**CRITICAL POINT EQUATION ON $N(k)$-CONTACT MANIFOLDS**

**Dibakar DEY**¹ and Pradip MAJHI*,²

**Abstract**

The object of the present paper is to characterize $N(k)$-contact metric manifolds satisfying the $*$-critical point equation. It is proved that, if $(g, \lambda)$ is a non-constant solution of the $*$-critical point equation of a non-compact $N(k)$-contact metric manifold, then (1) the manifold $M$ is locally isometric to the Riemannian product of a flat $(n + 1)$-dimensional manifold and an $n$-dimensional manifold of positive curvature 4 for $n > 1$ and flat for $n = 1$, (2) the manifold is $*$-Ricci flat and (3) the function $\lambda$ is harmonic. The result is also verified by an example.

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**Key words:** Critical point equation, $*$-Critical point equation, $N(k)$-contact manifolds.

**1 Introduction**

The Ricci tensor $S$ of a Riemannian manifold $M$ is a tensor field of type $(0, 2)$ and is given by

$$S(X, Y) = g(QX, Y) = \text{trace}\{Z \mapsto R(Z, X)Y\},$$

(1)

where $Q$ is the $(1, 1)$ Ricci operator.

In 1959, Tachibana [15] introduced the notion of $*$-Ricci tensor on almost Hermitian manifolds. Later in [10], Hamada defined the $*$-Ricci tensor of real hypersurfaces in non-flat complex space form by

$$S^*(X, Y) = g(Q^*X, Y) = \frac{1}{2}(\text{trace}\{\phi \circ R(X, \phi Y)\}),$$

(2)

where $Q^*$ is the $(1, 1)$ $*$-Ricci operator for any vector fields $X, Y$ on $M$. The $*$-scalar curvature is denoted by $r^*$ and is defined by $r^* = \text{trace}(S^*)$. In 2018, Majhi et. al.[11] studied $*$-Ricci solitons on Sasakian 3-manifolds.

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¹Department of Pure Mathematics, University of Calcutta, 35 Ballygunge Circular Road, Kol-700019, West Bengal, India, e-mail: deydibakar3@gmail.com

² Corresponding author, Department of Pure Mathematics, University of Calcutta, 35 Ballygunge Circular Road, Kol-700019, West Bengal, India, e-mail: mpradipmajhi@gmail.com
**Definition 1.** A Riemannian manifold $M$ is called $\ast\eta$-Einstein if there exist two smooth functions $\alpha$ and $\beta$ on $M$ which satisfy the relation
\[ S^\ast(X,Y) = \alpha g(X,Y) + \beta \eta(X)\eta(Y). \]

**Definition 2.** A Riemannian manifold $M$ is called $\ast$-Ricci flat if the $\ast$-Ricci tensor $S^\ast$ vanishes identically.

A Riemannian manifold $(M, g)$ of dimension $n \geq 3$ with constant scalar curvature and unit volume together with a non-constant smooth potential function $\lambda$ satisfying the equation
\[ \text{Hess}\lambda - (S - \frac{r}{n-1}g)\lambda = S - \frac{r}{n}g, \tag{3} \]

is called a critical point equation (in short, CPE) on $M$, where $S$ is the Ricci tensor, $r$ is the scalar curvature and $\text{Hess}\lambda$ is the Hessian of the smooth function $\lambda$.

Note that if $\lambda = 0$, then (3) becomes Einstein metric. In [2], Besse conjectured that the solution of the CPE is Einstein. Barros and Ribeiro [1] proved that the CPE conjecture is true for half conformally flat. In [9], Hwang proved that the CPE conjecture is also true under certain conditions on the bounds of the potential function $\lambda$. Very recently, Neto [12] deduced a necessary and sufficient condition on the norm of the gradient of the potential function for a CPE metric to be Einstein. Similar kind of critical metric was studied by Wang and Wang in [17].

In this paper the following notion is introduced

**Definition 3.** A Riemannian manifold $(M, g)$ of dimension $(2n+1) \geq 3$ with constant $\ast$-scalar curvature and unit volume together with a non-constant smooth potential function $\lambda$ satisfying the equation
\[ \text{Hess}\lambda - (S^\ast - \frac{r^\ast}{2n}g)\lambda = S^\ast - \frac{r^\ast}{2n+1}g, \tag{4} \]

is called a $\ast$-critical point equation (in short, $\ast$-CPE) on $M$, where $r^\ast$ is the $\ast$-scalar curvature.

In this paper, we consider the above notion of $\ast$-CPE in the framework of $(2n+1)$-dimensional $N(k)$-contact metric manifolds and prove the following:

**Theorem 1.** Let $M$ be a $(2n+1)$-dimensional non-compact $N(k)$-contact metric manifold. If $(g, \lambda)$ is a non-constant solution of the $\ast$-critical point equation, then

1. The manifold $M$ is locally isometric to the Riemannian product of a flat $(n+1)$-dimensional manifold and an $n$-dimensional manifold of positive curvature 4 for $n > 1$ and flat for $n = 1$.
2. The manifold $M$ is $\ast$-Ricci flat.
3. The function $\lambda$ is harmonic.
2 Preliminaries

By an almost contact structure on a \((2n+1)\)-dimensional manifold \(M\) we mean a structure \((\phi, \xi, \eta)\) satisfying

\[
\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi \xi = 0, \quad \eta \circ \phi = 0,
\]

where \(\phi\) is a \((1, 1)\)-tensor field, \(\xi\) is a unit vector field and \(\eta\) is a smooth 1-form dual to \(\xi\) with respect to the Riemannian metric \(g\). Let \(g\) be a compatible Riemannian metric with almost contact structure \((\phi, \xi, \eta)\), that is,

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).
\]

Then \(M\) becomes an almost contact metric manifold equipped with an almost contact metric structure \((\phi, \xi, \eta, g)\). Using (5), we can easily see from the above equation that

\[
g(\phi X, Y) = -g(X, \phi Y), \quad g(X, \xi) = \eta(X).
\]

An almost contact metric structure becomes a contact metric structure if \(g(X, \phi Y) = d\eta(X, Y)\) for all vector fields \(X, Y\) on \(M\). The \((1, 1)\)-tensor field \(h\) is defined by \(h = \frac{1}{2} \xi \phi\), where \(\xi\) denotes the Lie differentiation. The tensor field \(h\) is symmetric and satisfies

\[
h\phi = -\phi h, \quad \text{trace}(h) = \text{trace}(\phi h) = 0, \quad h\xi = 0.
\]

Also,

\[
\nabla_X \xi = -\phi X - \phi h X.
\]

The \(k\)-nullity distribution \(N(k)\) of a Riemannian manifolds is defined by [16]

\[
N(k) = \{ Z \in T(M) : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y] \},
\]

\(k\) being a real number and \(T(M)\) is the Lie algebra of all vector fields on \(M\). If the characteristic vector field \(\xi \in N(k)\), then we call a contact metric manifold as \(N(k)\)-contact metric manifold [16]. However, for a \(N(k)\)-contact metric manifold \(M\) of dimension \((2n + 1)\), we have ([4], [5])

\[
h^2 = (k - 1)\phi^2,
\]

\[
R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y],
\]

\[
R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X],
\]

\[
S(X, Y) = 2(n - 1)g(X, Y) + 2(n - 1)g(hX, Y) + [2(1 - n) + 2nk]\eta(X)\eta(Y), \quad n \geq 1.
\]
Theorem 1. ([3]) A contact metric manifold \( M^{2n+1} \) satisfying condition \( R(X,Y)\xi = 0 \) for all \( X, Y \) is locally isometric to the Riemannian product of a flat \((n+1)\)-dimensional manifold and an \( n \)-dimensional manifold of positive curvature \( 4 \), i.e., \( E^{n+1}(0) \times S^n(4) \) for \( n > 1 \) and flat for \( n = 1 \).

Lemma 2. ([8]) Let \( M^{2n+1}(\phi, \xi, \eta, g) \) be a contact metric manifold and \( \{\xi, e_j, \phi e_j\} \) a local orthonormal \( \phi \)-basis. Then
\[
\sum_{i=0}^{2n} g(R(X,Y)\phi e_i, e_i) = 0
\]
for any vector fields \( X, Y \) on \( M \).

Lemma 3. In a \((2n+1)\)-dimensional \( N(k) \)-contact metric manifold
\[
(\text{Div} R)(X,Y)\xi = -4kg(\phi X,Y)
\]
for any vector fields \( X, Y \) on \( M \), where “\( \text{Div} \)” stands for divergence.

Proof. It is well known that \([18]\)
\[
\]
Hence,
\[
(\text{Div} R)(X,Y)\xi = (\nabla_X S)(Y,\xi) - (\nabla_Y S)(X,\xi).
\] (16)

Now,
\[
(\nabla_Y S)(X,\xi) = \nabla_Y S(X,\xi) - S(\nabla_Y X,\xi) - S(X,\nabla_Y \xi).
\]
Using (9), (13)-(15) in the foregoing equation yields
\[
(\nabla_Y S)(X,\xi) = 2nk[g(\phi X,Y) + g(\phi X,\phi Y)] + S(X,\phi Y) + S(X,\phi h Y).
\] (17)
Making use of (17) in (16) and then using (8), (10) and (13) we get the desired result. \( \Box \)
Lemma 4. A \((2n+1)\)-dimensional \(N(k)\)-contact metric manifold is \(*\)-\(\eta\)-Einstein and the \(*\)-Ricci tensor \(S^*\) is given by
\[
S^*(X,Y) = -k[g(X,Y) - \eta(X)\eta(Y)].
\] (18)

Proof. Differentiating (11) covariantly along any vector field \(Z\) we have
\[
\nabla_Z R(X,Y)\xi = k[(\nabla_X \eta(Y))X + \eta(Y)\nabla_X X - (\nabla_Y \eta(X))Y - \eta(X)\nabla_Y Y].
\] (19)

Now,
\[
(\nabla_Z R)(X,Y)\xi = \nabla_Z R(X,Y)\xi - R(\nabla_X Y, Y)\xi - R(X, \nabla_Y Z)\xi - R(X, Y)\nabla_Z \xi.
\]
Using (9) and (11) in the foregoing equation we have
\[
\nabla_Z R(X,Y)\xi = (\nabla_Z R)(X,Y)\xi + k[\eta(\nabla_X Y)X - \eta(\nabla_X X)]
+ k[\eta(\nabla_Y X)X - \eta(X)\nabla_Y Y] - R(X,Y)\phi Z
- R(X,Y)\phi h Z.
\] (20)

Substituting (18) in (17) and then using (15) we infer that
\[
(\nabla_Z R)(X,Y)\xi - R(X,Y)\phi Z - R(X,Y)\phi h Z
= k[g(\phi Y, Z)X + g(h Z, \phi Y)X - g(\phi X, Z)Y - g(h Z, \phi X) Y].
\] (21)

Let \(\{e_i\}, i = 1, 2, \ldots, (2n+1)\) be an orthonormal basis of the tangent space at each point of the manifold. Now contracting \(Z\) in (19) and using (8) we obtain
\[
(\text{Div} R)(X,Y)\xi - g(R(X,Y)\phi e_i, e_i) - g(R(X,Y)\phi e_i, e_i) = -2kg(\phi X, Y).
\] (22)

Now using Lemma 2 and Lemma 3 in the above equation yields
\[
g(R(X,Y)\phi e_i, e_i) = -2kg(\phi X, Y).
\]
Substituting \(Y = \phi Y\) and using (2) in the foregoing equation we complete the proof.

Remark 1. The above Lemma can also be obtained from Lemma 5.3 of [8]. For completeness, we give the detailed proof here.

Proof of Theorem 1:

Tracing (18) we get \(r^* = -2nk\). Substituting the value of \(r^*\) and \(S^*\) from (18) in (4) we obtain
\[
\text{Hess} \lambda(X,Y) = -\frac{k}{2n+1}g(X,Y) + k(1 + \lambda)\eta(X)\eta(Y),
\]
which implies
\[
\nabla_X D\lambda = -\frac{k}{2n+1}X + k(1 + \lambda)\eta(X)\xi.
\] (23)
Differentiating (23) covariantly along any vector field $Y$ we get

$$\nabla_Y \nabla_X \lambda = -\frac{k}{2n+1} \nabla_Y X + k(Y\lambda)\eta(X)\xi + k(1 + \lambda)(\nabla_Y \eta(X))\xi + k(1 + \lambda)\eta(X)(-\phi Y - \phi h Y).$$  (24)

Interchanging $X$ and $Y$ in the above equation we obtain

$$\nabla_X \nabla_Y \lambda = -\frac{k}{2n+1} \nabla_X Y + k(X\lambda)\eta(Y)\xi + k(1 + \lambda)(\nabla_X \eta(Y))\xi + k(1 + \lambda)\eta(Y)(-\phi X - \phi h X).$$  (25)

Again from (23) we have

$$\nabla_{[X,Y]} \lambda = -\frac{k}{2n+1} (\nabla_X Y - \nabla_Y X) + k(1 + \lambda)\eta(\nabla_X Y - \nabla_Y X)\xi.$$  (26)

Now,

$$R(X,Y)D \lambda = \nabla_X \nabla_Y D \lambda - \nabla_Y \nabla_X D \lambda - \nabla_{[X,Y]} D \lambda.$$  (27)

Substituting (24)-(26) in (27) we get

$$R(X,Y)D \lambda = k(X\lambda)\eta(Y)\xi - k(Y\lambda)\eta(X)\xi - 2k(1 + \lambda)g(\phi X, Y) + k(1 + \lambda)(\eta(X)(\phi Y + \phi h Y) - \eta(Y)(\phi X + \phi h X)).$$  (28)

Putting $X = \xi$ in (28) yields

$$R(\xi,Y)D \lambda = k(\xi\lambda)\eta(Y)\xi - k(Y\lambda)\xi + k(1 + \lambda)(\phi Y + \phi h Y).$$  (29)

Taking inner product of the foregoing equation with $X$ we obtain

$$g(R(\xi,Y)D \lambda, X) = k(\xi\lambda)\eta(X)\eta(Y) - k(Y\lambda)\eta(X) + k(1 + \lambda)(g(\phi Y, X) + g(\phi h Y, X)).$$  (30)

Again, using (11) we obtain

$$g(R(\xi,Y)D \lambda, X) = -k g(X,Y)(\xi\lambda) + k \eta(X)(Y\lambda).$$  (31)

Hence, from the above two equations we have

$$-k g(X,Y)(\xi\lambda) + k \eta(X)(Y\lambda) = k(\xi\lambda)\eta(X)\eta(Y) - k(Y\lambda)\eta(X) + k(1 + \lambda)(g(\phi Y, X) + g(\phi h Y, X)).$$  (32)

Antisymmetrizing the above equation we infer that

$$k(Y\lambda)\eta(X) - k(X\lambda)\eta(Y) + k(1 + \lambda)g(\phi X, Y) = 0.$$  (33)

Replacing $X$, $Y$ by $\phi X$, $\phi Y$ respectively in the above equation we get

$$k(1 + \lambda)g(\phi X, Y) = 0,$$  (34)

which implies $k = 0$ as $\lambda$ is a non-constant smooth function. Hence, equation (11) and Lemma 1 proves (1). Again Lemma 4 shows that the manifold becomes $\ast$-Ricci flat, which proves (2). Now, putting $k = 0$ in (23) and taking inner product with $Y$ and then tracing it we obtain $\Delta \lambda = Div(\text{grad}\lambda) = 0$, where $\Delta$ is the Laplace operator. The non-compactness of the manifold is used here, as $\lambda$ is non-constant. Therefore, $\lambda$ is harmonic proving (3). This completes the proof.
4 Example

In [7], the authors have constructed an example of a 3-dimensional $N(1 - \alpha^2)$-contact metric manifold. In this example we can easily calculate that

$$S^*(e_1, e_1) = 0, \quad S^*(e_2, e_2) = S^*(e_3, e_3) = -(1 - \alpha^2).$$

Therefore, $r^* = S^*(e_1, e_1) + S^*(e_2, e_2) + S^*(e_3, e_3) = -2(1 - \alpha^2)$.

Now considering the dimension and tracing (4) we obtain $\Delta \lambda = (1 - \alpha^2)\lambda$, where $\Delta$ is the Laplace operator given by $\Delta \lambda = \text{Div}(\text{grad} \lambda)$. If we consider $\alpha = 1$, then the manifold reduces to a flat manifold and $(g, \lambda)$ is a solution of the *-critical point equation, where $\lambda$ satisfies the Laplace equation $\Delta \lambda = 0$. Thus, our Theorem 1 is verified.

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