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ON BESSEL AND RIESZ-FISHER SYSTEMS WITH RESPECT TO BANACH SPACE OF VECTOR-VALUED SEQUENCES

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Abstract

The paper deals with Bessel systems and Riesz-Fisher systems in Banach spaces with respect to some Banach space of vector-valued sequences. The notions of $\tilde{X}$-Bessel systems and $\tilde{X}$-Riesz-Fisher systems were introduced and the characterization of such systems were established. $\tilde{X}$-frames and $\tilde{X}$-Riesz g-bases are also studied in this paper. The relations between them are obtained.

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1 Introduction

In recent times, due to numerical applications, the interest in frames has increased greatly. The frames have found applications in signaling processes, data compression, information processing, when studying the features of surfaces of crystals and nano-objects and in other fields. The notion of frame was introduced by R.J.Duffin and A.C.Schaeffer [1] when studying the theory of nonharmonic Fourier series with respect to the perturbed system of exponents. In [1] the frames in abstract Hilbert spaces were also studied. Note that the non-empty system of non-zero elements of Hilbert spaces $\{f_n\}_{n\in N}$ is called a frame in H if there exist constants $A>0$ and $B>0$ such that for each $f\in H$

$$A\left\|f\right\|_H^2 \le \sum_{n=1}^{\infty}\left|(f,f_n)\right|^2 \le B\left\|f\right\|_H^2,$$

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where $\|\cdot\|_H$ is a norm in H generated by the scalar product $(\cdot,\cdot)$. If $\{f_n\}_{n\in N} \subset H$ is a frame, then there exists a frame $\{g_n\}_{n\in N} \subset H$ such that every $f \in H$ has a representation of the form

$$f = \sum_{n=1}^{\infty} (f, g_n) f_n = \sum_{n=1}^{\infty} (f, f_n) g_n.$$

Later, a lot of literature was devoted to frames (see, for example [2-7]). It is known that the frames in Hilbert spaces are the generalizations of the Riesz bases introduced and studied by N.K.Bari [8], i.e. the bases equivalent to orthonormal ones. The stability of frames is very important to obtain new frames. Similarly to theorems of stability of orthonormal bases, the stability of frames in Hilbert spaces was studied in [9-11]. One of the important directions of frames in Hilbert spaces is the study of generalizations of frames. The notions of g-frame and g-Riesz bases were introduced and several properties of frames and Riesz bases in Hilbert spaces were obtained in [12]. The g-frames were also considered in [13, 14]. Another generalization of frames in Hilbert spaces was studied in [15, 16], where the notion of t-frames was introduced and many properties and noetherian perturbation of such frames were studied. The Banach generalizations of frames were first studied by K. Gröchenig in [17]; in this paper, the notions of Banach frame and atomic decomposition were introduced. Banach frames, atomic decomposition and their stability were also studied in [18-20]. The important results were obtained in the case of spaces l_p, $p \neq 2$. In [21] the p-frames and atomic decompositions with respect to shift of subspaces L_p were obtained as generalizations of frames and Riesz bases in Banach spaces. The results with respect to p-frame decompositions and its relationship with q-Riesz bases were studied in [22]. Then the results of [23-25] were taken to general Banach spaces of number sequences (for these and other facts see [11]).

The Banach generalization of frames with respect to Banach spaces of vector-valued sequences were considered, the notions of $\tilde{X}$-frame and $\tilde{X}$-Riesz g-bases were introduced, necessary and sufficient condition theorems were established in [26, 27]. The present paper was devoted to the generalization of Bessel systems and Riesz-Fisher system with respect to space $\tilde{X}$. The notions of $\tilde{X}$-Bessel systems and $\tilde{X}$-Riesz-Fisher systems were introduced, the criteria of $\tilde{X}$-Bessel property of the system and $\tilde{X}$-Riesz-Fisher system were proved and their relations with $\tilde{X}$-Riesz g-bases were studied.

2 Auxiliary notions and results

Let X and Z be Banach spaces with appropriate norms $\|\cdot\|_X$ and $\|\cdot\|_Z$. The dual space of the to space X is denoted by X^*. By $\pi_X : X \to X^{**}$ we denote a canonical mapping of space X into space X^{**}, i.e.

$$\pi_X(x)(x^*) = x^*(x), \forall x \in X, \forall x^* \in X^*.$$

For a reflexive space X the canonical mapping is an isometric isomorphism. The image of the operator $T : X \to Z$ is denoted by ImT. The adjoint of an operator $T \in L(X, Z)$ is denoted by T^*, where $L(X, Z)$ denotes the space of all linear bounded operators acting from X to Z. We need the following theorem.

Theorem 1. *[28]. Let X and Z be Banach spaces and let $T \in L(X, Z)$. Then the equality $ImT = Z$ holds if and only if there is a number $c > 0$ such that*

$$\|T^* f\|_{X^*} \geq c \|f\|_{Z^*}, \forall f \in Z^*.$$

Let $\tilde{X}$ be some Banach space of sequences consisting of the vectors of X with coordinate vise linear operations such that operators $e_k : X \to \tilde{X}$, $e_k(x) = \{\delta_{ik} x\}_{i\in N}$ and $e_k^* : \tilde{X} \to X$, $e_k^*(\{x_n\}_{n\in N}) = x_k$ are continuous (KB-space). $\tilde{X}$ is said to be CB-space if for every $\tilde{x} = \{x_k\}_{k\in N} \in \tilde{X}$

$$\tilde{x} = \sum_{k=1}^{\infty} e_k(x_k).$$

For CB-space $\tilde{X}$, the conjugated space $\tilde{X}^*$ is isometrically isomorphic to the space $\tilde{Y}$ of sequences $\{x_k^*\}_{k\in N} \subset X^*$ with the finite norm

$$\left\|\{x_k^*\}_{k\in N}\right\| = \sup_{\|\tilde{x}\|=1} \left| \sum_{k=1}^{\infty} x_k^*(x_k) \right|.$$

Indeed, for $\forall \tilde{x}^* \in \tilde{X}^*$

$$\tilde{x}^*(\{x_k\}_{k\in N}) = \sum_{k=1}^{\infty} \tilde{x}^* e_k(x_k) = \sum_{k=1}^{\infty} x_k^*(x_k),$$

where $x_k^* = \tilde{x}^* e_k$. We have

$$\left\|\{x_k^*\}_{k\in N}\right\|_{\tilde{Y}} = \sup_{\|\tilde{x}\|=1} \left| \sum_{k=1}^{\infty} x_k^*(x_k) \right| = \sup_{\|\tilde{x}\|=1} |\tilde{x}^*(\tilde{x})| = \|\tilde{x}^*\|_{\tilde{X}^*}.$$

Vice versa, for each such sequence $\{x_k^*\}_{k\in N} \subset X^*$ a linear continuous functional $\tilde{x}^*$ is determined on $\tilde{X}$ by the formula $\tilde{x}^*(\tilde{x}) = \sum_{k=1}^{\infty} x_k^*(x_k), \tilde{x} = \{x_k\}_{k\in N} \in \tilde{X}$ and

$$\|\tilde{x}^*\|_{\tilde{X}^*} = \sup_{\|\tilde{x}\|=1} |\tilde{x}^*(\tilde{x})| = \sup_{\|\tilde{x}\|=1} \left| \sum_{k=1}^{\infty} x_k^*(x_k) \right| = \left\|\{x_k^*\}_{k\in N}\right\|_{\tilde{Y}}.$$

Throughout the paper, space $\tilde{X}^*$ is identified with space $\tilde{Y}$. Let us consider the system of operators $\{g_k\}_{k\in N} \subset L(Z, X)$. The following notion is the generalization of the notion of frame in Hilbert spaces.

Definition 1. *The system $\{g_k\}_{k\in N}$ is said to be $\tilde{X}$-frames in Z if there exist constants $A>0$ and $B>0$ such that*

$$A\left\|z\right\|_Z \le \left\|\{g_k(z)\}_{k\in N}\right\|_{\tilde{X}} \le B\left\|z\right\|_Z, \forall z\in Z. \tag{1}$$

Constants A and B are called $\tilde{X}$-frame bounds of $\{g_k\}_{k\in N}$. If there exists an operator $S\in L(\tilde{X},Z)$ such that $S\left(\{g_n(z)\}_{n\in N}\right)=z, \forall z\in Z$, then $\left(\{g_k\}_{k\in N}, S\right)$ is called Banach $\tilde{X}$-frame in Z.

Obviously, the system $\{g_k\}_{k\in N}$ is $\tilde{X}$-frame in Z if and only if the operator $U:Z\to\tilde{X}$, given by the formula

$$U(z)=\{g_k(z)\}_{k\in N}, \forall z\in Z \tag{2}$$

is bounded and has a bounded inverse on ImU.

By $L_{X^*}\left(\{g_n\}_{n\in N}\right)$ we denote the set of all possible finite sums of the form $\sum\limits_k x_k^* g_k$, where $x_k^*\in X^*$.

Definition 2. *The system $\{g_k\}_{k\in N}$ is said to be g-complete in Z^* if $\overline{L_{X^*}\left(\{g_n\}_{n\in N}\right)}=Z^*$ in the norm of Z^*. The system $\{g_k\}_{k\in N}$ is called g-biorthogonal in Z^* if there exists a system $\{\Lambda_j\}_{j\in N}\subset L(X,Z)$ such that*

$$g_k\Lambda_j=\delta_{kj}.$$

The system $\{g_k\}_{k\in N}$ is said to be g-biorthogonal system to $\{\Lambda_j\}_{j\in N}\subset L(X,Z)$. The system $\{g_k\}_{k\in N}$ is said to be g-minimal in Z^* if $\forall x^*\in X^*$, $x^*\neq 0$ and $\forall k\in N$ the relation $x^*g_k\notin\overline{L_{X^*}\left(\{g_n\}_{n\neq k}\right)}$ holds.

Lemma 1. *The g-minimal system $\{g_k\}_{k\in N}\subset L(Z,X)$ is minimal.*

Proof. Assume the contrary, i.e. let the system $\{g_k\}_{k\in N}$ be not minimal. Then $\exists k_0$ such that $g_{k_0}\in\overline{L\left(\{g_n\}_{n\neq k_0}\right)}$. Consequently,

$$g_{k_0}=\lim_{n\to\infty}\sum_{i=1,i\neq k_0}^{m_n}\alpha_i^{m_n}g_i.$$

Hence $\forall x^*\in X^*,\ x^*\neq 0$, we get

$$x^*g_{k_0}=\lim_{n\to\infty}\sum_{i=1,i\neq k_0}^{m_n}\alpha_i^{m_n}x^*g_i,$$

which means that $x^*g_{k_0}\in\overline{L_{X^*}\left(\{g_n\}_{n\neq k_0}\right)}$. This contradicts g-minimality of the system $\{g_k\}_{k\in N}$. The obtained contradiction proves the theorem. □

The following lemma holds as for an ordinary basis.

Lemma 2. *Let $\tilde{X}$ be a reflexive CB-space. Then for $\forall \tilde{x}^* = \{x_k^*\}_{k\in N} \in \tilde{X}^*$ the following expansion holds*

$$\tilde{x}^* = \sum_{k=1}^{\infty} x_k^* e_k^*.$$

Proof. Let us take $\forall \tilde{x}^* = \{x_k^*\}_{k\in N} \in \tilde{X}^*$ and $\varepsilon > 0$. By virtue of the fact that $\tilde{X}$ is CB-space, $\exists M > 0$ such that $\forall \tilde{x} = \{x_k\}_{k\in N} \in \tilde{X}$

$$\left\| \sum_{k=1}^{n} e_k(x_k) \right\|_{\tilde{X}} \le M \left\| \tilde{x} \right\|_{\tilde{X}}, \forall n \in N.$$

Taking this and $x_k^* = \tilde{x}^* e_k$ into account , we get

$$\left\| \sum_{k=1}^{n} x_k^* e_k^* \right\|_{\tilde{X}^*} = \sup_{\|\tilde{x}\|=1} \left| \sum_{k=1}^{n} x_k^* e_k^*(\tilde{x}) \right| =$$

$$= \sup_{\|\tilde{x}\|=1} \left| \tilde{x}^* \left(\sum_{k=1}^{n} e_k(x_k) \right) \right| \le M \left\| \tilde{x}^* \right\|_{\tilde{X}^*}. \qquad (3)$$

Let us show the validity of the relation $\overline{L_{X^*}\left(\{e_k^*\}_{k\in N}\right)} = \tilde{X}^*$. Assume the contrary. By virtue of reflexivity of $\tilde{X}$, there exists $\tilde{x} = \{x_k\}_{k\in N} \in \tilde{X}, \tilde{x} \neq 0$ such that

$$x^* e_k^*(\tilde{x}) = 0, \forall x^* \in X^*, \forall k \in N$$

i.e. $x^*(x_k) = 0$. So, $\tilde{x} = 0$. This contradicts the assumption. Therefore, $\exists \tilde{y}^* \in L_{X^*}\left(\{e_k^*\}_{k\in N}\right)$ such that

$$\left\| \tilde{x}^* - \tilde{y}^* \right\|_{\tilde{X}^*} < \frac{\varepsilon}{M+1}. \qquad (4)$$

Using (3) and (4) we get

$$\left\| \tilde{x}^* - \sum_{k=1}^{n} x_k^* e_k^* \right\|_{\tilde{X}^*} \le \left\| \tilde{x}^* - \tilde{y}^* \right\|_{\tilde{X}^*} + \left\| \sum_{k=1}^{n} (\tilde{x}^* - \tilde{y}^*) e_k e_k^* \right\|_{\tilde{X}^*} \le$$

$$\le (1 + M) \left\| \tilde{x}^* - \tilde{y}^* \right\|_{\tilde{X}^*} < \varepsilon.$$

□

Definition 3. *The system $\{g_k\}_{k\in N}$ is said to be $\tilde{X}^*$-Riesz g-basis in Z^* if $\{g_k\}_{k\in N}$ is g-complete in Z^* and there exist constants $A > 0$ and $B > 0$ such that*

$$A \left\| \tilde{x}^* \right\|_{\tilde{X}^*} \le \left\| \sum_{k=1}^{\infty} x_k^* g_k \right\|_{Z^*} \le B \left\| \tilde{x}^* \right\|_{\tilde{X}^*}, \forall \tilde{x}^* \in \tilde{X}^*. \qquad (5)$$

Constants A and B are called lower and upper bounds of the $\tilde{X}^*$-Riesz g-basis $\{g_k\}_{k\in N}$, recpectively.

It is easily shown that the system $\{g_k\}_{k\in N}$ forms $\tilde{X}^*$-Riesz g-basis in Z^* if and only if operator $T : \tilde{X}^* \to Z^*$, determined by the formula

$$T(\tilde{x}^*) = \sum_{k=1}^{\infty} x_k^* g_k, \forall \tilde{x}^* = \{x_k^*\}_{k\in N} \in \tilde{X}^*, \tag{6}$$

is bounded and bounded invertible.

3 $\tilde{X}$-Bessel systems and $\tilde{X}$-Riesz-Fisher systems

In this section we study Bessel systems and Riesz-Fisher systems in Banach space with respect to Banach space of the sequence of vectors. We establish their characterization and also relation between the frames and Riesz bases in Banach space with respect to CB-space $\tilde{X}$. Let X and Z be Banach spaces, $\tilde{X}$ be CB-space and $\{g_k\}_{k\in N} \subset L(Z, X)$.

The following notion is the generalization of Bessel sequence with respect to Banach space of scalars.

Definition 4. *The system $\{g_k\}_{k\in N}$ is said to be $\tilde{X}$-Bessel in Z if*

$$\{g_k(z)\}_{k\in N} \in \tilde{X}, \forall z \in Z.$$

We have the following criteria of $\tilde{X}$-Bessel property of the system.

Theorem 2. *The system $\{g_k\}_{k\in N}$ is $\tilde{X}$-Bessel in Z if and only if there exists an operator $U \in L(Z, X)$ such that*

$$e_n^* U = g_n, \forall n \in N. \tag{7}$$

Proof. Necessity. Let the system $\{g_k\}_{k\in N}$ be $\tilde{X}$-Bessel in Z. Let us consider operator $U_n : Z \to \tilde{X}, \forall n \in N$ defined by the formula

$$U_n(z) = \sum_{k=1}^{n} e_n g_k(z).$$

It is clear that $U_n \in L(Z, X)$. By condition, $\forall z \in Z$ relation $\{g_k(z)\}_{k\in N} \in \tilde{X}$ holds. So,

$$\lim_{n\to\infty} U_n(z) = U(z).$$

By the uniform boundedness principle, the sequence $\{\|U_k\|\}_{k\in N}$ is bounded. Therefore, $U \in L(Z, X)$. Furthermore

$$e_n^* U(z) = e_n^* \left(\{g_k(z)\}_{k\in N}\right) = g_n(z),$$

i.e. (7) holds.

Sufficiency. Let there exist an operator $U \in L(Z, \tilde{X})$ and let condition (7) be fulfilled. We take $\forall z \in Z$ and set $U(z) = \tilde{x}$. Then it follows from $g_n(z) = e_n^* U(z) = x_n$ that $\{g_k(z)\}_{k\in N} = \tilde{x}$, i.e. the system $\{g_k\}_{k\in N}$ is $\tilde{X}$-Bessel in Z. $\square$

The following statement follows from this theorem.

Corollary 1. *Let the system* $\{g_k\}_{k\in N}$ *be* $\tilde{X}$*-Bessel in* Z*. Then* $\exists B>0$*:*

$$\left\|\{g_k(z)\}_{k\in N}\right\| \leq B\,\|z\|_Z\,, z\in Z.$$

Proof. By Theorem 2 operator $U: Z\to\tilde{X}$ determined by formula (2) is bounded. Therefore $\forall z\in Z$ we have

$$\left\|\{g_k(z)\}_{k\in N}\right\| = \|U(z)\|_{\tilde{X}} \leq \|U\|\,\|z\|_Z\,.$$

Thus, the required number satisfies condition $\|U\|\leq B$. □

Theorem 3. *In order for the system* $\{g_k\}_{k\in N}$ *to be* $\tilde{X}$*-Bessel in* Z *it is necessary, and in the case of reflexivity of* $\tilde{X}$*, it is also sufficient that the following relation*

$$\left\|\sum_k x_k^* g_k\right\|_{Z^*} \leq B\,\|\{x_k^*\}\|_{\tilde{X}^*} \tag{8}$$

holds for any finite sequence $\{x_k^*\}\subset X^*$*.*

Proof. Necessity. Let the system $\{g_k\}_{k\in N}$ be $\tilde{X}$-Bessel in Z. Let us take an arbitrary finite sequence $\{x_k^*\}\subset X^*$. Taking into account Corollary 1 we get

$$\left\|\sum_k x_k^* g_k\right\|_{Z^*} = \sup_{\|z\|=1}\left|\sum_k x_k^* g_k(z)\right| \leq$$

$$\leq \|\{x_k^*\}\|_{\tilde{X}^*}\sup_{\|z\|=1}\left\|\{g_k(z)\}_{k\in N}\right\| \leq B\,\|\{x_k^*\}\|_{\tilde{X}^*}\,,$$

i.e. (8) holds.

Sufficiency. Assume that (8) holds for any finite sequence $\{x_k^*\}\subset X^*$. Take $\forall\,\{x_k^*\}_{k\in N}\in\tilde{X}^*$. Since $\tilde{X}$ is reflexive , by Lemma 2 the expansion $\tilde{x}^* = \sum\limits_{k=1}^{\infty} x_k^* e_k^*$ holds. Therefore,

$$\left\|\sum_{k=n}^{m} x_k^* e_k^*\right\|_{\tilde{X}^*} \to 0$$

for $m>n$ and $n\to\infty$. Then $\forall z\in Z$ for $m>n$ and $n\to\infty$, using (8), we get

$$\left|\sum_{k=n}^{m} x_k^* g_k(z)\right| \leq \left\|\sum_{k=n}^{m} x_k^* g_k\right\|_{Z^*}\|z\| \leq B\left\|\sum_{k=n}^{m} x_k^* e_k^*\right\|_{\tilde{X}^*}\|z\| \to 0.$$

Thus, $\forall z\in Z$ the series $\sum\limits_{k=1}^{\infty} x_k^* g_k(z)$ converges and so, $\{g_k(z)\}_{k\in N}\in\tilde{X}$ i.e. the system $\{g_k\}_{k\in N}$ is $\tilde{X}$-Bessel in Z. □

The following definition generalizes the notion of Riesz-Fisher system (Hilbert system) in Hilbert space (see [8, 29]).

Definition 5. *The system* $\{g_k\}_{k\in N}$ *is said to be* $\tilde{X}$*-Riesz-Fisher (* $\tilde{X}$*-Hilbert) in* Z *if*

$$\forall \{x_k\}_{k\in N} \in \tilde{X}, \exists z \in Z : g_k(z) = x_k.$$

We give characterization of the $\tilde{X}$-Riesz-Fisher system.

Theorem 4. *For the system* $\{g_k\}_{k\in N}$ *to be* $\tilde{X}$*-Riesz-Fisher in* Z*, it is sufficient, and in the case of* g*-completeness of* $\{g_k\}_{k\in N}$ *in* Z^* *it is also necessary that there be* $S \in L(\tilde{X}, Z)$ *such that*

$$g_n S = e_n^*, \forall n \in N. \tag{9}$$

Proof. Sufficiency. Let $\exists S \in L(\tilde{X}, Z)$ be such that (9) holds. We take $\forall \tilde{x} \in \tilde{X}$. Let $S(\tilde{x}) = z$. According to (9) we have

$$g_n(z) = g_n S(\tilde{x}) = e_n^*(\tilde{x}) = x_n,$$

i.e. the system $\{g_k\}_{k\in N}$ is the $\tilde{X}$-Riesz-Fisher system in Z.

Necessity. Let $\{g_k\}_{k\in N}$ be g-complete in Z^* and $\tilde{X}$-Riesz-Fisher system in Z. As $\{g_k\}_{k\in N}$ is $\tilde{X}$-Riesz-Fisher system in Z for $\forall \tilde{x} \in \tilde{X}, \exists z \in Z : g_n(z) = x_n$. Such an element is unique. Indeed, let $\exists z_1 \in Z : g_n(z_1) = x_n$. Then $g_n(z - z_1) = 0$ for $\forall n \in N$. By g-completeness in Z^* of the system $\{g_k\}_{k\in N}$ we have $f(z - z_1) = 0, f \in Z^*$. This is possible only if $z = z_1$. We define operator S assuming $S(\tilde{x}) = z$. Let us show the closeness of operator S. Suppose that

$$\lim_{n\to\infty} \tilde{x}_n = \tilde{x}, S(\tilde{x}_n) = z_n$$

and

$$\lim_{n\to\infty} S(\tilde{x}_n) = z.$$

Since $\{g_k\}_{k\in N}$ is $\tilde{X}$-Riesz-Fisher system in Z, there exist $z_n \in Z$ such that $\{g_k(z_n)\}_{k\in N} = \tilde{x}_n$. Let $\tilde{x}_n = \left\{x_k^{(n)}\right\}_{k\in N}$ and $\tilde{x} = \{x_k\}_{k\in N}$. Then $\lim\limits_{n\to\infty} g_k(z_n) = \lim\limits_{n\to\infty} x_k^{(n)} = x_k$. It follows from $\lim\limits_{n\to\infty} z_n = z$ that $\lim\limits_{n\to\infty} g_k(z_n) = g_k(z)$ and therefore $\{g_k(z)\}_{k\in N} = \tilde{x}$, i.e. $S(\tilde{x}) = z$, operator S is closed. By the closed graph theorem operator S is bounded. On the other hand, we have

$$g_n S(\tilde{x}) = g_n(z) = x_n = e_n^*(\tilde{x}), \forall \tilde{x} \in \tilde{X},$$

i.e. (9) holds. □

The following properties of g-complete $\tilde{X}$-Riesz-Fisher system follows from this theorem.

Corollary 2. *Let the system* $\{g_k\}_{k\in N}$ *be* g*-complete in* Z^* *and be* $\tilde{X}$*-Riesz-Fisher system in* Z*. Then* $\exists A > 0$*:*

$$A \|z\|_Z \leq \left\|\{g_k(z)\}_{k\in N}\right\|_{\tilde{X}},$$

for any $z \in Z$ *for which* $\{g_k(z)\}_{k\in N} \in \tilde{X}$*.*

Proof. Let $z \in Z$ be such that $\{g_k(z)\}_{k\in N} \in \tilde{X}$. By Theorem 4 there exists an operator $S \in L(\tilde{X}, Z)$ such that $S(\{g_k(z)\}_{k\in N}) = z$. Then

$$\|z\|_Z = \left\|S(\{g_k(z)\}_{k\in N})\right\|_Z \le \|S\| \left\|\{g_k(z)\}_{k\in N}\right\|_{\tilde{X}} .$$

Consequently, $A \le \frac{1}{\|S\|}$ is the required constant. □

Corollary 3. *Let the system $\{g_k\}_{k\in N}$ be g-complete in Z^* and be $\tilde{X}$-Riesz-Fisher system in Z. Then the system $\{g_k\}_{k\in N}$ is g-minimal.*

Proof. Assume the contrary, i.e. let the system $\{g_k\}_{k\in N}$ be not g-minimal. Then there exist $x^* \in X^*, x^* \neq 0$, and $k_0 \in N$ such that $x^* g_{k_0} \in \overline{L_{X^*}\left(\{g_n\}_{n\neq k_0}\right)}$. We have

$$x^* g_{k_0} = \lim_{n\to\infty} \sum_{i=1, i\neq k_0}^{m_n} x_i^{(m_n)*} g_i \tag{10}$$

By Theorem 4 there exists operator $S \in L(\tilde{X}, Z)$ such that (9) holds. From (10) we get

$$x^* e_{k_0}^* = x^* g_{k_0} S = \lim_{n\to\infty} \sum_{i=1, i\neq k_0}^{m_n} x_i^{(m_n)*} g_i S = \lim_{n\to\infty} \sum_{i=1, i\neq k_0}^{m_n} x_i^{(m_n)*} e_i^* .$$

Hence, for $\forall x \in X$ we get $x^*(x) = x^* g_{k_0} S e_{k_0}(x) = 0$. Therefore, $x^* = 0$. The obtained contradiction proves the theorem. □

In the case of reflexive space we have the following criterion for $\tilde{X}$-Riesz-Fisher system.

Theorem 5. *Let $\tilde{X}$ be a reflexive CB-space and Z be reflexive. The system $\{g_k\}_{k\in N}$ is $\tilde{X}$-Riesz-Fisher system in Z satisfying condition $\forall \tilde{x} = \{x_k\}_{k\in N} \in \tilde{X}, \exists z \in Z : g_k(z) = x_k$ and*

$$A \|z\|_Z \le \left\|\{x_k\}_{k\in N}\right\|_{\tilde{X}} \tag{11}$$

if and only if condition

$$A \|\{x_k^*\}\|_{\tilde{X}^*} \le \left\|\sum_k x_k^* g_k\right\|_{Z^*} , \tag{12}$$

holds for any finite sequence $\{x_k^*\} \subset X^*$, where A is some positive constant.

Proof. Necessity. Let $\{g_k\}_{k\in N}$ be $\tilde{X}$-Riesz-Fisher system in Z and condition (11) be fulfilled. Let us take an arbitrary finite system $\{x_k^*\} \in \tilde{X}^*$. By reflexivity of $\tilde{X}$ there exists $\{x_k\}_{k\in N} \in \tilde{X}$ such that $\left\|\{x_k\}_{k\in N}\right\|_{\tilde{X}} = 1$ and $\sum_k x_k^*(x_k) = \|\{x_k^*\}\|_{\tilde{X}^*}$.

Since $\{g_k\}_{k\in N}$ is $\tilde{X}$-Riesz-Fisher system in Z, there exists $z \in Z : g_k(z) = x_k$. Then

$$\|\{x_k^*\}\|_{\tilde{X}^*} = \left|\sum_k x_k^*(x_k)\right| = \left|\sum_k x_k^* g_k(z)\right| \leq$$

$$\leq \|z\|_Z \left\|\sum_k x_k^* g_k\right\|_{Z^*} \leq \frac{1}{A}\left\|\sum_k x_k^* g_k\right\|_{Z^*}.$$

Sufficiency. Let condition (12) be fulfilled. We take $\forall \{x_k\}_{k\in N} \in \tilde{X}$. Let us determine a linear functional φ on $L_{X^*}(\{g_k\}_{k\in N})$ by the formula:

$$\varphi\left(\sum_k x_k^* g_k\right) = \sum_k x_k^*(x_k).$$

Obviously, a fact that a functional is well defined follows from inequality (12). Using (12) we get

$$\left|\varphi\left(\sum_k x_k^* g_k\right)\right| = \left|\sum_k x_k^*(x_k)\right| \leq$$

$$\leq \|\{x_k^*\}\|_{\tilde{X}^*} \left\|\{x_k\}_{k\in N}\right\|_{\tilde{X}} \leq \frac{1}{A}\left\|\{x_k\}_{k\in N}\right\|_{\tilde{X}} \left\|\sum_k x_k^* g_k\right\|_{Z^*},$$

i.e. the functional φ is bounded on $L_{X^*}(\{g_k\}_{k\in N})$ and $\|\varphi\| \leq \frac{1}{A}\left\|\{x_k\}_{k\in N}\right\|_{\tilde{X}}$. By the Hahn-Banach Theorem we continue φ to the linear continuos functional on the whole Z^* preserving the norm. We denote this functional also by φ. Thus, $\varphi \in Z^{**}$ and

$$|\varphi(f)| \leq \frac{1}{A}\left\|\{x_k\}_{k\in N}\right\|_{\tilde{X}} \|f\|_{Z^*}, f \in Z^*. \tag{13}$$

By the reflexivity of Z there exists $z \in Z$ such that $\varphi(f) = f(z), f \in Z^*$ and $\|\varphi\| = \|z\|_Z$. It is clear that $\forall x^* \in X^*$

$$x^* g_k(z) = \varphi(x^* g_k) = x^*(x_k).$$

Hence, from the arbitrariness of $x^* \in X^*$ we get $g_k(z) = x_k$. Consequently, $\{g_k\}_{k\in N}$ is the $\tilde{X}$-Riesz-Fisher system in Z. Further, from (13) we have

$$\|z\|_Z = \|\varphi\| \leq \frac{1}{A}\left\|\{x_k\}_{k\in N}\right\|_{\tilde{X}}.$$

□

The following theorem establishes the relation between the $\tilde{X}$-Bessel system and $\tilde{X}$-Riesz-Fisher system.

Theorem 6. *Let $\tilde{X}$ be a reflexive CB-space and Z be reflexive. For the system $\{g_k\}_{k\in N}$ to be the $\tilde{X}$-Riesz-Fisher system in Z, it is sufficient, and in the case of g-completeness of $\{g_k\}_{k\in N}$ in Z^* it is also necessary that the following conditions hold:*

1) $\{g_k\}_{k\in N}$ has the g-biorthogonal system $\{\Lambda_k\}_{k\in N} \subset L(X, Z)$;

2) the system $\{\Lambda_k\}_{k\in N}$ is $\tilde{X}^*$-Bessel in Z^*.

Proof. Sufficiency. Let the system $\{g_k\}_{k\in N}$ be g-complete in Z^* and conditions 1) and 2) be fulfilled. We take $\forall \{x_k\}_{k\in N} \in \tilde{X}$. From $\tilde{X}^*$-Bessel property in Z^* of the system $\{\Lambda_k\}_{k\in N}$ it follows by Theorem 1 that the operator $V : Z^* \to \tilde{X}^*$ determined by the formula $V(f) = \left\{\Lambda_j^*(f)\right\}_{j\in N}$ is bounded. Thus, for $\forall x^* \in X^*$ we have

$$V(f)(\tilde{x}) = \sum_{j=1}^{\infty} \Lambda_j^*(f)(x_j) = \sum_{j=1}^{\infty} f\Lambda_j(x_j).$$

Then

$$V(x^* g_n)(\tilde{x}) = \sum_{j=1}^{\infty} x^* g_n \Lambda_j(x_j) = x^*(x_n).$$

On the other hand, we have

$$V(x^* g_n)(\tilde{x}) = \pi_{\tilde{X}}(\tilde{x})(V(x^* g_n)) =$$
$$= V^*(\pi_{\tilde{X}}(\tilde{x}))(x^* g_n) = x^* g_n(\pi_Z^{-1} V^* \pi_{\tilde{X}}(\tilde{x})).$$

So, from the obtained relations we have $x^*(x_n) = x^* g_n(\pi_Z^{-1} V^* \pi_{\tilde{X}}(\tilde{x}))$. Hence, by the arbitrariness of $x^* \in X^*$ we have

$$g_n(\pi_Z^{-1} V^* \pi_{\tilde{X}}(\tilde{x})) = x_n.$$

Assume that $\pi_Z^{-1} V^* \pi_{\tilde{X}}(\tilde{x}) = z$. Then $g_n(z) = x_n, \forall n \in N$, i.e. $\{g_k\}_{k\in N}$ is the $\tilde{X}$-Riesz-Fisher system in Z.

Necessity. Let $\{g_k\}_{k\in N}$ be g-complete in Z^* and be the $\tilde{X}$-Riesz-Fisher system in Z. By Theorem 4 there exists an operator $S \in L(\tilde{X}, Z)$ satisfying conditions $g_n S = e_n^*, \forall n \in N$. Assume that

$$\Lambda_j = S e_j, j \in N.$$

Then $\forall n, j \in N$ we get

$$g_n \Lambda_j = g_n S e_j = e_n^* e_j = \delta_{nj},$$

i.e. $\{g_k\}_{k\in N}$ has g-biorthogonal system $\{\Lambda_k\}_{k\in N} \subset L(X, Z)$.

Furter, $\forall \tilde{x} = \{x_k\}_{k\in N} \in \tilde{X}$ we have

$$S(\tilde{x}) = S(\sum_{j=1}^{\infty} e_j(x_j)) = \sum_{j=1}^{\infty} S e_j(x_j) = \sum_{j=1}^{\infty} \Lambda_j(x_j).$$

Consequently, $\forall f \in Z^*$

$$f S(\tilde{x}) = \sum_{j=1}^{\infty} f \Lambda_j(x_j) = \sum_{j=1}^{\infty} \Lambda_j^*(f)(x_j).$$

Thus, for $\forall \tilde{x} = \{x_k\}_{k\in N} \in \tilde{X}$ the series $\sum\limits_{j=1}^{\infty} \Lambda_j^*(f)(x_j)$ converges, and so $\left\{\Lambda_j^*(f)\right\}_{j\in N} \in \tilde{X}^*$, i.e. the system $\{\Lambda_k\}_{k\in N}$ is $\tilde{X}^*$-Bessel in Z^*. □

The following theorem is a criterion for $\tilde{X}^*$-Riesz g-bases property of systems.

Theorem 7. *Let $\tilde{X}$ be a reflexive CB-space and Z be reflexive. For the system $\{g_k\}_{k\in N}$ to be the $\tilde{X}^*$-Riesz g-bases in Z^*, it is necessary and sufficient that $\{g_k\}_{k\in N}$ be both $\tilde{X}$-Bessel system and $\tilde{X}$-Riesz-Fisher system in Z satisfying the inequality*

$$A\left\|z\right\|_Z \leq \left\|\{g_k(z)\}_{k\in N}\right\|_{\tilde{X}}, z \in Z, \tag{14}$$

where $A > 0$ is some constant.

Proof. Necessity. Let $\{g_k\}_{k\in N}$ be $\tilde{X}^*$-Riesz g-bases in Z^*. According to Theorem 3 and Theorem 5, $\{g_k\}_{k\in N}$ is $\tilde{X}$-Bessel system and $\tilde{X}$-Riesz-Fisher system in Z. The validity of the left hand side of inequality 9 remains to be shown. We take $\forall z \in Z$. By $\tilde{X}$-Bessel property of the system $\{g_k\}_{k\in N}$ we have $\{g_k(z)\}_{k\in N} \in \tilde{X}$. Then by condition 11 there exists $z_0 \in Z$ such that $g_k(z_0) = g_k(z)$ and $A\left\|z_0\right\| \leq \left\|\{g_k(z_0)\}_{k\in N}\right\|_{\tilde{X}}$. Consequently, $g_k(z_0-z) = 0$ and therefore $\forall \tilde{x}^* = \{x_k^*\}_{k\in N} \in \tilde{X}^*$ the relation $\sum\limits_k x_k^* g_k(z_0 - z) = 0$ holds. Then from g-completeness of system $\{g_k\}_{k\in N}$ in Z^* we have $f(z_0 - z) = 0$, $\forall f \in Z^*$ and hence $z = z_0$.

Sufficiency. Let $\{g_k\}_{k\in N}$ be $\tilde{X}$-Bessel system and $\tilde{X}$-Riesz-Fisher system in Z and the left hand side of inequality (1) be fulfilled. According to Theorem 3 and Theorem 5, (5) holds. It remains to show the completeness of $\{g_k\}_{k\in N}$ in Z^*. Assume that $\{g_k\}_{k\in N}$ is not g-complete in Z^*. By virtue of reflexivity of Z,

$$\exists z \in Z, z \neq 0 : \sum_k x_k^* g_k(z) = 0, x_k^* \in X^*.$$

Then for an arbitrary $\tilde{x}^* = \{x_k^*\}_{k\in N} \in \tilde{X}^*$ we get $\tilde{x}^*(\{g_k(z)\}_{k\in N}) = 0$. Hence $g_k(z) = 0$ and so from (14) we have $z = 0$. The obtained contradiction proves the theorem. □

The following theorem establishes the relation between $\tilde{X}^*$-Riesz g-basicity and $\tilde{X}$-frame property of a system.

Theorem 8. *Let $\tilde{X}$ be a reflexive CB-space and Z be reflexive. The system $\{g_k\}_{k\in N}$ forms $\tilde{X}^*$-Riesz g-basis in Z^* with bounds A and B if and only if*

1) $\{g_k\}_{k\in N}$ is $\tilde{X}$-frame in Z with the bounds A and B;

2) $\{g_k\}_{k\in N}$ is g-minimal in Z^*.

Proof. The necessity of the theorem follows immediately from Theorem 7 and corollaries 1, 2 and 3.

Sufficiency. Let conditions 1) and 2) be fulfilled. Then operator U determined by formula (2) is invertible on ImU and by Theorem 1 operator U^* is surjection in $\tilde{X}^*$. Let us consider operator T determined by formula (6). It is easy to show that $U^* = T$. Therefore, T is a surjection in $\tilde{X}^*$. Let us find $KerT$. Assume that $T\tilde{x}^* = \sum\limits_{k=1}^{\infty} x_k^* g_k = 0, \tilde{x}^* = \{x_k^*\}_{k\in N}$. If $\exists k_0 : x_{k_0}^* \neq 0$, then $x_{k_0}^* g_{k_0} = -\sum\limits_{k=1, k\neq k_0}^{\infty} x_k^* g_k$ and this contradicts g-minimality in Z^* of the system $\{g_k\}_{k\in N}$. So $\forall k$ we have $x_k^* = 0$, i.e. $KerT = \{0\}$ and therefore T is bounded invertible. Consequently, the system $\{g_k\}_{k\in N}$ forms $\tilde{X}^*$-Riesz g-basis in Z^*. □

The following theorem for $\tilde{X}$-Riesz Λ-basis is proved in the same way.

Theorem 9. *Let $\tilde{X}$ be a reflexive CB-space. The system $\{\Lambda_k\}_{k\in N}$ forms $\tilde{X}$-Riesz Λ-basis in Z with bounds A and B if and only if*
1) $\{\Lambda_k\}_{k\in N}$ is $\tilde{X}^*$-frame in Z^* with bounds A and B;
2) $\{\Lambda_k\}_{k\in N}$ is Λ-minimal in Z.

We now study expansion in spaces Z and Z^*.

Theorem 10. *Let $\left(\{g_k\}_{k\in N}, S\right)$ be a Banach $\tilde{X}$-frame in Z with bounds A and B. Then there exists $\tilde{X}^*$-frame in Z^* $\{\Lambda_k\}_{k\in N} \subset L(X,Z)$ with bounds $\frac{1}{B}$ and $\frac{1}{A}$ such that*

$$z = \sum_{k=1}^{\infty} \Lambda_k g_k(z), \forall z \in Z, \tag{15}$$

$$f = \sum_{k=1}^{\infty} \Lambda_k^*(f) g_k, \forall f \in Z^*. \tag{16}$$

Proof. Determine the operator $\Lambda_k \in L(X,Z)$ by the formula $\Lambda_k = Se_k$. Then $\forall z \in Z$ we have

$$z = SU(z) = S(\sum_{k=1}^{\infty} e_k g_k(z)) = \sum_{k=1}^{\infty} Se_k g_k(z) = \sum_{k=1}^{\infty} \Lambda_k g_k(z).$$

Equality (16) follows immediately from (15). Further we have

$$S^*(f) = \{\Lambda_k^*(f)\}_{k\in N}, \forall f \in Z^*. \tag{17}$$

Indeed, $\forall \tilde{x} = \{x_n\}_{n\in N} \in \tilde{X}$ we get

$$S^*(f)(\tilde{x}) = f(S(\tilde{x})) = f(\sum_{k=1}^{\infty} Se_k(x_k)) = \sum_{k=1}^{\infty} f(\Lambda_k(x_k)) = \sum_{k=}^{\infty} \Lambda_k^*(f)(x_k).$$

Then from (17) we have

$$\left\| \{\Lambda_k^*(f)\}_{k\in N} \right\|_{\tilde{X}^*} = \|S^* f\|_{\tilde{X}^*} \le \|S\| \, \|f\|_{Z^*} = \frac{1}{A} \|f\|_{Z^*} .$$

Using (15) we get

$$\|f\|_{Z^*} = \sup_{\|z\|=1} |f(z)| = \sup_{\|z\|=1} \left| \sum_{k=1}^{\infty} f(\Lambda_k(g_k(z))) \right| =$$

$$= \sup_{\|z\|=1} \left| \sum_{k=1}^{\infty} \Lambda_k^*(f)(g_k(z)) \right| \leq \left\| \{\Lambda_k^*(f)\}_{k\in N} \right\|_{\tilde{X}^*} \sup_{\|z\|=1} \left\| \{g_n(z)\}_{n\in N} \right\|_{\tilde{X}} \leq$$

$$\leq B \left\| \{\Lambda_k^*(f)\}_{k\in N} \right\|_{\tilde{X}^*},$$

i.e. $\frac{1}{B}\|f\| \leq \left\| \{\Lambda_k^*(f)\}_{k\in N} \right\|$. □

In the case of $\tilde{X}$-Riesz Λ-basis, the following theorem is valid.

Theorem 11. *Let $\tilde{X}$ be a reflexive CB-space and Z be reflexive. Suppose that $\{g_k\}_{k\in N}$ forms $\tilde{X}^*$-Riesz g-basis in Z^* with bounds A and B. Then there exist a unique $\tilde{X}$-Riesz Λ-basis $\{\Lambda_k\}_{k\in N} \subset L(X,Z)$ in Z with bounds $\frac{1}{B}$ and $\frac{1}{A}$ such that equalities (15) and (16) hold. Furthermore, $\{\Lambda_k\}_{k\in N}$ is a unique system g-biorthogonal to $\{g_k\}_{k\in N}$.*

Proof. By Theorem 8 the system $\{g_k\}_{k\in N}$ is $\tilde{X}$-frame in Z with bounds A and B. It is clear that operator U is bounded invertible and $\left\|U^{-1}\right\| = \frac{1}{A}$. Thus, $\left(\{g_k\}_{k\in N}, U^{-1}\right)$ is a Banach $\tilde{X}$-frame in Z. By Theorem 10 the system $\{\Lambda_k\}_{k\in N} \subset L(X,Z), \Lambda_k = U^{-1}e_k$ forms $\tilde{X}^*$-frame in Z^* with bounds $\frac{1}{B}$, $\frac{1}{A}$ and equalities (15) and (16) are fulfilled. It is clear that $g_k\Lambda_j = \delta_{kj}$. Indeed, if $\Lambda_j(x) = z$, then $e_j(x) = U(z)$, and so $g_k(z) = \delta_{kj}x$. The uniqueness of g-biorthogonal system $\{\Lambda_k\}_{k\in N}$ follows from expansion (10). Let us show Λ-minimality of the system $\{\Lambda_k\}_{k\in N}$. Assume the contrary, let $\{\Lambda_k\}_{k\in N}$ be not Λ-minimal in Z. Then $\exists x \in X, x \neq 0, \exists k_0 : \Lambda_{k_0}(x) = \overline{L_X\left(\{\Lambda_j\}_{j\neq k_0}\right)}$, i.e.

$$\Lambda_{k_0}(x) = \lim_{n\to\infty} \sum_{j=1, j\neq k_0}^{m_n} \Lambda_j(x_j^{(m_n)}).$$

Hence, we have

$$x = g_{k_0}(\Lambda_{k_0}(x)) = \lim_{n\to\infty} \sum_{j=1, k\neq k_0}^{m_n} g_{k_0}(\Lambda_j(x_j^{(m_n)})) = 0,$$

that contradicts the assumption $x \neq 0$. Therefore, by Theorem 9 the system $\{\Lambda_k\}_{k\in N}$ forms $\tilde{X}$-Riesz Λ-basis in Z. □

References

[1] Duffin R. J., Schaeffer A. C., *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341-366.

[2] Astafieva N.M. *Wavelet-analysis: fundamentals of theory and examples of the application*, The Successes of the Physical Sciences, **166** (1996), no. 1, 1-26.

[3] Dremin, I.M., Ivanov, O.V., Nechitailo, V.A., *Wavelets and their uses*, Usp. Phisics Nauk., **171** (2001), no. 6, 465-501.

[4] Chui, C., *Wavelets: A Tutorial in Theory and Applications.* Academic Press, Boston, 1992.

[5] Meyer S., *Wavelets and operators*, Herman, Paris, 1990.

[6] Daubechies, I., *Ten Lectures on Wavelets*, SIAM, Philadelphia, 1992.

[7] Malat S., *A wavelet tour of signal processing*, Academic Press, San Diego, 1999.

[8] Bari N. K., *Biorthogonal systems and bases in Hilbert space*, (Russian), Ucheniye Zapiski MGU, **158** (1951, no. 4, 69-107.

[9] Balan, R., *Stability theorems for Fourier frames and wavelet Riesz bases*, J. Fourier Anal. Appl., **3** (1997), no. 5, 499-504.

[10] Casazza P.G., *The art of frame theory*, Taiwanese J. Math., **4** (2000), no. 2, 129-201.

[11] Christensen O., *An Introduction to frames and Riesz bases*, Appl. Numer. Harmon. Anal., Birkhuser, Boston, MA, 2002.

[12] Sun W., *G-frames and G-Riesz bases*, J. Math. Anal. Appl., **322** (2006), 437-452.

[13] Sun W., *Stability of G-frames*, J. Math. Anal. Appl., **326** (2007), 858-868.

[14] Najati A., M. H. Faroughi and A. Rahimi, *G-frames and stability g-frames in Hilbert space*, Methods of Functional Analisis and Topology, (2008), 271-286.

[15] Bilalov B.T., Guliyeva F.A., *t-Frames and their Noetherian Perturbation*, Complex Anal. Oper. Theory, **8** (2014), no. 7, 1405-1418.

[16] Ismailov M.I., Guliyeva F.A., Nasibov Y.I., *On a generalization of the Hilbert frame generated by the bilinear mapping*, Jornal of Function Spaces, (2016), 1-8.

[17] Gröchenig K., *Describing functions: atomic decomposition versus frames*, Monatsh. Math. **112** (1991), no. 1, 1 - 41.

[18] Christensen O., Heil N., *Perturbations of Frames and Atomic Decompositions*, Math. Nachr. **185** (1997), 33-47.

[19] Casazza P.G., Han D. and Larson D.R., *Frames for Banach space*, Contemp. Math. **247** (1999), 149-182.

[20] Casazza P.G. and Christensen O., *Perturbation of operators and applications to frame theory*, J. Math. Anal. Appl. **307** (2005), 710-723.

[21] Aldroubi A., Sun Q. and Tang W., *p-frames and shift invariant subspaces of L_p*, J. Fourier Anal. Appl. **7** (2001), 1-21.

[22] Christensen O., Stoeva D.T., *p-frames in separable Banach spaces*, Adv. Comp. Math. **18** (2003), 117-126.

[23] Casazza P.G., Christensen O. and Stoeva D.T., *Frame expansions in separable Banach spaces*, J. Math. Anal. Appl. **307** (2005), no. 2, 710-723.

[24] Stoeva D.T., *X_d-frames in Banach spaces and their duals*, Int. J. Pure Appl. Math. **52** (2009), no. 1, 1-14.

[25] Abdollahpour M. R., Faroughi M. H. and Rahimi A. *PG-frames in Banach spaces*, Methods of functional Analysis and Topology, **13** (2007), no. 3, 201-210.

[26] Ismailov M.I., Jabrailova A. N., *On $\tilde{X}$-frames and conjugate systems in Banach spaces*, Sahand Communications in Mathematical Analysis (SCMA) (2014), 19-29.

[27] Ismailov M.I. and Nasibov Y.I., *On One generalization of Banach frame*, Azerbaijan Journal of Mathematics, **6** (2016), no. 2, 143-159.

[28] Rudin, W., *Functional Analysis*, Mc Graw-Hill, New York, 1974.

[29] Young R. *An introduction to nonharmonic Fourier series*, New York, 1980.