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REMARKS ON THE COEFFICIENTS OF TERNARY CYCLOTOMIC AND INVERSE CYCLOTOMIC POLYNOMIALS

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Abstract

Using a formula attributed to Von Sterneck, we present a method for deriving recursive relations for the coefficients of ternary cyclotomic polynomials $\Phi_n(x)$ and ternary inverse cyclotomic polynomials $\Psi_n(x)$. We consider $n = pqr$ with $p < q < r$ are primes. As an application, we determine exact values for the first $(p+1)$ coefficients. For an infinite subfamily, the first r coefficients are computed. In the case of ternary cyclotomic polynomials, we identify an explicit infinite family whose r-th coefficient is equal to -2. Additionally, we include a section on numerical simulations that demonstrates the computation of non-flat coefficients in specific cases for Φ_{105} and Ψ_{561}.

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1 Introduction

Cyclotomic polynomials play a fundamental role in various areas of mathematics, bridging classical and modern theory. Their study dates back to Gauss, whose work highlighted some deep connections of these polynomials to algebra and number theory. In algebra, for instance, cyclotomic polynomials appear in Witt's proof of Wedderburn's little theorem asserting that every finite division

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ring is a field. They also provide a foundation for the "cyclotomic criterion" used to study primitive divisors of Lucas and Lehmer sequences in number theory [11].

In addition to their purely theoretical significance, cyclotomic polynomials have also found applications in public-key cryptographic protocols. For instance, Lenstra showed that cyclotomic polynomials can be used to construct efficient discrete logarithm cryptosystems over finite fields [25]. Lately these polynomials have become foundational in lattice-based cryptography, where cyclotomic fields and polynomials play a key role in constructing lattices in which hard problems are believed to be resistant to quantum attacks. Specifically, Lyubashevsky, Peikert, and Regev's [26] and Langlois and Stehlé's [22] work on ideal lattices and learning with errors over rings demonstrated the use of cyclotomic fields in developing cryptographic systems that leverage the hardness of certain lattice problems.

The n-th inverse cyclotomic polynomial is defined as the quotient of x^n-1 by the n-th cyclotomic polynomial. The investigation of these polynomials originates in the work of Moree [28] with particular focus on their coefficients. This paper specifically investigates recursive techniques for determining the coefficients of cyclotomic and inverse cyclotomic polynomials. In subsections 1.1 and 1.2, we review essential properties of these polynomials that are used throughout.

1.1 Cyclotomic polynomials

For an integer $n \geq 1$, an n-th root ζ of the unity is called primitive if $\zeta^n = 1$, while $\zeta^d \neq 1$ for all $1 \leq d < n$. Denoting by $\zeta_n = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ the first root of order n of the unity, the n-th cyclotomic polynomial Φ_n is defined by

$$\Phi_n(z) = \prod_{\substack{1\leq k\leq n-1\\ \gcd(k,n)=1}} (z-\zeta_n^k) = \sum_{j=0}^{\varphi(n)} c_j^{(n)} z^j, \tag{1}$$

where φ is Euler's totient function, also the degree of the polynomial, which is monic and palindromic (i.e., $c_j^{(n)} = c_{\varphi(n)-j}^{(n)}$, $j = 0, \ldots, \varphi(n)$) and $c_0^{(n)} = c_{\varphi(n)}^{(n)} = 1$).

The term "cyclotomic" originates in the geometric property of the n-th roots of unity, which evenly divide the unit circle into n equal arcs. These are the vertices of a regular polygon inscribed within the unit circle, illustrating how the concept links both algebra and geometry. The explicit computation of the coefficients of Φ_n is very difficult (see, e.g., [10] or [17]), but many properties are known [12, 14, 15, 16, 20, 21, 23, 24, 31]. An explicit integral formula for the coefficients was established in the paper [5], using a special version of the Cauchy integral formula given in [4]. Specifically, the following result was proved in [5]:

$$c_j^{(n)} = \frac{2^{\varphi(n)}}{\pi}\int_0^{\pi} \Lambda_n(t)\cdot\cos(\varphi(n)-2j)t\,\mathrm{d}t, \quad j = 0, 1, \ldots, \varphi(n), \tag{2}$$

where for $n \geq 3$

$$\Lambda_n(t) = \prod_{\substack{1\leq k\leq n-1\\ \gcd(k,n)=1}} \sin\left(t - \frac{k\pi}{n}\right).$$

By the same techniques, similar formulae have been obtained for the coefficients of Gaussian [3], polygonal [1], and other families of polynomials [2].

1.2 The inverse cyclotomic polynomial Ψ_n

For an integer $n \geq 2$, the n-th inverse cyclotomic polynomial $\Psi_n(z)$ is defined by the formulae

$$\Psi_n(z) = \prod_{\substack{1\leq k\leq n\\ \gcd(k,n)>1}} \left(z - e^{\frac{2k\pi i}{n}}\right) = \frac{x^n-1}{\Phi_n(z)} = \sum_{j=0}^{n-\varphi(n)} d_j^{(n)} z^j. \tag{3}$$

The polynomial $\Psi_n(z)$ is of degree $n-\varphi(n)$ and integer coefficients, which are denoted here by $d_j^{(n)}$, $j = 0, 1, \ldots, n-\varphi(n)$. Clearly, if n is a prime, then $\Psi_n(z) = z-1$. By formula (3), the first inverse cyclotomic polynomials are given by

$$\begin{aligned}
&\Psi_1(z) = 1,\ \Psi_4(z) = z^2-1,\ \Psi_6(z) = z^4+z^3-z-1,\ \Psi_8(z) = z^4-1,\\
&\Psi_9(z) = z^3-1,\ \Psi_{10}(z) = z^6+z^5-z-1,\ \Psi_{12}(z) = z^8+z^6-z^2-1,\\
&\Psi_{14}(z) = z^8+z^7-z-1,\ \Psi_{15}(z) = z^7+z^6+z^5-z^2-z-1,\\
&\Psi_{16}(z) = z^8-1,\ \Psi_{18}(z) = z^{12}+z^9-z^3-1,\ \Psi_{20}(z) = z^{12}+z^{10}-z^2-1,\\
&\Psi_{21}(z) = z^9+z^8+z^7-z^2-z-1,\ \Psi_{22}(z) = z^{12}+z^{11}-z^2-1.
\end{aligned}$$

We recall some known properties of Ψ_n. For proofs, see [7], [8], or [28].
1° If $n = p^\alpha$ with p prime and $\alpha \geq 1$, then $\Psi_n(z) = z^{p^{\alpha-1}} - 1$.
2° For $n = p_1 \cdots p_k$ square-free, $\deg(\Psi_n) = p_1 \cdots p_k - (p_1-1)\cdots(p_k-1)$.
3° If $p < q$ are primes, then for $n = pq$ one has

$$\Psi_n(z) = \frac{(z^p-1)(z^q-1)}{z-1} = z^{p+q-1} + \cdots + z^{q+1} - z^{p-1} - \cdots - z^2 - z - 1.$$

4° $\Psi_{2n}(z) = (1-z^n)\,\Psi_n(-z)$, if n is odd.
5° $\Psi_{pn}(z) = \Psi_n(z^p)$, if $p \mid n$.
6° $\Psi_{pn}(z) = \Psi_n(z^p)\Phi_n(z)$, if $p \nmid n$.
7° Ψ_n is monic and antipalindromic, that is $d_j^{(n)} = -d_{n-\varphi(n)-j}^{(n)}$, $j = 0, \ldots, n-\varphi(n)$, and $d_0^{(n)} = -1$, while $d_{n-\varphi(n)}^{(n)} = 1$.
8° The number of positive coefficients of Ψ_n is equal to the number of negative coefficients.

This explicit integral formula for the coefficients of Ψ_n was given in [9]:

$$d_j^{(n)} = (-1)^{n+1}\frac{2^{n-\varphi(n)}}{\pi}\int_0^\pi \Gamma_n(t)\cdot\sin\left(n-\varphi(n)-2j\right)t\,\mathrm{d}t, \tag{4}$$

where

$$\Gamma_n(t) = \prod_{\substack{1\leq k\leq n\\ \gcd(k,n)>1}} \sin\left(t - \frac{k\pi}{n}\right).$$

2 Recursive formula for the coefficients of Φ_n and Ψ_n in terms of Ramanujan sums

Recall the Ramanujan sums, denoted as $\rho(n,j)$, which are fascinating constructs due to their unique properties and applications. These sums, first introduced by S. Ramanujan, have been studied extensively for their intriguing behavior in various mathematical contexts. For positive integers n and j, the Ramanujan sum is defined as

$$\rho(n,j) = \sum_{\gcd(a,n)=1} e^{2\pi i \frac{a}{n} j}, \tag{5}$$

where the summation extends over all integers a satisfying $1 \leq a \leq n$ and $\gcd(a,n) = 1$. This concept plays a vital role in many areas of mathematics, as evidenced in works such as [18] or [27], and the recursive formulas involving Ramanujan sums presented in [7]. It is noteworthy that in some texts, including Ramanujan's original work [30], the notation $c_n(j)$ is used for these sums. However, in our discourse, we adopt the notation $\rho(n,j)$, a choice made to better highlight the multiplicative nature of these sums in the variable n, a property that is central to our discussion. This choice of notation, while differing from the original, is made with the utmost respect for the historical context.

The Möbius function μ is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 p_2 \cdots p_k, \\ 0 & \text{if } n = p^2 m, \end{cases}$$

where p is prime and $p_1, \ldots, p_k$ are distinct prime numbers. This function captures the essence of the multiplicative structure of integers in a compact form.

The connection with the Ramanujan sum $\rho(n,j)$ is known as the Kluyver's formula [18]:

$$\rho(n,j) = \sum_{d \mid \gcd(n,j)} d\mu\left(\frac{n}{d}\right) \text{ for all } n \in \mathbb{N}^*, \tag{6}$$

which follows from the application of Möbius inversion.

If n is prime, one can deduce from (6) that

$$\rho(n,j) = \begin{cases} -1, \text{ if } n \nmid j \\ n-1, \text{ if } n \mid j \end{cases}$$

Note that, since n is prime, $\varphi(n) = n-1$. Similarly, from Kluyver's formula it follows that when $n = p^k$, where p is prime and $k > 1$ the following formulas hold

$$\rho(n,j) = \begin{cases} 0, \text{ if } p^{k-1} \nmid j \\ -1, \text{ if } p^{k-1} \mid j \text{ but } p^k \nmid j \\ \varphi(p^k), \text{ if } n \mid j \end{cases}.$$

The following recursive formulae for the coefficients of Φ_n and Ψ_n are proved in [6], [8, Section 8.5], and [7], respectively:

Theorem 1. *The following relation holds for every $k = 2, \ldots, \varphi(n)$:*

$$c_k^{(n)} = -\frac{1}{k}\left[\rho(n,k) + \rho(n,k-1)c_1^{(n)} + \cdots + \rho(n,1)c_{k-1}^{(n)}\right]. \tag{7}$$

Theorem 2. *The coefficients of Ψ_n satisfy the following relation:*

$$d_k^{(n)} = \frac{1}{k}\left[-\rho(n,k) + \rho(n,k-1)d_1^{(n)} + \cdots + \rho(n,1)d_{k-1}^{(n)}\right]. \tag{8}$$

3 Applications to ternary polynomials Φ_n and Ψ_n

The following formula holds

$$\rho(n,j) = \frac{\mu\left(\frac{n}{\gcd(n,j)}\right)\varphi(n)}{\varphi\left(\frac{n}{\gcd(n,j)}\right)}. \tag{9}$$

The fact that for fixed j, $\rho(\cdot, j) : \mathbb{N} \to \mathbb{R}$ is multiplicative can be seen for instance on page 16 of [32]. In addition, we remark that for fixed j, the functions $n \mapsto \mu\left(\frac{n}{\gcd(n,j)}\right)$, $n \mapsto \varphi\left(\frac{n}{\gcd(n,j)}\right)$ are multiplicative and so is the Euler totient function φ. Hence, it follows that the formula (9) holds for every positive integers n and j. The right-hand side of (9) often appears under the name of Von Sterneck's function and the first proof of the equality in (9) is due to Hölder (see the discussion on page 243 of [19]).

3.1 The ternary polynomial Φ_n

An interesting application of formula (9) is inspired by Theorem 1, from which one can observe that the coefficient $c_k^{(n)}$ of Φ_n can be computed recursively as

$$c_k^{(n)} = -\frac{\varphi(n)}{k}\left[\frac{\mu\left(\frac{n}{\gcd(n,k)}\right)}{\varphi\left(\frac{n}{\gcd(n,k)}\right)} + \frac{\mu\left(\frac{n}{\gcd(n,k-1)}\right)}{\varphi\left(\frac{n}{\gcd(n,k-1)}\right)}c_1^{(n)} + \cdots + \frac{\mu\left(\frac{n}{\gcd(n,1)}\right)}{\varphi\left(\frac{n}{\gcd(n,1)}\right)}c_{k-1}^{(n)}\right]. \tag{10}$$

Using the multiplicative properties of the functions featuring in the right hand side of (10) it is easy to show that this formula is equivalent to one first published in [17] and which is also recalled in the survey [31]. Therefore, one can argue that the use of Von Sterneck's formula for reinterpreting Ramanujan sums provides a concise proof of the previously discovered recursive formula.

We present the formula (10) in the special case when n is a product of three distinct primes. Suppose $n = pqr$, where $p < q < r$ are primes. In this case, the polynomial Φ_n said to be ternary and has the following apparently simple form

$$\Phi_{pqr}(z) = \frac{(1-z^{pqr})(1-z^r)(1-z^q)(1-z^p)}{(1-z^{qr})(1-z^{pr})(1-z^{pq})(1-z)}.$$

If $k < p$, then $\gcd(n, j) = 1$ for all $1 \leq j \leq k$, hence formula (10) becomes

$$c_k^{(n)} = \frac{1}{k}\left[1 + c_1^{(n)} + \cdots + c_{k-1}^{(n)}\right]. \tag{11}$$

For $k = p$, we have

$$c_p^{(n)} = \frac{1}{p}\left[-p + 1 + c_1^{(n)} + \cdots + c_{p-1}^{(n)}\right]. \tag{12}$$

From the two relation above, one can inductively prove the following result.

Proposition 1. *Suppose $n = pqr$, where $p < q < r$ are primes. Then*

1. $c_k^{(n)} = 1$, *for all* $1 \leq k \leq p-1$;
2. $c_p^{(n)} = c_{p+1}^{(n)} = 0$.

Proof. Indeed, in this case we know that $c_1^{(n)} = -\mu(n) = 1$. Now, we prove the first claimed result using strong induction on k. Suppose $c_j^{(n)} = 1$ for all $1 \leq j \leq k$, where $k \leq p-2$. Then, using the recurrence in (11), we get that

$$c_{k+1}^{(n)} = \frac{1}{k+1}\left[1 + c_1^{(n)} + \ldots + c_k^{(n)}\right] = \frac{1}{k+1} \cdot (k+1) = 1,$$

completing the induction.

Using what we just proved in (12) we obtain

$$c_p^{(n)} = \frac{1}{p}\left[-p + 1 + c_1^{(n)} + \cdots + c_{p-1}^{(n)}\right] = \frac{1}{p} \cdot (-p + 1 + p - 1) = 0.$$

To compute the $(p+1)$-th coefficient of Φ_n we apply (10), using $\gcd(n, p+1) = 1$, $\gcd(n, p) = p$ and $\gcd(n, k) = 1$ for all $1 \leq k \leq p-1$, and the hypotheses $\mu(n) = -1$ and $\mu(qr) = 1$. Plugging all these in formula (10), we obtain

$$c_{p+1}^{(n)} = -\frac{\varphi(n)}{p+1}\left[-\frac{1}{\varphi(n)} + \frac{1}{\varphi(qr)}c_1^{(n)} - \frac{1}{\varphi(n)}c_2^{(n)} + \ldots - \frac{1}{\varphi(n)}c_{p-1}^{(n)} - \frac{1}{\varphi(n)}c_p^{(n)}\right],$$

which after replacing the previously known coefficients gives

$$c_{p+1}^{(n)} = -\frac{1}{p+1}\left[-(p-1) + \varphi(p)\right] = 0,$$

completing the proof of the second claim. □

In general, for $1 \leq j \leq k$, we have $\gcd(n, k-j) \in \{1, p, q, r, pq, rp, qr\}$, as $k \leq n - \varphi(n) < n$. Substituting the required values for φ and since $\mu(n) = -1$,

$\mu(p) = \mu(q) = \mu(r) = -1$, and $\mu(pq) = \mu(pr) = \mu(qr) = 1$, we derive the formula

$$\begin{aligned} c_k^{(n)} = &-\frac{(p-1)(q-1)(r-1)}{k} \frac{\mu\left(\frac{n}{\gcd(n,k)}\right)}{\varphi\left(\frac{n}{\gcd(n,k)}\right)} + \frac{1}{k} \sum_{\gcd(n,k-j)=1} c_j^{(n)} \\ &- \frac{p-1}{k} \sum_{\gcd(n,k-j)=p} c_j^{(n)} - \frac{q-1}{k} \sum_{\gcd(n,k-j)=q} c_j^{(n)} - \frac{r-1}{k} \sum_{\gcd(n,k-j)=r} c_j^{(n)} \\ &+ \frac{(p-1)(q-1)}{k} \sum_{\gcd(n,k-j)=pq} c_j^{(n)} + \frac{(p-1)(r-1)}{k} \sum_{\gcd(n,k-j)=pr} c_j^{(n)} \\ &+ \frac{(q-1)(r-1)}{k} \sum_{\gcd(n,k-j)=qr} c_j^{(n)}, \end{aligned} \tag{13}$$

where all the sums above run through the values $1 \leq j \leq k-1$ satisfying the given condition regarding the greatest common divisor.

In what follows, we introduce the definition of a particular triple of prime numbers that will be used throughout the paper.

Definition 1. *We call a triple of natural numbers (p, q, r) a Ramanujan triple if p, q, r are primes and $p < q < r < 2p$.*

In his landmark 1919 paper [30], Ramanujan presented a novel proof of Bertrand's postulate, together with a generalisation. An important consequence is that for all $x \geq 11$ one has

$$\pi(x) - \pi\left(\frac{x}{2}\right) \geq 3,$$

where π represents the prime-counting function. In consequence, by considering arbitrarily large values of x, there exist infinitely many distinct Ramanujan triples.

We note that for $n = pqr$, where (p, q, r) forms a Ramanujan triple, the recurrence formula above simplifies significantly, allowing us to obtain the following results, complementing the ones obtained in Proposition 1.

Proposition 2. *For every $n = pqr$ where (p, q, r) is a Ramanujan triple, we have*

1. $c_k^{(n)} = 0$ *for every* $p < k < q$;
2. $c_k^{(n)} = -1$ *for every* $q \leq k < r$;
3. $c_r^{(n)} = -2$.

Proof. First note that for $p < k < q$ and for all $j \in \{1, 2, \ldots, k-1\} \setminus \{k-p\}$ we have $\gcd(n, k-j) = 1$. Also, $\gcd(n, p) = p$. Using Proposition 1 in (10) we get

$$\begin{aligned} c_k^{(n)} &= -\frac{\varphi(n)}{k} \left[-\frac{k-p}{\varphi(n)} + \frac{p-1}{\varphi(n)} \cdot c_{k-p}^{(n)} - \frac{1}{\varphi(n)} \sum_{j=k-p+1}^{k-1} c_j^{(n)} \right] \\ &= -\frac{\varphi(n)}{k} \left[\frac{2p-1-k}{\varphi(n)} + \frac{k+1-2p}{\varphi(n)} - \frac{1}{\varphi(n)} \sum_{j=p}^{k-1} c_j^{(n)} \right] = \frac{1}{k} \sum_{j=p}^{k-1} c_j^{(n)}. \end{aligned}$$

Note that in the second equality we used that $0 < k-p < p$ for $k < q < 2p$, hence $c_{k-p}^{(n)} = 1$. Then, by Proposition 1 we had $c_p^{(n)} = 0$, hence the formula above implies $c_k^{(n)} = 0$ for all $p < k < q$, proving the first claim. One can also compute

$$c_q^{(n)} = -\frac{\varphi(n)}{q}\left[\frac{\varphi(q)}{\varphi(n)} - \frac{1}{\varphi(n)}\sum_{k=1}^{q-p-1} c_k^{(n)} + \frac{\varphi(p)}{\varphi(n)}c_{q-p}^{(n)} - \sum_{k=q-p+1}^{q-1} c_k^{(n)}\right]$$

$$= -\frac{1}{q}\Bigg[(q-1)+(p+1)-q+(p-1)-\sum_{k=q-p+1}^{p-1} 1 - \underbrace{\sum_{k=p}^{q-1} c_k^{(n)}}_{=0}\Bigg] = -\frac{1}{q}\cdot q = -1.$$

We now compute the terms $c_k^{(n)}$ where $q < k < r$. By the formula (10) we get

$$c_k^{(n)} = -\frac{1}{k}\Bigg[(-1)\sum_{j=0}^{k-q-1} 1 + \varphi(q) + (-1)\sum_{j=k-q+1}^{k-p-1} 1 + \varphi(p) + (-1)\sum_{j=k-p+1}^{p-1} 1 +$$

$$+ (-1)\sum_{j=p}^{q-1} c_j^{(n)} + (-1)\sum_{j=q}^{k-1} c_j^{(n)}\Bigg]$$

$$= -\frac{1}{k}\left(q-k+q-1+p+1-q+p-1+k+1-2p-\sum_{j=q}^{k-1} c_j^{(n)}\right)$$

$$= -\frac{1}{k}\left(q-\sum_{j=q}^{k-1} c_j^{(n)}\right).$$

Now since we saw that $c_q^{(n)} = -1$, we obtain $c_k^{(n)} = -1$ for all $q < k < r$, which proves the second claim.

Similarly, we will compute the coefficient $c_r^{(n)}$ and for such n. We obtain

$$c_r^{(n)} = -\frac{1}{r}\Bigg[\varphi(r) + (-1)\sum_{j=1}^{r-q-1} 1 + \varphi(q) + (-1)\sum_{j=r-q+1}^{r-p-1} 1 + \varphi(p)$$

$$+ (-1)\sum_{j=r-p+1}^{p-1} 1 + (-1)\sum_{j=p}^{q-1} 0 + (-1)\sum_{j=q}^{r-1}(-1)\Bigg] = -\frac{1}{r}\cdot 2r = -2.$$

□

Combined with Ramanujan's result, this shows that there is an infinite family of n of the form $n = pqr$ for which $c_r^{(n)} = -2$. We note that the first Ramanujan triple is $(7, 11, 13)$, in which case $n = 1001$. This polynomial has $c_{13}^{(1001)} = -2$ but we also saw experimentally that there are other coefficients that are not flat, such as $c_{199}^{(1001)} = -2$. In this case, the polynomial has no coefficients of absolute value strictly grater than 2. However, for the Ramanujan tripe $(17, 19, 29)$ and $n = 9367$ there are coefficients of larger absolute value such as $c_{3107}^{(9367)} = -4$.

3.2 The ternary polynomial Ψ_n

Combining formula (9) with Theorem 2, it follows that the coefficient $d_k^{(n)}$ of Ψ_n can be obtained recursively as

$$d_k^{(n)} = \frac{\varphi(n)}{k}\left[-\frac{\mu\left(\frac{n}{\gcd(n,k)}\right)}{\varphi\left(\frac{n}{\gcd(n,k)}\right)} + \frac{\mu\left(\frac{n}{\gcd(n,k-1)}\right)}{\varphi\left(\frac{n}{\gcd(n,k-1)}\right)}d_1^{(n)} + \cdots + \frac{\mu\left(\frac{n}{\gcd(n,1)}\right)}{\varphi\left(\frac{n}{\gcd(n,1)}\right)}d_{k-1}^{(n)}\right]. \tag{14}$$

To complement the theoretical recurrence formula presented in (14), one can easily compute the coefficients of the n-th inverse cyclotomic polynomial using a computer algebra system such as Magma. By utilizing a simple recursive approach, as outlined in the code snippet below, Magma efficiently handles the necessary calculations, providing a practical calculation of these coefficients.

```
// Define the function to compute d_k^{(n)}
function InverseCyclotomicCoefficient(n, k)
    // Define the Euler totient function and the Mobius function
    phi := EulerPhi(n);
    mu_n := MoebiusMu(n);

    // Initialize the array to store the coefficients d_1^{(n)}, ..., d_k^{(n)}
    d := [0 : i in [1..k]];

    // Compute d_1^{(n)} = -mu(n)
    d[1] := -mu_n;

    // Recurrence to compute d_i^{(n)} for i >= 2
    for i in [2..k] do
        // Start with the first term
        gcd_ni := GCD(n, i);
        mu_term_i := MoebiusMu(n div gcd_ni);
        phi_term_i := EulerPhi(n div gcd_ni);

        // k * d_k^{(n)} starts with the first term
        i_di := - (phi / phi_term_i) * mu_term_i;

        // Add the sum for j = 1 to i-1
        for j in [1..i-1] do
            gcd_nij := GCD(n, i-j);
            mu_term_ij := MoebiusMu(n div gcd_nij);  // M\"obius term for this
                gcd
            phi_term_ij := EulerPhi(n div gcd_nij);
            i_di +:= (phi / phi_term_ij) * mu_term_ij * d[j];  // Include M\"
                obius term
        end for;

        // Now compute d_i^{(n)} by dividing by i
        d[i] := i_di / i;
    end for;

    // Return the result for d_k^{(n)}
    return d[k];
end function;
```

As in the previous subsection, suppose $n = pqr$, where $p < q < r$ are primes. In this case, the polynomial Ψ_n is said to be ternary and has the following form

$$\Psi_n(z) = \frac{(z^{pq}-1)(z^{qr}-1)(z^{rp}-1)(z-1)}{(z^p-1)(z^q-1)(z^r-1)}.$$

Nonetheless, despite its seemingly simple form, the structure of the coefficients of this polynomial remains quite intricate and is not yet fully understood.

If $k < p$, then $\gcd(n, j) = 1$ for all $1 \leq j \leq k$, hence formula (14) becomes

$$d_k^{(n)} = -\frac{1}{k}\left[-1 + d_1^{(n)} + \cdots + d_{k-1}^{(n)}\right]. \tag{15}$$

For $k = p$, we have

$$d_p^{(n)} = -\frac{1}{p}\left[p - 1 + d_1^{(n)} + \cdots + d_{p-1}^{(n)}\right]. \tag{16}$$

Building upon the recursive formula (14), we now focus on deriving explicit formulas for the coefficients of the inverse cyclotomic polynomial Ψ_n. In particular, we aim to compute the exact values of the coefficients $d_k^{(n)}$ up to the $(p+1)$-th term. This result is the analogue of Proposition 1, where we previously determined the corresponding coefficients for the cyclotomic polynomial Φ_n.

As we will show, the structure of the coefficients for Ψ_n exhibits differences, starting with $d_1^{(n)} = 1$, and continuing with specific values of $d_k^{(n)}$ for $2 \leq k \leq p+1$, which we state in the following proposition.

Proposition 3. *Suppose $n = pqr$, where $p < q < r$ are primes. Then*

1. *$d_1^{(n)} = 1$ and $d_k^{(n)} = 0$, for all $2 \leq k \leq p-1$;*
2. *$d_p^{(n)} = -1$ and $d_{p+1}^{(n)} = 1$.*

Proof. The proof follows the same lines as the one used for Proposition 1. First we note that $d_1^{(n)} = -\mu(n) = 1$, for such n. It is easy to see that using (15), we get $d_2^{(n)} = -1/2\,(-1+1) = 0$. Now, the fact that $d_k^{(n)} = 0$, for all $2 \leq k \leq p-1$ follows from an immediate inductive argument and the same recurrence formula.

Using (16) we get

$$d_p^{(n)} = -\frac{1}{p}\left[p - 1 + 1 + 0 + \cdots + 0\right] = -1,$$

proving the third claim.

To compute $d_{p+1}^{(n)}$ we will proceed by applying directly the formula (14). First, we note that for such chosen n, we have $\mu(n) = -1$, $\mu(n/p) = 1$, $\gcd(n, p+1) = 1$, $\gcd(n, p) = p$ and $\gcd(n, k) = 1$ for all $1 \leq k \leq p-1$. We get

$$d_{p+1}^{(n)} = \frac{\varphi(n)}{p+1}\left[\frac{1}{\varphi(n)} + \frac{1}{\varphi(qr)}d_1^{(n)} - \frac{1}{\varphi(n)}d_2^{(n)} - \ldots - \frac{1}{\varphi(n)}d_{p-1}^{(n)} - \frac{1}{\varphi(n)}d_p^{(n)}\right],$$

from where we obtain

$$d_{p+1}^{(n)} = \frac{1}{p+1}\left[1+\varphi(p)-\underbrace{0-\ldots-0}_{p-2 \text{ terms}}+1\right] = 1,$$

completing the proof of the last claim. □

In general, for $1 \le j \le k$, we have $\gcd(n,j) \in \{1,p,q,r,pq,rp,qr\}$, because $k \le n-\varphi(n) < n$, hence using that $\mu(n) = -1$, $\mu(p) = \mu(q) = \mu(r) = -1$, $\mu(pq) = \mu(pr) = \mu(qr) = 1$, we derive

$$\begin{aligned} d_k^{(n)} = &-\frac{(p-1)(q-1)(r-1)}{k}\frac{\mu\left(\frac{n}{\gcd(n,k)}\right)}{\varphi\left(\frac{n}{\gcd(n,k)}\right)} - \frac{1}{k}\sum_{\gcd(n,j)=1} d_j^{(n)} \\ &+ \frac{p-1}{k}\sum_{\gcd(n,j)=p} d_j^{(n)} + \frac{q-1}{k}\sum_{\gcd(n,j)=q} d_j^{(n)} + \frac{r-1}{k}\sum_{\gcd(n,j)=r} d_j^{(n)} \\ &- \frac{(p-1)(q-1)}{k}\sum_{\gcd(n,j)=pq} d_j^{(n)} - \frac{(p-1)(r-1)}{k}\sum_{\gcd(n,j)=pr} d_j^{(n)} \\ &- \frac{(q-1)(r-1)}{k}\sum_{\gcd(n,j)=qr} d_j^{(n)}, \end{aligned} \tag{17}$$

where all the sums above run through the values $1 \le j \le k-1$ satisfying the given condition regarding the greatest common divisor.

When (p,q,r) is a Ramanujan triple and $n = pqr$, formula (17) can be simplified and we obtain the following result, complementing Proposition 3.

Proposition 4. *For every $n = pqr$ where (p,q,r) is a Ramanujan triple, we have*

1. $d_k^{(n)} = 0$ *for all* $p+2 \le k \le q-1$ *and* $d_q^{(n)} = -1$;
2. $d_{q+1}^{(n)} = 1$ *and* $d_k^{(n)} = 0$ *for all* $q+2 \le k < r$;
3. $d_r^{(n)} = -1$.

Proof. We saw above that $d_1^{(n)} = 1$, $d_k^{(n)} = 0$ for all $2 \le k \le p-1$, $d_p^{(n)} = -1$ and $d_{p+1}^{(n)} = 1$. If $p+1 < k < q$, then we have

$$d_k^{(n)} = \frac{1}{k}\left[1+(-1)-d_p^{(n)}-d_{p+1}^{(n)}-\sum_{j=p+2}^{k-1} d_j^{(n)}\right] = \frac{1}{k}\sum_{j=p+2}^{k-1} d_j^{(n)},$$

which implies that $d_k^{(n)} = 0$ for all $p+2 \le k \le q-1$.

Similarly, for $k = q$ one shows that

$$d_q^{(n)} = \frac{1}{q}\left[-\varphi(q)+(-1)+(-1)\sum_{j=p}^{q-1} d_j^{(n)}\right] = \frac{1}{q}(1-q-1) = -1,$$

completing the proof of the first statement. To justify the second equality we note that all but the first two terms in the sum are equal to 0.

Similarly, it is easy to show that $d_{q+1}^{(n)} = 1$. When $q+2 \leq k < r$, one gets $d_k^{(n)} = \frac{1}{k}\left[\sum_{j=q+2}^{k} d_k^{(n)}\right]$ which is equal to 0, which proves the second statement.

Finally, when $k = r$ one obtains

$$\begin{aligned} d_r^{(n)} &= \frac{1}{r}\left[-\varphi(r) + (-1) + (-1)d_p^{(n)} + (-1)d_{p+1}^{(n)} + (-1)d_q^{(n)} + (-1)d_{q+1}^{(n)}\right] \\ &= \frac{1}{r}\left(-r + 1 - 1\right) = -1. \end{aligned}$$

This ends the proof. □

We note that, combined with Ramanujan's result on the infinitude of such triples, Propositions 3 and 4 give precise values for the first r terms in an infinite family of inverse cyclotomic polynomials. One would be mislead to think that for $n = pqr$, where (p, q, r) is a Ramanujan triple, the inverse cyclotomic polynomials are flat. Indeed, when $n = 11 \cdot 13 \cdot 19$, we computed using Magma and observed that $d_{53}^{(n)} = 2$. Similar computations were carried out for the Ramanujan triple $(101, 103, 109)$ and $n = 1133927$ where we calculated that $d_{15651}^{(n)} = -16$.

4 Numerical simulations

In this section, we present numerical simulations for two specific cases of interest, namely the 105-th cyclotomic polynomial Φ_{105} and the 561-th inverse cyclotomic polynomial Ψ_{561}, the first instances where the polynomials are non-flat. The goal of the following subsections is to illustrate, step by step, how the first non-flat coefficients can be computed for each polynomial using the recurrence formulas and methods discussed in the previous section. These computations provide insight into the structure of the coefficients and suggest a pattern where larger coefficients tend to concentrate toward the center of the polynomials.

4.1 Results for Φ_{105}

When $n = 3 \cdot 5 \cdot 7 = 105$, we have $c_1^{(n)} = -\mu(n) = 1$ and from Proposition (1), we know that $c_2^{(n)} = 1$, $c_3^{(n)} = 0$.

Now, using the recurrence formula (10) we obtain

$$\begin{aligned} c_4^{(n)} &= -\frac{\varphi(105)}{4}\left[\frac{\mu(105)}{\varphi(105)} + \frac{\mu(5\cdot 7)}{\varphi(5\cdot 7)}c_1^{(n)} + \frac{\mu(105)}{\varphi(105)}c_2^{(n)} + \frac{\mu(105)}{\varphi(105)}c_3^{(n)}\right] \\ &= -\frac{1}{4}\left[-1 + 2\cdot 1 - 1\cdot 1 - 1\cdot 0\right] = 0. \end{aligned}$$

Similarly, $c_5^{(n)} = -1$ and $c_6^{(n)} = -1$. Now, we use the formula to get

$$\begin{aligned} c_7^{(n)} &= -\frac{\varphi(105)}{7}\left[\frac{\mu(15)}{\varphi(15)} + \frac{\mu(35)}{\varphi(35)}c_1^{(n)} + \frac{\mu(21)}{\varphi(21)}c_2^{(n)} + \frac{\mu(105)}{\varphi(105)}c_3^{(n)} + \frac{\mu(35)}{\varphi(35)}c_4^{(n)}\right. \\ &\quad \left. + \frac{\mu(105)}{\varphi(105)}c_5^{(n)} + \frac{\mu(105)}{\varphi(105)}c_6^{(n)}\right] \\ &= -\frac{1}{7}\left[6 + 2\cdot 1 + 4\cdot 1 + (-1)\cdot 0 + 2\cdot 0 + (-1)\cdot(-1) + (-1)\cdot(-1)\right] = -2. \end{aligned}$$

This confirms that -2 appears as the coefficient of z^7 in

$$\begin{aligned} \Phi_{105}(z) &= z^{48} + z^{47} + z^{46} - z^{43} - z^{42} - 2z^{41} - z^{40} - z^{39} + z^{36} + z^{35} + z^{34} \\ &\quad + z^{33} + z^{32} + z^{31} - z^{28} - z^{26} - z^{24} - z^{22} - z^{20} + z^{17} + z^{16} + z^{15} \\ &\quad + z^{14} + z^{13} + z^{12} - z^9 - z^8 - 2z^7 - z^6 - z^5 + z^2 + z + 1. \end{aligned}$$

While this result is in line with Proposition 2.3 which stated that $c_r^{(n)} = -2$, we notice that $(p, q, r) = (3, 5, 7)$ is not a Ramanujan triple.

4.2 Results for Ψ_{561}

For numerical simulations we focus on some instances where Ψ_n is not flat. It is known that the first non flat inverse cyclotomic polynomial is Ψ_{561}. As listed in Table 1 of [28], -3 first appears in Ψ_{1155} as the coefficient of z^{33}, while 4 first appears in Ψ_{2145} as the coefficient of z^{44}. These are the three cases we focus on for the moment, and for which we compute the coefficients explicitly until we get the first coefficient which is not 0, 1 or -1.

Let us focus on the case $n = 561 = 3 \cdot 11 \cdot 17$.

Since $\mu(n) = -1$, by the argument used previously we get $d_1^{(n)} = -\mu(n) = 1$. From Proposition 3 we also know that $d_2^{(n)} = 0$ and $d_3^{(n)} = -1$, so by (14), we get

$$\begin{aligned} d_4^{(n)} &= \frac{\varphi(561)}{4}\left[-\frac{\mu(561)}{\varphi(561)} + \frac{\mu(187)}{\varphi(187)}d_1^{(n)} + \frac{\mu(561)}{\varphi(561)}d_2^{(n)} + \frac{\mu(561)}{\varphi(561)}d_3^{(n)}\right] \\ &= \frac{1}{4}\left[1 + 2\cdot 1 + (-1)\cdot 0 + (-1)\cdot(-1)\right] = 1. \end{aligned}$$

Similarly, one obtains $d_5^{(n)} = 0$, $d_6^{(n)} = -1$, $d_7^{(n)} = 1$, $d_8^{(n)} = 0$, $d_9^{(n)} = -1$, $d_{10}^{(n)} = 1$,

$d_{11}^{(n)}=-1$, $d_{12}^{(n)}=0$, $d_{13}^{(n)}=1$, $d_{14}^{(n)}=-1$, $d_{15}^{(n)}=0$, $d_{16}^{(n)}=1$, from where

$$\begin{aligned}
d_{17}^{(n)} &= \frac{\varphi(561)}{17}\left[-\frac{\mu(33)}{\varphi(33)}+\frac{\mu(561)}{\varphi(561)}d_1^{(n)}+\frac{\mu(187)}{\varphi(187)}d_2^{(n)}+\frac{\mu(51)}{\varphi(51)}d_3^{(n)}\right.\\
&+\frac{\mu(561)}{\varphi(561)}d_4^{(n)}+\frac{\mu(187)}{\varphi(187)}d_5^{(n)}+\frac{\mu(51)}{\varphi(51)}d_6^{(n)}+\frac{\mu(561)}{\varphi(561)}d_7^{(n)}+\frac{\mu(187)}{\varphi(187)}d_8^{(n)}+\\
&+\frac{\mu(561)}{\varphi(561)}d_9^{(n)}+\frac{\mu(51)}{\varphi(51)}d_{10}^{(n)}\frac{\mu(187)}{\varphi(187)}d_{11}^{(n)}+\frac{\mu(561)}{\varphi(561)}d_{12}^{(n)}+\frac{\mu(561)}{\varphi(561)}d_{13}^{(n)}+\\
&\left.+\frac{\mu(187)}{\varphi(187)}d_{14}^{(n)}+\frac{\mu(561)}{\varphi(561)}d_{15}^{(n)}+\frac{\mu(561)}{\varphi(561)}d_{16}^{(n)}\right]\\
&=\frac{1}{17}\left[-16+(-1)\cdot 1+2\cdot 0+(-1)\cdot(-1)+(-1)\cdot 1+2\cdot 0+10\cdot(-1)\right]+\\
&+\frac{1}{17}\left[(-1)\cdot 1+2\cdot 0+(-1)\cdot(-1)+(-1)\cdot 1+2\cdot(-1)+(-1)\cdot 0\right]+\\
&+\frac{1}{17}\left[(-1)\cdot 1+2\cdot(-1)+(-1)\cdot 0+(-1)\cdot 1\right]=-2.
\end{aligned}$$

This confirms that -2 appears as the coefficient of z^{17} in

$$\begin{aligned}
\Psi_{561}(z) &= z^{241}-z^{240}+\cdots+2z^{224}+\cdots+z^{18}+\\
&-2z^{17}+z^{16}-z^{14}+z^{13}-z^{11}+z^{10}-z^9+z^7-z^6+z^4-z^3+z-1.
\end{aligned}$$

This argument then allows the calculation of further coefficients, and suggests why the larger coefficients of inverse cyclotomic polynomials are moving towards the centre (also using the fact that the polynomial is antipalindromic).

Note that in Proposition 4.3 we had $d_r^{(n)}=-1$, so the condition that (p,q,r) is a Ramanujan triple is in fact necessary, as $(3,11,17)$ is clearly not Ramanujan.

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