alpha}{\tau^-}\int_\Omega |\ {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u|^{\tau(x)}dx + \frac{\beta}{2(\tau^-)^2}\left(\int_\Omega |\ {}^{\mathrm{H}}\mathrm{D}_{0^+}^{\gamma,\kappa;\psi}u|^{\tau(x)}dx\right)^2 \\
&\quad - \frac{\xi}{r^+}\int_\Omega |u|^{r(x)}dx + \frac{1}{\tau^-}\int_\Omega h(x)|u|^{\tau(x)}dx \\
&\leq c_1\Big(\|u\|^{\tau^-} + \|u\|^{2\tau^-}\Big) - c_2\|u\|^{r^+}_{L^{r(x)}} + \frac{1}{\tau^-}\|h\|_{L^\infty}\|u\|^{\tau^-} \\
&\leq c_3\Big(\|u\|^{\tau^-} + \|u\|^{2\tau^-}\Big) - c_4\|u\|^{r^+} \\
&= \|u\|^{r^+}\Big[c_3\Big(\|u\|^{\tau^- - r^+} + \|u\|^{2\tau^- - r^+}\Big) - c_4\Big].
\end{aligned}$$

There exists $\lambda \in (0,1)$ sufficiently small such that $\lambda^{r^+} < 1$ and $c_3\lambda^{\tau^- - r^+} + c_3\lambda^{2\tau^- - r^+} \leq \frac{c_4}{2}$. Let consider $\mathbf{S}_\lambda^k := \Big\{u \in Y_k |\ \|u\| = \lambda\Big\}$. We have $\mathfrak{E}(u) \leq \lambda^{\tau^+}\Big(c_3\lambda^{\tau^- - r^+} + c_3\lambda^{2\tau^- - r^+} - c_4\Big)$ for all $u \in \mathbf{S}_\lambda^k$. Thus

$$\sup_{u\in\mathbf{S}_\lambda^k} \mathfrak{E}(u) \leq \left(\frac{c_4}{2} - c_4\right) = -\frac{c_4}{2} < 0 = \mathfrak{E}(0).$$

Because Y_k and $\mathbb{R}^k$ are isomorphic, then $\mathbf{S}_\lambda^k$ and $\mathbb{S}^{k-1}$ are homeomorphic, thus $\mathfrak{G}\left(\mathbf{S}_\lambda^k\right) = k$. According to Theorem 2, $\mathfrak{E}$ has least k pairs of different critical points.

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