Bulletin of the *Transilvania* University of Braşov • Vol 12(61), No. 2 - 2019
Series III: Mathematics, Informatics, Physics, 265-274
https://doi.org/10.31926/but.mif.2019.12.61.2.6

$*$-RICCI SOLITONS ON $(\epsilon)-$PARA SASAKIAN 3-MANIFOLDS

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Abstract

In the present paper we study $*$-Ricci solitons in $(\epsilon)-$para Sasakian manifolds and prove that if an $(\epsilon)-$para Sasakian 3-manifold with constant scalar curvature admits a $*$-Ricci soliton, then the $*$-Ricci soliton is steady if and only if $\pounds_V \xi$ is g-orthogonal to ξ provided $a =$Trϕ is constant. Beside these, we study gradient $*$-Ricci solitons on $(\epsilon)-$para Sasakian 3-manifolds.

2000 *Mathematics Subject Classification:* $53C15$, $53C25$.
Key words: $(\epsilon)-$para Sasakian manifolds, $*$-Ricci solitons, gradient $*$-Ricci solitons, $*$-Einstein manifold.

1 Introduction

In this paper, we introduce a new type of Ricci solitons, called *$*$-Ricci solitons in $(\epsilon)-$para Sasakian manifolds* with indefinite metric which play a functional role in contemporary mathematics. The properties of a manifold solely depend on the nature of the metric defined on it. With the help of *indefinite metric*, A. Bejancu and K. L. Duggal [1] introduced $(\epsilon)-$*Sasakian manifolds*. Also, Xufeng and Xiaoli [18] showed that *every* $(\epsilon)-$*Sasakian manifold must be a real hypersurface of some indefinite Kähler manifold.* In 2010, Tripathi et.al[14] studied $(\epsilon)-$*almost paracontact manifolds*, and in particular, $(\epsilon)-$*para Sasakian manifolds*. They introduced the notion of an $(\epsilon)-$para Sasakian structure. Since Sasakian manifolds with indefinite metric play significant role in physics [8], our natural trend is to study various contact manifolds with indefinite metric.

A Ricci soliton is a generalization of an Einstein metric. We recall the notion of Ricci soliton according to [5]. On manifold M, a Ricci soliton is a triple (g, V, λ) with g a Riemannian metric, V a vector field, called potential vector field and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1}$$

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where $\pounds$ is the Lie derivative. Metrics satisfying (1) are interesting and useful in physics and are often referred as quasi-Einstein ([6],[7]).

The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive respectively. Ricci solitons have been studied by several authors such as ([21],[20],[12],[13]) and many others.

Ricci solitons have been generalized in several ways, such as almost Ricci solitons ([9],[16],[10]), η-Ricci solitons ([2],[3]), generalized Ricci soliton, $*$-Ricci solitons and many others.

As a generalization of Ricci soliton, Tachibana [17] introduced the notion of $*$-*Ricci tensor* on almost Hermitian manifolds. Later, in [11] Hamada studied $*$-Ricci flat real hypersurfaces in non-flat complex space forms and Blair [4] defined $*$-Ricci tensor in contact metric manifolds by

$$S^*(X,Y) = g(Q^*X,Y) = Trace\{\phi \circ R(X,\phi Y)\}, \tag{2}$$

where Q^* is called the $*$-*Ricci operator*.

Definition 1. *[15] A Riemannian (or semi-Riemannian) metric g on M is called $*$-Ricci soliton if*

$$\pounds_V g + 2S^* + 2\lambda g = 0, \tag{3}$$

where λ is a constant.

Definition 2. *[15] A Riemannian (or semi-Riemannian) metric g on M is called gradient $*$-Ricci soliton if*

$$\nabla\nabla f + S^* + \lambda g = 0, \tag{4}$$

where $\nabla\nabla f$ denotes the hessian of the smooth function f on M with respect to g and λ is a constant.

Definition 3. *A contact metric manifold of dimension $n > 2$ is called $*$-Einstein if the $*$-Ricci tensor S^* of type $(0,2)$ satisfies the relation*

$$S^* = \lambda g, \tag{5}$$

where λ is a constant.

If an $(\epsilon)-$para Sasakian manifold M satisfies relation (3), then we say that M admits a $*$-Ricci soliton.

The present paper focuses on the study of $(\epsilon)-$para Sasakian 3-manifolds M admitting a $*$-Ricci soliton. More precisely, the following theorems are proved.

Theorem 1. *If an $(\epsilon)-$para Sasakian 3-manifold $(M,\phi,\xi,\eta,g,\epsilon)$ with constant scalar curvature admits a $*$-Ricci soliton, then the $*$-Ricci soliton is steady if and only if $\pounds_V\xi$ is g-orthogonal to ξ provided $a = Tr\phi$ is constant.*

Theorem 2. *A gradient $*$-Ricci soliton with potential vector field of gradient type, $V = Df$, satisfying $\pounds_\xi f = 0$ on an $(\epsilon)-$para Sasakian 3-manifold $(M,\phi,\xi,\eta,g,\epsilon)$ is $*$-Einstein provided $a = Tr\phi$ is constant.*

2 Preliminaries

A $(2n+1)-$dimensional smooth manifold M together with a $(1,1)$-tensor field ϕ, a vector field ξ, a 1-form η and a semi-Riemannian metric g is called an $(\epsilon)-$*almost paracontact metric manifold* if

$$\phi^2 X = X - \eta(X)\xi, \;\; \eta(\xi) = 1, \tag{6}$$

$$g(\xi,\xi) = \epsilon, \;\; \eta(X) = \epsilon g(X,\xi), \tag{7}$$

$$g(\phi X, \phi Y) = g(X,Y) - \epsilon\eta(X)\eta(Y), \tag{8}$$

where ϵ is 1 or -1 accordingly as ξ is spacelike or timelike, and the rank of ϕ is $2n$. It is important to mention that in the above definition, ξ is never a lightlike vector field. It follows that $\phi\xi = 0$, $\eta \circ \phi = 0$ and $g(X,\phi Y) = g(\phi X, Y)$, for any $X, Y \in \chi(M)$.

If moreover, the manifold satisfies

$$(\nabla_X \phi)Y = -g(\phi X, \phi Y)\xi - \epsilon\eta(Y)\phi^2 X, \tag{9}$$

where ∇ denotes the Riemannian connection of g, then we shall call the manifold an $(\epsilon)-$*para Sasakian manifold* .

On an $(\epsilon)-$para Sasakian manifold $(M, \phi, \xi, \eta, g, \epsilon)$, the following relations hold [14]:

$$\nabla_X \xi = \epsilon\phi X, \tag{10}$$

$$(\nabla_X \eta)Y = \epsilon g(Y, \phi X), \tag{11}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{12}$$

$$\eta(R(X,Y)Z) = \epsilon[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)], \tag{13}$$

$$S(X,\xi) = -2n\eta(X), \tag{14}$$

$$Q\xi = -\epsilon 2n\xi, \tag{15}$$

where ∇, R, S and Q denote respectively, the Riemannian connection, the curvature tensor of type $(1,3)$, the Ricci tensor of type $(0,2)$ and the Ricci operator of type $(1,1)$.

Lemma 1. *In an* $(\epsilon)-$*para Sasakian manifold* $(M, \phi, \xi, \eta, g, \epsilon)$*, we have*

$$\begin{aligned} R(X,Y)\phi Z &= \phi R(X,Y)Z - \epsilon[g(Y,Z)\phi X - g(X,Z)\phi Y] \\ &+ \epsilon[g(Y,\phi Z)X - g(X,\phi Z)Y] \\ &+ 2\epsilon[g(X,\phi Z)\eta(Y) - g(Y,\phi Z)\eta(X)]\xi \\ &+ 2[\eta(Y)\phi X - \eta(X)\phi Y]\eta(Z). \end{aligned} \tag{16}$$

Proof. To prove the above Lemma we shall use equation (9).

Now

$$
\begin{aligned}
& R(X,Y)\phi Z \\
= \;& \nabla_X\nabla_Y\phi Z - \nabla_Y\nabla_X\phi Z - \nabla_{[X,Y]}\phi Z \\
= \;& \nabla_X(\phi(\nabla_Y Z) - g(Y,Z)\xi - \epsilon\eta(Z)Y + 2\epsilon\eta(Y)\eta(Z)\xi) \\
- \;& \nabla_Y(\phi(\nabla_X Z) - g(X,Z)\xi - \epsilon\eta(Z)X + 2\epsilon\eta(X)\eta(Z)\xi) \\
- \;& (\phi(\nabla_{[X,Y]}Z) - g([X,Y],Z)\xi - \epsilon\eta(Z)[X,Y] + 2\epsilon\eta([X,Y])\eta(Z)\xi) \\
= \;& \phi(\nabla_X\nabla_Y Z) - g(X,\nabla_Y Z)\xi - \epsilon\eta(\nabla_Y Z)X + 2\epsilon\eta(X)\eta(\nabla_Y Z)\xi \\
- \;& \nabla_X g(Y,Z)\xi - \epsilon g(Y,Z)\phi X - \epsilon\nabla_X\eta(Z)Y - \epsilon\eta(Z)\nabla_X Y \\
+ \;& 2\epsilon\nabla_X\eta(Y)\eta(Z)\xi + 2\epsilon\eta(Y)\nabla_X\eta(Z)\xi + 2\epsilon\eta(Y)\eta(Z)\phi X \\
- \;& \phi(\nabla_Y\nabla_X Z) + g(X,\nabla_X Z)\xi + \epsilon\eta(\nabla_X Z)Y - 2\epsilon\eta(Y)\eta(\nabla_X Z)\xi \\
+ \;& \nabla_Y g(X,Z)\xi + \epsilon g(X,Z)\phi Y + \epsilon\nabla_Y\eta(Z)X + \epsilon\eta(Z)\nabla_Y X \\
- \;& 2\epsilon\nabla_Y\eta(X)\eta(Z)\xi - 2\epsilon\eta(X)\nabla_Y\eta(Z)\xi - 2\epsilon\eta(X)\eta(Z)\phi Y \\
- \;& \phi(\nabla_{[X,Y]}Z) + g(\nabla_X Y,Z)\xi - g(\nabla_Y X,Z)\xi \\
+ \;& \epsilon\eta(Z)\nabla_X Y - \epsilon\eta(Z)\nabla_Y X - 2\epsilon\eta(\nabla_X Y)\eta(Z)\xi + 2\epsilon\eta(\nabla_Y X)\eta(Z)\xi \\
= \;& \phi R(X,Y)Z - \epsilon[g(Y,Z)\phi X - g(X,Z)\phi Y] \\
+ \;& \epsilon[g(Y,\phi Z)X - g(X,\phi Z)Y] \\
+ \;& 2\epsilon[g(X,\phi Z)\eta(Y) - g(Y,\phi Z)\eta(X)]\xi \\
+ \;& 2[\eta(Y)\phi X - \eta(X)\phi Y]\eta(Z).
\end{aligned}
$$

This completes the proof. □ □

Lemma 2. *In an* $(\epsilon)-$*para Sasakian manifold* $(M,\phi,\xi,\eta,g,\epsilon)$*, we have*

$$
\begin{aligned}
\tilde{R}(X,Y,\phi Z,\phi W) \;=\;& \tilde{R}(X,Y,Z,W) + 2g(Y,Z)\eta(X)\eta(W) - 2g(X,Z)\eta(Y)\eta(W) \\
-\;& \epsilon[g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\
-\;& g(X,\phi W)g(Y,\phi Z) + g(Y,\phi W)g(X,\phi Z)] \\
+\;& 2g(X,W)\eta(Y)\eta(Z) - 2g(Y,W)\eta(X)\eta(Z), \qquad (17)
\end{aligned}
$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$*, for* $X,Y,Z,W \in \chi(M)$*.*

Proof. To prove the above Lemma we shall use equations (8),(13) and (16).

Now

$$
\begin{aligned}
& \tilde{R}(X,Y,\phi Z,\phi W) \\
= \;& g(\phi R(X,Y)Z,\phi W)-\epsilon[g(Y,Z)g(\phi X,\phi W) \\
- \;& g(X,Z)g(\phi Y,\phi W)]-\epsilon[g(X,\phi Z)g(Y,\phi W)-g(Y,\phi Z)g(X,\phi W)] \\
- \;& 2\epsilon[g(Y,\phi Z)\eta(X)-g(X,\phi Z)\eta(Y)]\eta(\phi W) \\
+ \;& 2[g(\phi X,\phi W)\eta(Y)-g(\phi Y,\phi W)\eta(X)]\eta(Z) \\
= \;& g(R(X,Y)Z,W)-\epsilon\eta(R(X,Y)Z)\eta(W)+\epsilon[g(Y,Z)g(X,W) \\
- \;& \epsilon g(Y,Z)\eta(X)\eta(W)-g(X,Z)g(Y,W)+\epsilon g(X,Z)\eta(Y)\eta(W) \\
- \;& \epsilon[g(X,\phi Z)g(Y,\phi W)-g(Y,\phi Z)g(X,\phi W)] \\
+ \;& 2[g(X,W)\eta(Y)-g(Y,W)\eta(X)]\eta(Z) \\
= \;& \tilde{R}(X,Y,Z,W)+2g(Y,Z)\eta(X)\eta(W)-2g(X,Z)\eta(Y)\eta(W) \\
- \;& \epsilon[g(Y,Z)g(X,W)-g(X,Z)g(Y,W) \\
- \;& g(X,\phi W)g(Y,\phi Z)+g(Y,\phi W)g(X,\phi Z)] \\
+ \;& 2g(X,W)\eta(Y)\eta(Z)-2g(Y,W)\eta(X)\eta(Z).
\end{aligned}
$$

This completes the proof. □ □

For a 3-dimensional $(\epsilon)-$para Sasakian manifold $(M,\phi,\xi,\eta,g,\epsilon)$, we have

$$
\begin{aligned}
R(X,Y)Z \;=\;& g(Y,Z)QX-g(X,Z)QY+S(Y,Z)X-S(X,Z)Y \\
& -\frac{r}{2}[g(Y,Z)X-g(X,Z)Y], \qquad (18)
\end{aligned}
$$

for any $X,Y,Z\in\chi(M)$, where Q is the Ricci operator, that is, $g(QX,Y)=S(X,Y)$ and r is the scalar curvature of the manifold.

Putting $Z=\xi$ in (18) and using (12) we have

$$\eta(Y)QX-\eta(X)QY=\left(\frac{r}{2}+\epsilon\right)[\eta(Y)X-\eta(X)Y]. \qquad (19)$$

Again replacing Y by ξ in the foregoing equation and using (15), we get

$$QX=\left(\frac{r}{2}+\epsilon\right)X-\left(\frac{r}{2}+3\epsilon\right)\eta(X)\xi, \qquad (20)$$

which implies

$$S(X,Y)=\left(\frac{r}{2}+\epsilon\right)g(X,Y)-\left(\frac{r}{2}+3\epsilon\right)\epsilon\eta(X)\eta(Y). \qquad (21)$$

Now we prove the following Lemma which will be used later.

Lemma 3. *In an $(\epsilon)-$para Sasakian 3-manifold $(M,\phi,\xi,\eta,g,\epsilon)$, the $*$-Ricci tensor is given by*

$$S^*(X,Y)=S(X,Y)-[\epsilon g(X,Y)+ag(X,\phi Y)]+3\eta(X)\eta(Y), \qquad (22)$$

where S and S^* are the Ricci tensor and the $*$-Ricci tensor of type $(0,2)$, respectively and a=$Tr\phi$.

Proof. Let $\{e_i\}$, $i = 1, 2, 3$ be an orthonormal basis of the tangent space at each point of the manifold. From (2) and using (17), we infer

$$\begin{aligned}
S^*(Y,Z) &= \sum_{i=1}^{3} \tilde{R}(e_i, Y, \phi Z, \phi e_i) \\
&= \sum_{i=1}^{3} \{\tilde{R}(e_i, Y, Z, e_i) + 2g(Y,Z)\eta(e_i)\eta(e_i) - 2g(e_i, Z)\eta(Y)\eta(e_i) \\
&- \epsilon[g(Y,Z)g(e_i,e_i) - g(e_i,Z)g(Y,e_i) \\
&- g(e_i,\phi e_i)g(Y,\phi Z) + g(Y,\phi e_i)g(e_i,\phi Z)] \\
&+ 2g(e_i,e_i)\eta(Y)\eta(Z) - 2g(Y,e_i)\eta(e_i)\eta(Z)\} \\
&= S(Y,Z) - \epsilon[g(Y,Z) + ag(Y,\phi Z)] + 3\eta(Y)\eta(Z).
\end{aligned}$$

Hence, the $*$-Ricci tensor is

$$S^*(Y,Z) = S(Y,Z) - \epsilon[g(Y,Z) + ag(Y,\phi Z)] + 3\eta(Y)\eta(Z),$$

for any $Y, Z \in \chi(M)$. This completes the proof. □ □

From the above Lemma, the $(1,1)$ $*$-Ricci operator Q^* and the $*$-scalar curvature r^* are given by

$$Q^*X = QX - \epsilon(X + a\phi X) + 3\epsilon\eta(X)\xi, \tag{23}$$

$$r^* = r - 4\epsilon a^2. \tag{24}$$

Hereafter, unless otherwise stated, let us assume that $a =$Trϕ is constant.

3 Proof of the main theorems

In view of equation (21), the $*$-Ricci tensor is given by

$$S^*(X,Y) = \frac{r}{2}g(X,Y) - \frac{r}{2}\epsilon\eta(X)\eta(Y) - a\epsilon g(X,\phi Y). \tag{25}$$

Again from the equation of $*$-Ricci soliton we have

$$\begin{aligned}
(\pounds_V g)(X,Y) &= -2S^*(X,Y) - 2\lambda g(X,Y) \\
&= -(r + 2\lambda)g(X,Y) + r\epsilon\eta(X)\eta(Y) + 2a\epsilon g(X,\phi Y).
\end{aligned} \tag{26}$$

Taking the covariant derivative with respect to Z, we get

$$\begin{aligned}
(\nabla_Z \pounds_V g)(X,Y) = &-(Zr)g(\phi X, \phi Y) \\
&+ r[g(\phi X, Z)\eta(Y) + g(\phi Y, Z)\eta(X)].
\end{aligned} \tag{27}$$

Following Yano ([19], pp. 23), the following formula holds

$$\begin{aligned}
&(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y,Z) \\
&\qquad = -g((\pounds_V \nabla)(X,Y), Z) - g((\pounds_V \nabla)(X,Z), Y),
\end{aligned}$$

for any $X, Y, Z \in \chi(M)$. As g is parallel with respect to the Levi-Civita connection ∇, the above relation becomes

$$(\nabla_X \pounds_V g)(Y, Z) = g((\pounds_V \nabla)(X, Y), Z) + g((\pounds_V \nabla)(X, Z), Y), \tag{28}$$

for any $X, Y, Z \in \chi(M)$. Since $\pounds_V \nabla$ is a symmetric tensor of type $(1, 2)$, that is, $(\pounds_V \nabla)(X, Y) = (\pounds_V \nabla)(Y, X)$, then it follows from (28) that

$$\begin{aligned} &g((\pounds_V \nabla)(X, Y), Z) \\ &= \frac{1}{2}(\nabla_X \pounds_V g)(Y, Z) + \frac{1}{2}(\nabla_Y \pounds_V g)(X, Z) - \frac{1}{2}(\nabla_Z \pounds_V g)(X, Y). \end{aligned} \tag{29}$$

Using (27) in (29) yields

$$\begin{aligned} 2g((\pounds_V \nabla)(X, Y), Z) = &-(Xr)g(\phi Y, \phi Z) \\ &+ r[g(X, \phi Y)\eta(Z) + g(X, \phi Z)\eta(Y)] \\ &- (Yr)g(\phi X, \phi Z) \\ &+ r[g(\phi X, Y)\eta(Z) + g(Y, \phi Z)\eta(X) \\ &+ (Zr)g(\phi X, \phi Y) \\ &- r[g(\phi X, Z)\eta(Y) + g(\phi Y, Z)\eta(X). \end{aligned} \tag{30}$$

Removing Z from (30), it follows that

$$\begin{aligned} 2(\pounds_V \nabla)(X, Y) = &-(Xr)\{Y - \epsilon\eta(Y)\xi\} \\ &+ r[g(X, \phi Y)\xi + \phi X\eta(Y)] \\ &- (Yr)\{X - \epsilon\eta(X)\xi\} \\ &+ r[g(\phi X, Y)\xi + \phi Y\eta(X) \\ &+ (Dr)g(\phi X, \phi Y) \\ &- r[\phi X\eta(Y) + \phi Y\eta(X), \end{aligned} \tag{31}$$

where $(X\alpha) = g(D\alpha, X)$, for D the gradient operator with respect to g. Substituting $Y = \xi$ in the foregoing equation and using $r =$ constant (hence, $(Dr) = 0$ and (ξr)=0), we have

$$(\pounds_V \nabla)(X, \xi) = 0. \tag{32}$$

Taking the covariant derivative of (32) with respect to Y, we infer

$$(\nabla_Y \pounds_V \nabla)(X, \xi) = 0. \tag{33}$$

Again from [19]

$$(\pounds_V R)(X, Y, Z) = (\nabla_X \pounds_V \nabla)(Y, Z) - (\nabla_Y \pounds_V \nabla)(X, Z). \tag{34}$$

Therefore (33) and (34) yield

$$(\pounds_V R)(X, Y, \xi) = 0, \tag{35}$$

for any $X, Y \in \chi(M)$. Setting $Y = \xi$ in (26) it follows that $(\pounds_V g)(X, \xi) = -2\lambda\epsilon\eta(X)$. Lie-differentiating the equation (7) along V and by virtue of the last equation we have

$$(\pounds_V \eta)(X) - \epsilon g(\pounds_V \xi, X) + 2\lambda\eta(X) = 0. \tag{36}$$

Putting $X = \xi$ in the foregoing equation gives

$$\lambda = \eta(\pounds_V \xi). \tag{37}$$

Thus, we can say that the $*$-Ricci soliton is steady if and only if $\pounds_V \xi$ is g-orthogonal to ξ. This completes the proof of Theorem 1.1. □

Let $(M, \phi, \xi, \eta, g, \epsilon)$ be an $(\epsilon)-$para Sasakian 3-manifold with g as a gradient $*$-Ricci soliton. Then equation (4) can be written as

$$\nabla_X Df + Q^* X + \lambda X = 0, \tag{38}$$

for any $X \in \chi(M)$, where D denotes the gradient operator with respect to g. From (38) it follows that

$$R(X, Y)Df = (\nabla_Y Q^*)X - (\nabla_X Q^*)Y, \quad X, Y \in \chi(M). \tag{39}$$

Using (12), we have

$$g(R(\xi, X)Df, \xi) = \eta(X)(\xi f) - \epsilon(Xf). \tag{40}$$

With the help of (25), we have

$$\begin{aligned}(\nabla_X Q^*)Y = &\frac{(Xr)}{2}[Y - \epsilon\eta(Y)\xi] \\ &- \frac{r}{2}[g(X, \phi Y)\xi + \eta(Y)\phi X] \\ &+ a\epsilon[g(X, Y)\xi - 2\epsilon\eta(X)\eta(Y)\xi - \epsilon\eta(Y)X].\end{aligned} \tag{41}$$

Interchanging X and Y, we have

$$\begin{aligned}(\nabla_Y Q^*)X = &\frac{(Yr)}{2}[X - \epsilon\eta(X)\xi] \\ &- \frac{r}{2}[g(Y, \phi X)\xi + \eta(X)\phi Y] \\ &+ a\epsilon[g(X, Y)\xi - 2\epsilon\eta(X)\eta(Y)\xi - \epsilon\eta(X)Y].\end{aligned} \tag{42}$$

Making use of (41) and (42) we get

$$\begin{aligned}(\nabla_Y Q^*)X - (\nabla_X Q^*)Y = &\frac{(Xr)}{2}[Y - \epsilon\eta(Y)\xi] - \frac{(Yr)}{2}[X - \epsilon\eta(X)\xi] \\ &- \frac{r}{2}[\eta(Y)\phi X - \eta(X)\phi Y] + a[\eta(Y)X - \eta(X)Y].\end{aligned} \tag{43}$$

Putting $X = \xi$ in (43) and taking inner product with ξ, we infer that

$$g((\nabla_Y Q^*)\xi - (\nabla_\xi Q^*)Y, \xi) = 0, \tag{44}$$

for any $Y \in \chi(M)$. From (40) and (44) we get

$$\epsilon(Xf) = \eta(X)(\xi f), \tag{45}$$

for any $X \in \chi(M)$. Therefore, $Df = (\xi f)\xi$. Taking the covariant derivative with respect to X and using (38) it follows that

$$S^*(X, Y) = -[\lambda + (\xi f)\xi]g(X, Y) - \epsilon(\xi f)g(\phi X, Y), \tag{46}$$

for any $X, Y \in \chi(M)$. This completes the proof of Theorem 1.2. □

Also remark that if we assume $\pounds_\xi f = 0$, from (45) we obtain that f is a constant function.

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