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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A NONLINEAR EQUATION WITH CONVECTION TERM

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Abstract

In this paper, we consider the existence and uniqueness of weak solutions of a nonlinear elliptic equation with a variable exponent, a monotonic type operator and a convection term. With the topological degree theory, we prove the existence of at least one weak solution under some Leray-Lions and growth conditions. Moreover, we obtain the uniqueness of the solution of the problem under some additional assumptions. Our results generalize and improve existing results with another approach.

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1 Introduction

In the present paper, we focus on the existence and uniqueness of a weak solution for a class of nonlinear elliptic boundary value problems in the framework of the Lebesgue and Sobolev spaces with variable exponent. The use of such spaces is justified by the study of several materials that present inhomogeneities and for which the context given by classical spaces is not adequate. Indeed, for such materials, the exponents involved in the constitutive law could be variable. The great attention on this topic is due to the many and various applications concerning thermorheological fluids [6], image restoration [8], electrorheological fluids [17, 18] and elastic materials [22].

Consider the following problem with a Neuman boundary condition

$$\begin{cases} -\text{div}\, a(x, \nabla u) + \lambda |u|^{q(x)-2}u = f(x, u), & x \in \Omega \quad , \\ a(x, \nabla u).\eta = 0, & x \in \partial\Omega \quad , \end{cases} \tag{1}$$

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where $\lambda \in \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ is a bounded domain, a generates a nonlinear operator of Leray-Lions A from $W_0^{1,p(\cdot)}(\Omega)$ to its dual $W^{-1,p'(\cdot)}(\Omega)$ which is defined by

$$A(u) = -\text{div } a(x, \nabla u),$$

$p(\cdot)$ and $q(\cdot)$ are two variable exponents satisfying some conditions to be seen in the paper suite, f is a Carathéodory function satisfying a growth condition with a variable exponent that is suitably controlled by $p(\cdot)$ and η is the outward unit normal to $\partial\Omega$.

In [1], the authors showed the existence and uniqueness of a weak solution to the problem

$$-\Delta_p u + m(x)|u|^{p-2}u = f(x, u) \text{ in } \mathbb{R}^N$$

which involves the p-Laplacian through Browder's theorem, where
$1 < p < N$, $N \geq 3$ and under some conditions for the functions m and f.
When function f is null and $a(x, \nabla u) = |\nabla u|^{q(x)-2}\nabla u$, the problem (1) has been treated as an eigenvalue and eigenvector problem [12].
Zhao et al. [21] have established the existence and the multiplicity of weak solutions of the boundary-value problem

$$\begin{cases} -\text{div } a(x, \nabla u) + |u|^{p-2}u = \lambda f(x, u), & x \in \Omega \quad , \\ u(x) = \text{constant}, & x \in \partial\Omega \quad , \\ \int_{\partial\Omega} a(x, \nabla u).n \, ds = 0. & \end{cases}$$

They found one nontrivial solution by the mountain pass lemma in [16], when the nonlinearity has a $(p-1)$-superlinear growth at infinity, and two nontrivial solutions by minimization and mountain pass when the nonlinear term has a $(p-1)$-sublinear growth at infinity.
For $\lambda = 0$ and $a(x, \nabla u) = |\nabla u|^{q(x)-2}\nabla u$, Iliaş in [13] and Fan and Zhang in [10] gave several sufficient conditions for the existence of weak solutions for that problem under Dirichlet's boundary condition.

In this paper, we study problem (1), on the one hand as a kind of generalisation of the few previous works, and on the other hand, using another method based on the Topological Degree Theory developed by Berkovits [7] for some classes of operators in Banach reflexive spaces (For some applications of this degree, the reader can see [2, 4, 5, 3, 15]).

This document is organised as follows: Section 2 is reserved for some mathematical preliminaries. In section 3, we give our basic assumptions, some technical lemmas and we give and prove our results of existence and uniqueness.

2 Mathematical preliminaries

2.1 Some classes of operators

Let us start with a short reminder of some classes of operators and an important proposition which will be the key to proving the existence of at least one weak solution of problem (1).

Consider X as a real separable and reflexive Banach space with dual X^* and continuous pairing $\langle . \, , \, . \rangle$ and let $\Omega \subset X$ nonempty. The symbols $\to$ and $\rightharpoonup$ signify strong and weak convergence, respectively.
Let us recall that a mapping $F : \Omega \subset X \to X^*$ is of class (S_+), if for any sequence (u_n) in Ω such that $u_n \rightharpoonup u$ and $\limsup \langle Fu_n, u_n - u \rangle \leq 0$, we have $u_n \to u$. F is said to be quasimonotone, if for a weakly convergent sequence as above, $\limsup \langle Fu_n, u_n - u \rangle \geq 0$.

Proposition 1. *[3, Proposition 1.1] Let $S : X \to X^*$ and $T : X^* \to X$ be two operators bounded and continuous such that S is quasimonotone and T is an homeomorphism, strictly monotone and of class (S_+). If*

$$\Lambda := \{v \in X^* / v + tSoTv = 0 \text{ for some } t \in [0, 1]\}$$

is bounded in X^*, then the equation

$$v + SoTv = 0$$

admits at lest one solution in X^*.

2.2 Functional framework

In the sequel, Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 2)$.

In this section, we introduce the notation and functional framework used in our paper. These are the variable exponent Lebesgue and Sobolev spaces. We also recall the essential properties of these spaces. For more details, we refer the reader to [9, 11, 14].

Set

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : \min_{x \in \overline{\Omega}} p(x) > 1\}.$$

For any $p \in C_+(\overline{\Omega})$, we define

$$p^- = \inf_{x \in \overline{\Omega}} p(x) \text{ and } p^+ = \sup_{x \in \overline{\Omega}} p(x).$$

Let $p \in C_+(\overline{\Omega})$. We define the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) = \{u : \Omega \to \mathbb{R};\ u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\, dx < \infty\},$$

endowed with the *Luxemburg norm* defined by

$$\|u\|_{p(\cdot)} = \inf\{\mu > 0; \int_\Omega \Big| \frac{u(x)}{\mu} \Big|^{(p(x)} \leq 1\}.$$

Equipped with this norm, $L^{p(\cdot)}(\Omega)$ becomes a Banach space. If $1 < p^- \leq p^+ < \infty$, then $L^{p(\cdot)}(\Omega)$ is a reflexive uniformly convex Banach space. Moreover, for any measurable bounded exponent $p(\cdot)$, the space $L^{p(\cdot)}(\Omega)$ is separable.

If $p_1(\cdot),\ p_2(\cdot) \in C_+(\overline{\Omega})$ such that $p_1(x) \leq p_2(x)$ a.e. in Ω, there exists the continuous embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$.
We denote by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the Hölder inequality holds:

$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \leq 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}. \tag{2}$$

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the *modular* of the $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx.$$

We have the proposition

Proposition 2. *If $u \in L^{p(\cdot)}(\Omega)$, then the following relations hold:*

1. $\|u\|_{p(\cdot)} > 1 \quad \Rightarrow \quad \|u\|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^+}$,
2. $\|u\|_{p(\cdot)} < 1 \quad \Rightarrow \quad \|u\|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^-}$,
3. $\|u\|_{p(\cdot)} \leq \rho_{p(\cdot)}(u) + 1$,
4. $\rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^-} + \|u\|_{p(\cdot)}^{p^+}$.

The variable exponent Sobolev spaces $W^{1,p(\cdot)}(\Omega)$ contains all functions $u \in L^{p(\cdot)}(\Omega)$ such that ∇u exists almost everywhere and belongs to $L^{p(\cdot)}(\Omega)^N$. $W^{1,p(\cdot)}(\Omega)$ is a separable and reflexive Banach space with respect to the norm

$$\|u\|_{W^{1,p(\cdot)}} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

We suppose that p is *logarithmic Hölder continuous*, that is, there exists $M > 0$ such that

$$|p(x) - p(y)| \leq -\frac{M}{\log(|x-y|)} \text{ for all } x, y \in \Omega \text{ with } x \neq y \text{ and } |x-y| \leq \frac{1}{2}, \tag{3}$$

then the smooth functions are dense in $W^{1,p(\cdot)}(\Omega)$. Let $W_0^{1,p(\cdot)}(\Omega)$ denote the Sobolev space of functions with zero boundary values under the norm $\|\cdot\|_{W^{1,p(\cdot)}}$. Furthermore, if p satisfies (3), then $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$ and the Poincaré inequality is satisfied, that is, there exists a constant $C > 0$ depending only on Ω and the function p such that

$$\|u\|_{p(\cdot)} \leq C\|\nabla u\|_{p(\cdot)}, \ \ \forall u \in W_0^{1,p(\cdot)}(\Omega), \tag{4}$$

In particular, the space $W_0^{1,p(\cdot)}(\Omega)$ has a norm given by

$$\|u\|_{1,p(\cdot)} = \|\nabla u\|_{p(\cdot)},$$

which is equivalent to the norm $\|\cdot\|_{W^{1,p(\cdot)}}$. Moreover, the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is compact.
The space $(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)})$ is also a Banach space separable and reflexive .
The dual space of $W_0^{1,p(\cdot)}(\Omega)$ is denoted by $W^{-1,p'(\cdot)}(\Omega)$.

3 Basic assumptions and main results

Let $q \in C_+(\bar{\Omega})$, $1 < q^- \le q(x) \le q^+ < p^- \le p(x) \le p^+ < \infty$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a real-valued function such that:

(f_1) f satisfies the Carathéodory condition, i.e. $f(.,\eta)$ is measurable on Ω for all $\eta \in \mathbb{R}$ and $f(x,.)$ is continuous on $\mathbb{R}$ for a.e. $x \in \Omega$.

(f_2) f has the growth condition

$$|f(x,\eta)| \le c_1(k(x) + |\eta|^{r(x)-1})$$

for a.e. $x \in \Omega$ and all $\eta \in \mathbb{R}$, where c_1 is a positive constant, $k \in L^{p'(x)}(\Omega)$, $k(x) \ge 0$ and $r \in C_+(\bar{\Omega})$ with $1 < r^- \le r(x) \le r^+ < p^-$.

Let $A : W_0^{1,p(\cdot)}(\Omega) \to W^{-1,p'(\cdot)}(\Omega)$ be the nonlinear operator of Leray-Lions, which is defined by

$$A(u) = -\mathrm{div} a(x, \nabla u).$$

i.e.

$$\langle A(u), v\rangle = \int_\Omega a(x,\nabla u).\nabla v \, dx \text{ for all } v \in W_0^{1,p(.)}(\Omega). \tag{5}$$

The function a is assumed to satisfy the conditions:

(A_1) $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function,i.e., $a(.,\xi)$ is measurable on Ω for all $\xi \in \mathbb{R}^N$ and $a(x,.)$ is continuous on $\mathbb{R}^N$ for a.e. $x \in \Omega$.

(A_2) There exist a positive function $b(x)$ in $L^{p'(\cdot)}(\Omega)$ and constant $c > 0$ such that $|a(x,\xi)| \le b(x) + c|\xi|^{p(x)-1}$ for all $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

(A_3) $(a(x,\xi) - a(x,\xi')).(\xi - \xi') > 0$ a.e. $x \in \Omega$, for all $\xi, \xi' \in \mathbb{R}^N, \xi \ne \xi'$.

(A_4) There exist a constant $c' > 0$ such that $a(x,\xi).\xi \ge c'|\xi|^{p(x)}$.

Lemma 1. *([5, Lemma 4.3] Under assumptions (f_1) and (f_2), the operator $S : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ setting by*

$$\langle Su, v\rangle = \int_\Omega (\lambda|u|^{q(x)-2}u - f(x,u))v dx, \quad \forall u, v \in W_0^{1,p(x)}(\Omega)$$

is compact.

Lemma 2. *Assume that the conditions $(A_1)-(A_4)$ hold. Then the operator A defined by (5) is bounded, continuous, of type (S_+) and coercive i.e.* $\lim_{\|v\|_{1,p(\cdot)}\to\infty} \dfrac{< Av, v >}{\|v\|_{1,p(\cdot)}} = \infty.$

Proof. See the proof of Lemma 3.2 in [2] taking the weight function $w \equiv 1$. □

Let us first define the weak solution of problem (1).

Definition 1. *We call that $u \in W_0^{1,p(\cdot)}(\Omega)$ is a weak solution of (1) if*

$$\int_\Omega a(x, \nabla u)\nabla v\, dx = \int_\Omega -\lambda|u|^{q(x)-2}uv\, dx + \int_\Omega f(x, u)v\, dx, \ \ \forall v \in W_0^{1,p(\cdot)}(\Omega).$$

Theorem 1. *We suppose that the assumptions (f_1), (f_2) and $(A_1)-(A_4)$ hold true, then there exists at least one weak solution of the problem (1) in $W_0^{1,p(\cdot)}(\Omega)$.*

Proof. Using the operators A and S seen above, and according to the last definition, we can define the weak solutions of (1) to be the solutions of the equation

$$u \in W_0^{1,p(\cdot)}(\Omega), \ \ Au = -Su \tag{6}$$

Due to assumption (A_3), Lemma 2 and in consideration of the Theorem of Minty-Browder [20, Theorem 26A], the inverse operator T of A is also continuous, bounded and of type (S_+). Note in addition that the operator S is continuous, bounded, and quasimonotone (see Lemma 1). Equation (6) is therefore equivalent to

$$u \in W_0^{1,p(\cdot)}(\Omega), \ \ u = Tv \text{ and } v + SoTv = 0. \tag{7}$$

To resolve equation (7), we will use Proposition 1. It is simply a matter of showing that the set

$$\Lambda := \{v \in W^{-1,p'(\cdot)}(\Omega)|v + tSoTv = 0 \text{ for some } t \in [0, 1]\}$$

is bounded.

Note that if $v \in \Lambda$ and $u = Tv$, then $\|Tv\|_{1,p(\cdot)} = \|\nabla u\|_{p(\cdot)}$.

If $\|\nabla u\|_{p(\cdot)} \leq 1$, then $\|Tv\|_{1,p(\cdot)}$ is bounded.

If $\|\nabla u\|_{p(\cdot)} > 1$, then we have by Proposition 2

$$\|Tv\|_{1,p(\cdot)}^{p^-} = \|\nabla u\|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(\nabla u). \tag{8}$$

By the assumption (A_4), we have

$$a(x, \nabla u).\nabla u \geq c'|\nabla u|^{p(x)}.$$

Then

$$\begin{aligned}\rho_{p(\cdot)}(\nabla u) &= \int_\Omega |\nabla u|^{p(x)}dx \\ &\leq \frac{1}{c'}\int_\Omega a(x,\nabla u).\nabla u \\ &= \frac{1}{c'}\langle Au,u\rangle \\ &= \frac{1}{c'}\langle v,Tv\rangle \\ &= \frac{-t}{c'}\langle SoTv,Tv\rangle.\end{aligned}$$

This implies that

$$\rho_{p(\cdot)}(\nabla u) \leq \frac{t}{c'}\int_\Omega -\lambda|u|^{q(x)} + f(x,u)udx. \tag{9}$$

We get, by the inequalities (8), (9), the growth condition (f_2), the Hölder inequality (2), the inequality (4) of Proposition 2 and the Young inequality the estimate

$$\begin{aligned}\|Tv\|^{p^-}_{1,p(\cdot)} &\leq const(\lambda\rho_{q(\cdot)}(u) + \int_\Omega |k(x)u(x)|\,dx + \rho_{r(\cdot)}(u)) \\ &\leq const(\|u\|^{q^-}_{q(\cdot)} + \|u\|^{q^+}_{q(\cdot)} + \|k\|_{p'(\cdot)}\|u\|_{p(\cdot)} + \|u\|^{r^-}_{r(\cdot)} + \|u\|^{r^+}_{r(\cdot)}) \\ &\leq const(\|u\|^{q^-}_{q(\cdot)} + \|u\|^{q^+}_{q(\cdot)} + \|u\|_{p(\cdot)} + \|u\|^{r^-}_{r(\cdot)} + \|u\|^{r^+}_{r(\cdot)}).\end{aligned}$$

From the Poincaré inequality (4) and the continuous embedding $L^{p(x)} \hookrightarrow L^{q(x)}$ and $L^{p(x)} \hookrightarrow L^{r(x)}$, we can deduct the estimate

$$\|Tv\|^{p^-}_{1,p(\cdot)} \leq const(\|Tv\|^{q^+}_{1,p(\cdot)} + \|Tv\|_{1,p(\cdot)} + \|Tv\|^{r^+}_{1,p(\cdot)}).$$

It follows that $\{Tv|v\in\Lambda\}$ is bounded.
Since the operator S is bounded, it is obvious from (7) that the set Λ is bounded in $W^{-1,p'(\cdot)}(\Omega)$.
Hence, in virtu of Proposition 1, the equation $v+SoTv$ have at lest one non trivial solution $\bar{v}$ in $W^{-1,p'(\cdot)}(\Omega)$. So,

$$\bar{u} = T\bar{v}$$

is a weak solution of (1). □

Next, we consider the uniqueness of solutions of (1). To this end, we also need the following hypothesis on the convection term:

(f_3) There exists $c_2 \geq 0$ such that

$$(f(x,t)-f(x,s))(t-s) \leq c_2|t-s|^{q(x)}$$

for a.e. $x\in\Omega$ and all $t,s\in\mathbb{R}$.

Our uniqueness result reads as follows.

Theorem 2. *Assume that* $(f_1) - (f_3)$ *and* $(A_1) - (A_4)$ *hold. If, in addition,* $q(x) \geq 2$ *for all* $x \in \overline{\Omega}$, *then the weak solution of* (1) *is unique provided*

$$\frac{2^{q^+} c_2}{\lambda} < 1.$$

Proof. Let $u_1, u_2 \in W_0^{1,p(\cdot)}(\Omega)$ be two weak solutions of (1). by choosing $v = u_1 - u_2$ in the Definition 1, we have

$$\int_\Omega a(x, \nabla u_1)\nabla(u_1 - u_2)\, dx + \int_\Omega \lambda |u_1|^{q(x)-2} u_1 (u_1 - u_2)\, dx = \int_\Omega f(x, u_1)(u_1 - u_2)\, dx$$

and

$$\int_\Omega a(x, \nabla u_2)\nabla(u_1 - u_2)\, dx + \int_\Omega \lambda |u_2|^{q(x)-2} u_2 (u_1 - u_2)\, dx = \int_\Omega f(x, u_2)(u_1 - u_2)\, dx$$

Subtracting the above two equations, we have

$$\int_\Omega (a(x, \nabla u_1) - a(x, \nabla u_2))\nabla(u_1 - u_2)\, dx + \int_\Omega \lambda(|u_1|^{q(x)-2} u_1 - |u_2|^{q(x)-2} u_2)(u_1 - u_2)\, dx$$

$$= \int_\Omega (f(x, u_1) - f(x, u_2))(u_1 - u_2)\, dx.$$

By assumption (A_3), we have

$$(a(x, \nabla u_1) - a(x, \nabla u_2))\nabla(u_1 - u_2) > 0.$$

Moreover, since $q(x) \geq 2$, then we have the following inequality (see [19]):

$$(|u_1|^{q(x)-2} u_1 - |u_2|^{q(x)-2} u_2)(u_1 - u_2) \geq (\frac{1}{2})^{q(x)} |u_1 - u_2|^{q(x)}.$$

So, by using assumption (f_3), we have

$$\begin{aligned} \lambda(\frac{1}{2})^{q^+} \int_\Omega |u_1 - u_2|^{q(x)}\, dx &\leq \int_\Omega (f(x, u_1) - f(x, u_2))(u_1 - u_2)\, dx \\ &\leq c_2 \int_\Omega |u_1 - u_2|^{q(x)}\, dx. \end{aligned}$$

Consequently, when $\frac{2^{q^+} c_2}{\lambda} < 1$, it follows that $u_1 = u_2$ and so the solution of (1) is unique. □

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