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FIXED POINT THEOREMS EXTENDED TO SPACES WITH TWO METRICS

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Abstract

In this paper, we obtain generalizations on some classical fixed point theorems which will be defined in spaces that have two metrics. We will, also, obtain some methods of construction of the majorant metric.

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1 Introduction

In this section we will present some classical fixed point theorems.

Theorem 1 (Kannan)**.** *Let (X, d) be a complete metric space and T be a mapping of X into X. If there exists $a \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq a \cdot [d(x, Tx) + d(y, Ty)],\ \forall x, y \in X,$$

then T has a unique fixed point in X.

Theorem 2 (Reich)**.** *Let (X, d) be a complete metric space and T be a mapping of X into X such that*

$$d(Tx, Ty) \leq a \cdot d(x, Tx) + b \cdot d(y, Ty) + c \cdot d(x, y),\ \forall x, y \in X,$$

where $a, b, c \geq 0$ and $a + b + c < 1$. Then T has a unique fixed point in X.

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Definition 1. *Let (X, d) be a metric space. A mapping $T : X \to X$ is called asymptotically regular if*

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0,\ \forall x \in X,$$

where $T^n x$ are elements of Picard iteration $T^n x$ which is defined as follows: for $x_0 \in X$,

$$T^0 x_0 = x_0 \text{ and } T^n x_0 = T(T^{n-1}x_0).$$

The following theorems are consequences of Reich theorem.

Theorem 3. *Let (X, d) be a complete metric space and T be an asymptotically regular mapping of X into X. If there exists $a \in (0, 1)$ such that*

$$d(Tx, Ty) \leq a \cdot [d(x, Tx) + d(y, Ty)],\ \forall x, y \in X,$$

then T has a unique fixed point in X.

Theorem 4. *Let (X, d) be a complete metric space and T be an asymptotically regular mapping of X into X. If there exists $M < 1$ such that*

$$d(Tx, Ty) \leq M \cdot [d(x, Tx) + d(y, Ty) + d(x, y)],\ \forall x, y \in X,$$

then T has a unique fixed point in X.

Another important fixed point theorem is:

Theorem 5 (Chatterjea)**.** *Let (X, d) be a complete metric space and T be a mapping of X into X. If there exists $b \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq b[\cdot(x, Ty) + d(y, Tx)],\ \forall x, y \in X,$$

then T has a unique fixed point in X.

Definition 2. *Let (X, d) be a metric space. A mapping $T : X \to X$ is called contraction if there exists $c \in (0, 1)$ such that*

$$d(Tx, Ty) \leq c \cdot d(x, y),\ \forall x, y \in X.$$

The following theorem states the conditions for a contraction to have a unique fixed point:

Theorem 6. *(Banach) Let (X, d) be a complete metric space and T a mapping from X to X and a contraction with respect to d. Then T has a unique fixed point.*

Maria Grazia Maia [4] obtained a generalization of Banach's theorem:

Theorem 7. *Let (X, d, δ) be a space with two metrics such that $d(x, y) \leq \delta(x, y)$ for all $x, y \in X$, let (X, d) be a complete metric space and T a continuous mapping with respect to d and a contraction with respect to δ. Then T has a unique fixed point in X.*

In the following theorem M.G. Maia [4] obtained a method of construction for the majorant metric.

Theorem 8. *Let (X, d) be a metric space and T be a mapping of X into X. Let*

$$\sum_{0}^{\infty} \lambda^n d(T^n x, T^n y) \tag{1}$$

be a power series. Assuming that for a certain $\lambda > 1$ series (1) is converging for all $x, y \in X$ then δ is defined as follows:

$$\delta(x, y) = \sum_{0}^{\infty} \lambda^n d(T^n x, T^n y).$$

With the previous construction:

1. *δ is a metric on X and $d(x, y) \leq \delta(x, y)$, $\forall x, y\ inX$;*
2. *T is a contraction with respect to δ.*

2 Main results

In this section we will give generalizations of the previously mentioned results.

Theorem 9. *Let (X, d, δ) be a space with two metrics such that $d(x, y) \leq \delta(x, y)$ for all $x, y \in X$. Let X be a complete space with respect to metric d and suppose there exists $a \in [0, \frac{1}{2})$ and T a continuous mapping of X into X with respect to the metric d such that:*

$$\delta(Tx, Ty) \leq a[\delta(x, Tx) + \delta(y, Ty)]. \tag{2}$$

Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be a point and we define the sequence $\{x_n\}_{n \geq 1}$ as follows: $x_n = T^n x_0$ – the Picard iteration. By applying (2) with $x = x_{n-1}$ and $y = x_n$ we get: $\delta(Tx_{n-1}, Tx_n) = \delta(x_n, x_{n+1}) \leq a[\delta(x_{n-1}, x_n) + \delta(x_n, x_{n+1})]$. Then we have:

$$\delta(x_n, x_{n+1}) \leq \frac{a}{1-a} \delta(x_{n-1}, x_n), \tag{3}$$

where by supposing $\frac{a}{1-a} < 1$ we obtain condition $a < \frac{1}{2}$. From (3) we have:

$$\delta(x_n, x_{n+1}) \leq \frac{a}{1-a} \delta(x_{n-1}, x_n) \leq \cdots \leq \left(\frac{a}{1-a}\right)^n \delta(x_0, x_1).$$

Therefore, $\{x_n\}$ is a Cauchy sequence with respect to δ. But δ is a majorant metric for d so $\{x_n\}$ is a Cauchy sequence with respect to d and by completeness of X with respect to d we have that $\{x_n\}$ is a converging sequence. Now suppose

$$\lim_{n \to \infty} x_n = x^*,$$

we prove that x^* is a fixed point for T:

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n) = Tx^*.$$

Uniqueness: Suppose there are two fixed points x^* and y^*. By applying (2) we get:

$$\delta(Tx^*, Ty^*) \leq a[\delta(x^*, Tx^*) + \delta(y^*, Ty^*)],$$

which is

$$\delta(x^*, y^*) \leq 0,$$

so $\delta(x^*, y^*) = 0$ which implies $x^* = y^*$. □

Remark 1. *It is not necessary for T to be a continuous mapping.*

Proof. By applying (2) with $x = x^*$ and $y = x_n$:

$$\delta(Tx^*, Tx_n) \leq a[\delta(x^*, Tx^*) + \delta(x_n, Tx_n)],$$

$$\Leftrightarrow \delta(Tx^*, x_{n+1}) \leq a[\delta(x^*, Tx^*) + \delta(x_n, x_{n+1})].$$

For $n \to \infty$ we get:

$$\delta(Tx^*, x^*) \leq a\delta(x^*, Tx^*),$$

and because $a < \frac{1}{2}$ we have $\delta(Tx^*, x^*) = 0$ so $x^* = Tx^*$ which means x^* is a fixed point. □

Theorem 10. *Let (X, d, δ) be a space with two metrics such that $d(x, y) \leq \delta(x, y)$ for all $x, y \in X$. Let X be a complete space with respect to metric d and T a continuous mapping of X into X with respect to metric d such that:*

$$\delta(Tx, Ty) \leq a\delta(x, Tx) + b\delta(y, Ty) + c\delta(x, y), \tag{4}$$

where $a, b, c \geq 0$ and $a + b + c < 1$. Then T has a unique fixed point in X.

Proof. Because of the symmetry of δ we have:

$$\delta(Tx, Ty) \leq a\delta(x, Tx) + b\delta(y, Ty) + c\delta(x, y),$$

$$\delta(Ty, Tx) \leq a\delta(Tx, x) + b\delta(Ty, y) + c\delta(y, x),$$

and by summing these relations we get:

$$\delta(Tx, Ty) \leq \frac{a+b}{2}\delta(x, Tx) + \frac{a+b}{2}\delta(y, Ty) + c\delta(x, y),$$

so we can choose $a = b$, which means that relation (4) becomes:

$$\delta(Tx, Ty) \leq a[\delta(x, Tx) + \delta(y, Ty)] + c\delta(x, y),$$

with $2a + c < 1$. Further, we consider $x_0 \in X$ a point and we define the sequence $\{x_n\}_{n\geq 1}$ as follows:

$$x_n = T^n x_0 - \text{the Picard iteration.}$$

By applying (4) with $x = x_{n-1}$ and $y = x_n$ we get:

$$\delta(Tx_{n-1}, Tx_n) = \delta(x_n, x_{n+1}) \le a[\delta(x_{n-1}, x_n) + \delta(x_n, Tx_{n+1})] + c\delta(x_{n-1}, x_n).$$

Then we have:

$$\delta(x_n, x_{n+1}) \le \frac{c+a}{1-a}\delta(x_{n-1}, x_n), \tag{5}$$

where $\frac{c+a}{1-a} < 1$ because $2a + c < 1$. From (5) we obtain:

$$\delta(x_n, x_{n+1}) \le \frac{c+a}{1-a}\delta(x_{n-1}, x_n) \le \cdots \le \left(\frac{c+a}{1-a}\right)^n \delta(x_0, x_1),$$

which means that $\{x_n\}$ is a Cauchy sequence with respect to δ. But δ is a majorant metric for d so $\{x_n\}$ is a Cauchy sequence with respect to d and by completeness of the space (X, d) we get that $\{x_n\}$ is a converging sequence. Now supposing

$$\lim_{n\to\infty} x_n = x^*,$$

we will prove that x^* is a fixed point for T:

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n) = Tx^*.$$

Uniqueness. Let x^*, y^* be two fixed points. Applying (4) with $x = x^*$ and $y = y^*$ we get:

$$\delta(Tx^*, Ty^*) \le a[\delta(x^*, Tx^*) + \delta(y^*, Ty^*)] + c\delta(x^*, y^*),$$

which is

$$\delta(x^*, y^*) \le c\delta(x^*, y^*),$$

where $c < 1$. Therefore $\delta(x^*, y^*) = 0$, which means $x^* = y^*$. □

Remark 2. *It is not necessarily for T to be a continuous mapping. The proof is similar to the one in Remark* 1.

In the following result we use the concept of asymptotically regular mapping.

Theorem 11. *Let (X, d, δ) be a space with two metrics such that $d(x, y) \le \delta(x, y)$ for all $x, y \in X$ and (X, d) is a complete metric space. Let T be an asymptotically regular continuous mapping of X into X with respect to metric d. Assuming there exists $a \in (0, 1)$ such that:*

$$\delta(Tx, Ty) \le a[\delta(Tx, x) + \delta(Ty, y)], \ \forall x, y \in X. \tag{6}$$

Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be a point, we define the sequence $\{x_n\}_{n\ge1}$ as follows:

$$x_n = T^n x_0 - \text{the Picard iteration.}$$

By applying (6) with $x = x_{n-1}$ and $y = x_{n+p-1}$ we get:

$$\delta(Tx_{n-1}, Tx_{n+p-1}) = \delta(x_n, x_{n+p}) \leq a[\delta(x_{n-1}, x_n) + \delta(x_{n+p-1}, x_{n+p})] =$$
$$a[\delta(T^{n-1}x_0, T^n x_0) + \delta(T^{n+p-1}x_0, T^{n+p}x_0)].$$

Using the hypothesis that T is an asymptotically regular mapping we get:

$$\delta(x_n, x_{n+p}) \to 0. \tag{7}$$

Therefore, we have that $\{x_n\}$ is a Cauchy sequence with respect to δ, and because d is majorated by δ, then $\{x_n\}$ is a Cauchy sequence with respect to d and by completeness of the space (X, d) we get that $\{x_n\}$ is a converging sequence, and let x^* be its converging point.

Now we prove that x^* is a fixed point for T:

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n) = Tx^*.$$

Uniqueness. Suppose there are two fixed points x^* and y^*. By applying (6) we get:

$$\delta(Tx^*, Ty^*) \leq a[\delta(x^*, Tx^*) + \delta(y^*, Ty^*)],$$

which is

$$\delta(x^*, y^*) \leq 0,$$

so $\delta(x^*, y^*) = 0$ which implies $x^* = y^*$. □

Remark 3. *It is not necessary for T to be a continuous mapping. The proof is similar to the one in Remark* 1.

Theorem 12. *Let (X, d, δ) be a space with two metrics such that $d(x, y) \leq \delta(x, y)$ for all $x, y \in X$. Let X be a complete space with respect to metric d and T a continuous mapping with respect to metric d and an asymptotically regular mapping of X into X with respect to metric δ such that:*

$$\delta(Tx, Ty) \leq M[\delta(x, Tx) + \delta(y, Ty) + \delta(x, y)], \tag{8}$$

where $M < 1$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be a point, we define the sequence $\{x_n\}_{n\geq 1}$ as follows: $x_n = T^n x_0$ – the Picard iteration. By applying (6) with $x = x_{n-1}$ and $y = x_{n+p-1}$ we get: $\delta(Tx_{n-1}, Tx_{n+p-1}) = \delta(x_n, x_{n+p}) \leq M[\delta(x_{n-1}, x_n) + \delta(x_{n+p-1}, x_{n+p}) + \delta(x_{n-1}, x_{n+p-1})]$. Using the triangle inequality we get: $\delta(x_n, x_{n+p}) \leq M[\delta(x_{n-1}, x_n) + \delta(x_{n+p-1}, x_{n+p}) + \delta(x_{n-1}, x_n) + \delta(x_n, x_{n+p}) + \delta(x_{n+p}, x_{n+p-1})]$, which is

$$\delta(x_n, x_{n+p}) \leq \frac{M}{1-M}[\delta(x_{n-1}, x_n) + \delta(x_{n+p-1}, x_{n+p}) + \delta(x_{n-1}, x_n) + \delta(x_{n+p}, x_{n+p-1})].$$

Now using the hypothesis that T is an asymptotically regular mapping and if $n \to \infty$ we get:

$$\delta(x_n, x_{n+p}) \to 0. \tag{9}$$

By (9) we have that $\{x_n\}$ is a Cauchy sequence with respect to δ, and because d is majorated by δ, then $\{x_n\}$ is a Cauchy sequence with respect to d and by completeness of space (X, d) we get that $\{x_n\}$ is a converging sequence, and let x^* be its converging point.

Now we prove that x^* is a fixed point for T:

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n) = Tx^*.$$

Uniqueness. Suppose there are two fixed points x^* and y^*. By applying (6) we get:

$$\delta(Tx^*, Ty^*) \le M[\delta(x^*, Tx^*) + \delta(y^*, Ty^*) + \delta(x^*, y^*)],$$

which is

$$\delta(x^*, y^*) \le M\delta(x^*, y^*).$$

Because $M < 1$ we have $\delta(x^*, y^*) = 0$ which implies $x^* = y^*$. □

Remark 4. *It is not necessary for T to be a continuous mapping. The proof is similar to the one in Remark 1.*

The following result is a generalization of Chatterjea theorem.

Theorem 13. *Let (X, d, δ) be a space with two metrics such that $d(x, y) \le \delta(x, y)$ for all $x, y \in X$ and (X, d) is a complete metric space. Let T be a continuous mapping of X into X with respect to metric d. Assuming there exists $b \in (0, \frac{1}{2})$ such that:*

$$\delta(Tx, Ty) \le b[\delta(x, Ty) + \delta(y, Tx)]. \tag{10}$$

Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be a point, we define the sequence $\{x_n\}_{n\ge1}$ as follows:

$$x_n = T^n x_0 - \text{the Picard iteration.}$$

By applying (10) with $x = x_{n-1}$ and $y = x_n$ we get:

$$\delta(Tx_{n-1}, Tx_n) = \delta(x_n, x_{n+1}) \le b[\delta(x_{n-1}, Tx_n) + \delta(x_n, Tx_{n-1})],$$

which is

$$\delta(x_n, x_{n+1}) \le \frac{b}{1-b}\delta(x_{n-1}, x_n),$$

where $\frac{b}{1-b} < 1$. We obtain:

$$\delta(x_n, x_{n+1}) \le \frac{b}{1-b}\delta(x_{n-1}, x_n) \le \cdots \le \left(\frac{b}{1-b}\right)^n \delta(x_1, x_0),$$

which implies that $\{x_n\}$ is a Cauchy sequence with respect to δ. But, d is majorated by δ and (X, d) is a complete metric space, therefore $\{x_n\}$ is a Cauchy converging sequence with respect to d. Let x^* be its converging point. We now prove that x^* is a fixed point for T.

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n) = Tx^*.$$

Uniqueness. Suppose there are two fixed points x^* and y^*. By applying (10) we get:

$$\delta(Tx^*, Ty^*) \leq b[\delta(x^*, Ty^*) + \delta(y^*, Tx^*)],$$

which is

$$\delta(x^*, y^*) \leq 2b\delta(x^*, y^*).$$

Because $b < \frac{1}{2}$ we have $2b < 1$, so $\delta(x^*, y^*) = 0$ which implies $x^* = y^*$. $\square$

Remark 5. *It is not necessary for T to be a continuous mapping. The proof is similar to the one in Remark* 1.

The following result gives a way to construct the δ metric.

Theorem 14. *Let (X, d) be a metric space and T a mapping of X into X. Let*

$$\sum_{n=0}^{\infty} \lambda^n [d(T^n x, T^{n+1}x) + d(T^n y, T^{n+1}y)] \tag{11}$$

be a power series. Assuming there exists $\lambda > 1$ such that the series (11) *is converging for all $x, y \in X$ and $d(x, y) \leq d(x, Tx) + d(y, Ty)$, we define:*

$$\delta(x, y) = \begin{cases} \sum_{n=0}^{\infty} \lambda^n [d(T^n x, T^{n+1}x) + d(T^n y, T^{n+1}y)], & x \neq y \\ 0, & x = y \end{cases}. \tag{12}$$

Then:

(i) δ is a majorant metric for d,

(ii) T is a Kannan type operator with respect to δ,

(iii) T is a Chatterjea type operator with respect to δ.

Proof. (i) By definition (12), δ is a metric. Now we have

$$\begin{aligned} \delta(x, y) &= \sum_{n=0}^{\infty} \lambda^n [d(T^n x, T^{n+1}x) + d(T^n y, T^{n+1}y)] \\ &= d(x, Tx) + d(y, Ty) + \sum_{n=1}^{\infty} \lambda^n [d(T^n x, T^{n+1}x) + d(T^n y, T^{n+1}y)] \\ &\geq qd(x, Tx) + d(y.Ty) \geq d(x, y), \end{aligned}$$

which is true because series (12) has positive terms, so δ is a majorant metric for d.

(ii) We consider the following relations:

$$\delta(x,Tx)=\sum_{n=0}^{\infty}\lambda^n[d(T^nx,T^{n+1}x)+d(T^{n+1}x,T^{n+2}x)],$$

$$\delta(y,Ty)=\sum_{n=0}^{\infty}\lambda^n[d(T^ny,T^{n+1}y)+d(T^{n+1}y,T^{n+2}y)],$$

and by summing these relations we get:

$$\begin{aligned}\delta(x,Tx)+\delta(y,Ty) \quad &= \quad \sum_{n=0}^{\infty}\lambda^n[d(T^nx,T^{n+1}x)+d(T^ny,T^{n+1}y)]\\ &\quad +\sum_{n=0}^{\infty}\lambda^n[d(T^{n+1}x,T^{n+2}x)+d(T^{n+1}y,T^{n+2}y)].\end{aligned}$$

Which is:

$$\delta(x,Tx)+\delta(y,Ty)=\delta(x,y)+\delta(Tx,Ty). \tag{13}$$

Hence,

$$\delta(x,y)=d(x,Tx)+d(y,Ty)+\lambda\sum_{n=0}^{\infty}\lambda^n[d(T^{n+1}x,T^{n+2}x)+d(T^{n+1}y,T^{n+2}y)]=$$

$$=d(x,Tx)+d(y,Ty)+\lambda\delta(Tx,Ty).$$

Therefore, we have:

$$\delta(Tx,Ty)=\frac{1}{\lambda}(\delta(x,y)-d(x,Tx)-d(y,Ty)\leq\frac{1}{\lambda}\delta(x,y).$$

So by summation with $\frac{1}{\lambda}\delta(Tx,Ty)$ in the last inequality we get:

$$\Big(1+\frac{1}{\lambda}\Big)\delta(Tx,Ty)\leq\frac{1}{\lambda}(\delta(x,y)+\delta(Tx,Ty)).$$

Now by using relation (13) we obtain:

$$\delta(Tx,Ty)\leq\frac{1}{\lambda+1}[\delta(x,Tx)+\delta(y,Ty)]$$

But, because $\lambda>1$ then $\frac{1}{\lambda+1}<\frac{1}{2}$, so T is a Kannan type operator with respect to δ.
(iii) To prove that T is a Chatterjea type operator with respect to δ we consider:

$$\delta(x,Ty)=\sum_{n=0}^{\infty}\lambda^n[d(T^nx,T^{n+1}x)+d(T^{n+1}y,T^{n+2}y)]$$

and

$$\delta(y,Tx)=\sum_{n=0}^{\infty}\lambda^n[d(T^ny,T^{n+1}y)+d(T^{n+1}x,T^{n+2}x)].$$

Then, by summation we get:

$$\delta(x, Ty) + \delta(y, Tx) = \delta(x, y) + \delta(Tx, Ty). \tag{14}$$

But,

$$\delta(x, y) = d(x, Tx) + d(y, Ty) + \lambda \sum_{n=0}^{\infty} \lambda^n [d(T^{n+1}x, T^{n+2}x) + d(T^{n+1}y, T^{n+2}y)] =$$

$$= d(x, Tx) + d(y, Ty) + \lambda\delta(Tx, Ty).$$

Which means that:

$$\delta(Tx, Ty) = \frac{1}{\lambda}(\delta(x, y) - d(x, Tx) - d(y, Ty) \leq \frac{1}{\lambda}\delta(x, y).$$

So by summation with $\frac{1}{\lambda}\delta(Tx, Ty)$ in the last inequality we get:

$$\Big(1 + \frac{1}{\lambda}\Big)\delta(Tx, Ty) \leq \frac{1}{\lambda}(\delta(x, y) + \delta(Tx, Ty)).$$

Now by using relation (14) we obtain:

$$\delta(Tx, Ty) \leq \frac{1}{\lambda + 1}[\delta(x, Ty) + \delta(y, Tx)].$$

From hypothesis $\frac{1}{\lambda+1} < \frac{1}{2}$, T is a Chatterjea type operator. □

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