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$*$-CRITICAL POINT EQUATION ON $N(k)$-CONTACT MANIFOLDS

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Abstract

The object of the present paper is to characterize $N(k)$-contact metric manifolds satisfying the $*$-critical point equation. It is proved that, if (g, λ) is a non-constant solution of the $*$-critical point equation of a non-compact $N(k)$-contact metric manifold, then (1) the manifold M is locally isometric to the Riemannian product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive curvature 4 for $n > 1$ and flat for $n = 1$, (2) the manifold is $*$-Ricci flat and (3) the function λ is harmonic. The result is also verified by an example.

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1 Introduction

The Ricci tensor S of a Riemannian manifold M is a tensor field of type $(0, 2)$ and is given by

$$S(X, Y) = g(QX, Y) = trace\{Z \mapsto R(Z, X)Y\}, \tag{1}$$

where Q is the $(1, 1)$ Ricci operator.

In 1959, Tachibana [15] introduced the notion of $*$-Ricci tensor on almost Hermitian manifolds. Later in [10], Hamada defined the $*$-Ricci tensor of real hypersurfaces in non-flat complex space form by

$$S^*(X, Y) = g(Q^*X, Y) = \frac{1}{2}(trace\{\phi \circ R(X, \phi Y)\}), \tag{2}$$

where Q^* is the $(1, 1)$ $*$-Ricci operator for any vector fields X, Y on M. The $*$-scalar curvature is denoted by r^* and is defined by $r^* = trace(S^*)$. In 2018, Majhi et. al.[11] studied $*$-Ricci solitons on Sasakian 3-manifolds.

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Definition 1. *A Riemannian manifold M is called $*$-η-Einstein if there exist two smooth functions α and β on M which satisfy the relation*

$$S^*(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y).$$

Definition 2. *A Riemannian manifold M is called $*$-Ricci flat if the $*$-Ricci tensor S^* vanishes identically.*

A Riemannian manifold (M,g) of dimension $n \geq 3$ with constant scalar curvature and unit volume together with a non-constant smooth potential function λ satisfying the equation

$$Hess\lambda - (S - \frac{r}{n-1}g)\lambda = S - \frac{r}{n}g, \tag{3}$$

is called a critical point equation (in short, CPE) on M, where S is the Ricci tensor, r is the scalar curvature and $Hess\lambda$ is the Hessian of the smooth function λ.
Note that if $\lambda = 0$, then (3) becomes Einstein metric. In [2], Besse conjectured that the solution of the CPE is Einstein. Barros and Ribeiro [1] proved that the CPE conjecture is true for half conformally flat. In [9], Hwang proved that the CPE conjecture is also true under certain conditions on the bounds of the potential function λ. Very recently, Neto [12] deduced a necessary and sufficient condition on the norm of the gradient of the potential function for a CPE metric to be Einstein. Similar kind of critical metric was studied by Wang and Wang in [17].

In this paper the following notion is introduced

Definition 3. *A Riemannian manifold (M,g) of dimension $(2n+1) \geq 3$ with constant $*$-scalar curvature and unit volume together with a non-constant smooth potential function λ satisfying the equation*

$$Hess\lambda - (S^* - \frac{r^*}{2n}g)\lambda = S^* - \frac{r^*}{2n+1}g, \tag{4}$$

is called a $*$-critical point equation (in short, $*$-CPE) on M, where r^* is the $*$-scalar curvature.

In this paper, we consider the above notion of $*$-CPE in the framework of $(2n+1)$-dimensional $N(k)$-contact metric manifolds and prove the following:

Theorem 1. *Let M be a $(2n+1)$-dimensional non-compact $N(k)$-contact metric manifold. If (g,λ) is a non-constant solution of the $*$-critical point equation, then*

(1) The manifold M is locally isometric to the Riemannian product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive curvature 4 for $n > 1$ and flat for $n = 1$.

(2) The manifold M is $*$-Ricci flat.

(3) The function λ is harmonic.

2 Preliminaries

By an almost contact structure on a $(2n+1)$-dimensional manifold M we mean a structure (ϕ, ξ, η) satisfying

$$\phi^2 = -I + \eta \otimes \xi, \;\; \eta(\xi) = 1, \;\; \phi\xi = 0, \;\; \eta \circ \phi = 0, \tag{5}$$

where ϕ is a $(1,1)$-tensor field, ξ is a unit vector field and η is a smooth 1-form dual to ξ with respect to the Riemannian metric g. Let g be a compatible Riemannian metric with almost contact structure (ϕ, ξ, η), that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{6}$$

Then M becomes an almost contact metric manifold equiped with an almost contact metric structure (ϕ, ξ, η, g). Using (5), we can easily see from the above equation that

$$g(\phi X, Y) = -g(X, \phi Y), \;\; g(X, \xi) = \eta(X). \tag{7}$$

An almost contact metric structure becomes a contact metric structure if $g(X, \phi Y) = d\eta(X, Y)$ for all vector fields X, Y on M. The $(1,1)$-tensor field h is defined by $h = \frac{1}{2}\pounds_\xi \phi$, where $\pounds$ denotes the Lie differentiation. The tensor field h is symmetric and satisfies

$$h\phi = -\phi h, \;\; trace(h) = trace(\phi h) = 0, \;\; h\xi = 0. \tag{8}$$

Also,

$$\nabla_X \xi = -\phi X - \phi h X. \tag{9}$$

The k-nullity distribution $N(k)$ of a Riemannian manifolds is defined by [16]

$$N(k) = \{Z \in T(M) : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\},$$

k being a real number and $T(M)$ is the Lie algebra of all vector fields on M. If the characteristic vector field $\xi \in N(k)$, then we call a contact metric manifold as $N(k)$-contact metric manifold [16]. However, for a $N(k)$-contact metric manifold M of dimension $(2n+1)$, we have ([4], [5])

$$h^2 = (k-1)\phi^2, \tag{10}$$

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y], \tag{11}$$

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X], \tag{12}$$

$$\begin{aligned} S(X, Y) \;\; = \;\; & 2(n-1)g(X, Y) + 2(n-1)g(hX, Y) \\ & + [2(1-n) + 2nk]\eta(X)\eta(Y), \qquad n \geq 1. \end{aligned} \tag{13}$$

$$S(X,\xi) = 2nk\eta(X), \tag{14}$$

$$(\nabla_X \eta)(Y) = g(X + hX, \phi Y), \tag{15}$$

for any vector fields X, Y where R is the Riemannian curvature tensor and S is the Ricci tensor. $N(k)$-contact metric manifolds have been studied by several authors such as ([6], [13], [14]) and many others.

3 $*$-Critical point equation

To prove the main theorem we need to state the following lemmas.

Lemma 1. *([3]) A contact metric manifold M^{2n+1} satisfying condition $R(X,Y)\xi = 0$ for all X, Y is locally isometric to the Riemannian product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive curvature 4, i.e., $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$.*

Lemma 2. *([8]) Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a contact metric manifold and $\{\xi, e_j, \phi e_j\}$ a local orthonormal ϕ-basis. Then*

$$\sum_{i=0}^{2n} g(R(X,Y)\phi h e_i, e_i) = 0$$

for any vector fields X, Y on M.

Lemma 3. *In a $(2n+1)$-dimensional $N(k)$-contact metric manifold*

$$(DivR)(X,Y)\xi = -4kg(\phi X, Y)$$

for any vector fields X, Y on M, where "Div" stands for divergence.

Proof. It is well known that [18]

$$(DivR)(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z).$$

Hence,

$$(DivR)(X,Y)\xi = (\nabla_X S)(Y,\xi) - (\nabla_Y S)(X,\xi). \tag{16}$$

Now,

$$(\nabla_Y S)(X,\xi) = \nabla_Y S(X,\xi) - S(\nabla_Y X,\xi) - S(X,\nabla_Y \xi).$$

Using (9), (13)-(15) in the foregoing equation yields

$$(\nabla_Y S)(X,\xi) = 2nk[g(\phi X,Y) + g(\phi X, hY)] + S(X,\phi Y) + S(X,\phi hY). \tag{17}$$

Making use of (17) in (16) and then using (8), (10) and (13) we get the desired result. □

Lemma 4. *A $(2n+1)$-dimensional $N(k)$-contact metric manifold is $*$-η-Einstein and the $*$-Ricci tensor S^* is given by*

$$S^*(X,Y) = -k[g(X,Y) - \eta(X)\eta(Y)]. \tag{18}$$

Proof. Differentiating (11) covariantly along any vector field Z we have

$$\nabla_Z R(X,Y)\xi = k[(\nabla_Z \eta(Y))X + \eta(Y)\nabla_Z X - (\nabla_Z \eta(X))Y - \eta(X)\nabla_Z Y]. \tag{19}$$

Now,

$$(\nabla_Z R)(X,Y)\xi = \nabla_Z R(X,Y)\xi - R(\nabla_Z X,Y)\xi - R(X,\nabla_Z Y)\xi - R(X,Y)\nabla_Z \xi.$$

Using (9) and (11) in the foregoing equation we have

$$\begin{aligned} \nabla_Z R(X,Y)\xi &= (\nabla_Z R)(X,Y)\xi + k[\eta(Y)\nabla_Z X - \eta(\nabla_Z X)Y] \\ &\quad +k[\eta(\nabla_Z Y)X - \eta(X)\nabla_Z Y] - R(X,Y)\phi Z \\ &\quad -R(X,Y)\phi hZ. \end{aligned} \tag{20}$$

Substituting (18) in (17) and then using (15) we infer that

$$\begin{aligned} &(\nabla_Z R)(X,Y)\xi - R(X,Y)\phi Z - R(X,Y)\phi hZ \\ =\ & k[g(\phi Y,Z)X + g(hZ,\phi Y)X - g(\phi X,Z)Y - g(hZ,\phi X)Y]. \end{aligned} \tag{21}$$

Let $\{e_i\}$, $i = 1,2,\ldots,(2n+1)$ be an orthonormal basis of the tangent space at each point of the manifold. Now contracting Z in (19) and using (8) we obtain

$$(DivR)(X,Y)\xi - g(R(X,Y)\phi e_i, e_i) - g(R(X,Y)\phi e_i, e_i) = -2kg(\phi X,Y). \tag{22}$$

Now using Lemma 2 and Lemma 3 in the above equation yields

$$g(R(X,Y)\phi e_i, e_i) = -2kg(\phi X,Y).$$

Substituting $Y = \phi Y$ and using (2) in the foregoing equation we complete the proof. □

Remark 1. *The above Lemma can also be obtained from Lemma 5.3 of [8]. For completeness, we give the detailed proof here.*

Proof of Theorem 1:

Tracing (18) we get $r^* = -2nk$. Substituting the value of r^* and S^* from (18) in (4) we obtain

$$Hess\lambda(X,Y) = -\frac{k}{2n+1}g(X,Y) + k(1+\lambda)\eta(X)\eta(Y),$$

which implies

$$\nabla_X D\lambda = -\frac{k}{2n+1}X + k(1+\lambda)\eta(X)\xi. \tag{23}$$

Differentiating (23) covariantly along any vector field Y we get

$$\nabla_Y\nabla_X D\lambda = -\frac{k}{2n+1}\nabla_Y X + k(Y\lambda)\eta(X)\xi + k(1+\lambda)(\nabla_Y\eta(X))\xi + k(1+\lambda)\eta(X)(-\phi Y - \phi hY). \tag{24}$$

Interchanging X and Y in the above equation we obtain

$$\nabla_X\nabla_Y D\lambda = -\frac{k}{2n+1}\nabla_X Y + k(X\lambda)\eta(Y)\xi + k(1+\lambda)(\nabla_X\eta(Y))\xi + k(1+\lambda)\eta(Y)(-\phi X - \phi hX). \tag{25}$$

Again from (23) we have

$$\nabla_{[X,Y]}D\lambda = -\frac{k}{2n+1}(\nabla_X Y - \nabla_Y X) + k(1+\lambda)\eta(\nabla_X Y - \nabla_Y X)\xi. \tag{26}$$

Now,

$$R(X,Y)D\lambda = \nabla_X\nabla_Y D\lambda - \nabla_Y\nabla_X D\lambda - \nabla_{[X,Y]}D\lambda. \tag{27}$$

Substituting (24)-(26) in (27) we get

$$R(X,Y)D\lambda = k(X\lambda)\eta(Y)\xi - k(Y\lambda)\eta(X)\xi - 2k(1+\lambda)g(\phi X, Y) + k(1+\lambda)(\eta(X)(\phi Y + \phi hY) - \eta(Y)(\phi X + \phi hX)). \tag{28}$$

Putting $X = \xi$ in (28) yields

$$R(\xi,Y)D\lambda = k(\xi\lambda)\eta(Y)\xi - k(Y\lambda)\xi + k(1+\lambda)(\phi Y + \phi hY). \tag{29}$$

Taking inner product of the foregoing equation with X we obtain

$$g(R(\xi,Y)D\lambda, X) = k(\xi\lambda)\eta(X)\eta(Y) - k(Y\lambda)\eta(X) + k(1+\lambda)(g(\phi Y, X) + g(\phi hY, X)). \tag{30}$$

Again, using (11) we obtain

$$g(R(\xi,Y)D\lambda, X) = -kg(X,Y)(\xi\lambda) + k\eta(X)(Y\lambda). \tag{31}$$

Hence, from the above two equations we have

$$-kg(X,Y)(\xi\lambda) + k\eta(X)(Y\lambda) = k(\xi\lambda)\eta(X)\eta(Y) - k(Y\lambda)\eta(X) + k(1+\lambda)(g(\phi Y, X) + g(\phi hY, X)). \tag{32}$$

Antisymmetrizing the above equation we infer that

$$k(Y\lambda)\eta(X) - k(X\lambda)\eta(Y) + k(1+\lambda)g(\phi X, Y) = 0. \tag{33}$$

Replacing X, Y by ϕX, ϕY respectively in the above equation we get

$$k(1+\lambda)g(\phi X, Y) = 0, \tag{34}$$

which implies $k = 0$ as λ is a non-constant smooth function. Hence, equation (11) and Lemma 1 proves (1). Again Lemma 4 shows that the manifold becomes $*$-Ricci flat, which proves (2). Now, putting $k = 0$ in (23) and taking inner product with Y and then tracing it we obtain $\Delta\lambda = Div(grad\lambda) = 0$, where Δ is the Laplace operator. The non-compactness of the manifold is used here, as λ is non-constant. Therefore, λ is harmonic proving (3). This completes the proof.

4 Example

In [7], the authors have constructed an example of a 3-dimensional $N(1-\alpha^2)$-contact metric manifold. In this example we can easily calculate that

$$S^*(e_1, e_1) = 0, \;\; S^*(e_2, e_2) = S^*(e_3, e_3) = -(1-\alpha^2).$$

Therefore, $r^* = S^*(e_1, e_1) + S^*(e_2, e_2) + S^*(e_3, e_3) = -2(1-\alpha^2)$.
Now considering the dimension and tracing (4) we obtain $\Delta\lambda = (1-\alpha^2)\lambda$, where Δ is the Laplace operator given by $\Delta\lambda = Div(grad\lambda)$. If we consider $\alpha = 1$, then the manifold reduces to a flat manifold and (g, λ) is a solution of the $*$-critical point equation, where λ satisfies the Laplace equation $\Delta\lambda = 0$. Thus, our Theorem 1 is verified.

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