Bulletin of the *Transilvania* University of Braşov • Vol 12(61), No. 2 - 2019
Series III: Mathematics, Informatics, Physics, 247-264
https://doi.org/10.31926/but.mif.2019.12.61.2.5

ON THE GEOMETRY OF THE TANGENT BUNDLE WITH VERTICAL RESCALED GENERALIZED CHEEGER-GROMOLL METRIC

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Abstract

Let (M, g) be an n-dimensional smooth Riemannian manifold. In the present paper, we introduce a new class of natural metrics denoted by G^f and called the vertical rescaled generalized Cheeger-Gromoll metric on the tangent bundle TM. We calculate its Levi-Civita connection and Riemannian curvature tensor. We study the geometry of (TM, G^f) .

2000 *Mathematics Subject Classification:* 58A03, 58A05.
Key words: horizontal lift, vertical lift, Cheeger-Gromoll metric, tangent bundle, harmonic map.

1 Introduction

We recall some basic facts about the geometry of the tangent bundle. In the present paper, we denote by $\Gamma(TM)$ the space of all vector fields of a Riemannian manifold (M, g). Let (M, g) be an n-dimensional Riemannian manifold and (TM, π, M) be its tangent bundle. A local chart $(U, x^i)_{i=1...n}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=1...n}$ on TM. Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.
We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by

$$\begin{aligned}\mathcal{V}_{(x,u)} &= \ker(d\pi_{(x,u)}) \\ &= \left\{a^i \frac{\partial}{\partial y^i}_{|(x,u)}; a^i \in \mathbb{R}\right\}\end{aligned}$$

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$$\mathcal{H}_{(x,u)} = \Big\{\frac{\partial}{\partial x^i}|_{(x,u)} - a^i u^j \Gamma^k_{ij}\frac{\partial}{\partial y^k}|_{(x,u)}; a^i \in \mathbb{R}\Big\}$$

where $(x,u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.
Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i} \tag{1}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i\Big\{\frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij}\frac{\partial}{\partial y^k}\Big\} \tag{2}$$

Consequently, we have $\Big(\frac{\partial}{\partial x^i}\Big)^H = \frac{\delta}{\delta x^i}$ and $\Big(\frac{\partial}{\partial x^i}\Big)^V = \frac{\partial}{\partial y^i}$, then $\Big(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\Big)_{i=1..n}$ is a local adapted frame in TTM. The tangent bundle TM of a Riemannian manifold (M,g) can be endowed in a natural way with a Riemannian metric g^s, the Sasaki metric, depending only on the Riemannian structure g of the base manifold M. It is uniquely determined by

$$\begin{aligned} g^s(X^H,Y^H) &= g(X,Y)\circ\pi \\ g^s(X^H,Y^V) &= 0 \\ g^s(X^V,Y^V) &= g(X,Y)\circ\pi \end{aligned} \tag{3}$$

for all vector fields X and Y on M. More intuitively, the metric g^s is constructed in such a way that the vertical and horizontal sub bundles are orthogonal and the bundle map $\pi : (TM, g^s) \longrightarrow (M, g)$ is a Riemannian submersion.
The geometry of the tangent bundle TM equipped with Sasaki metric has been studied by many authors K. Yano and S. Ishihara [23], A. Salimov, A. Gezer and N. Cengiz (see [4], [14], [15], [21]) etc. The rigidity of Sasaki metric have incited some geometers to construct and study other metrics on TM. J. Cheeger and D. Gromoll have introduced the notion of Cheeger-Gromoll [5]. It is uniquely determined by

$$\begin{aligned} g_{CG}(X^H,Y^H) &= g(X,Y)\circ\pi \\ g_{CG}(X^H,Y^V) &= 0 \\ g_{CG}(X^V,Y^V) &= \frac{1}{\alpha}\{g(X,Y)+g(X,u)g(Y,u)\}\circ\pi \end{aligned} \tag{4}$$

Where $X, Y \in \Gamma(TM)$, $(x,u) \in TM$, $\alpha = 1 + g_x(u,u)$.
M. Benyounes, E. Loubeau, and C. M. Wood in [3] introduced the geometry of the tangent bundle equipped with a two-parameter family of Riemannian metrics is called generalized Cheeger-Gromoll metric given by

$$\begin{aligned} G(X^H,Y^H)_{(x,u)} &= g_x(X,Y) \\ G(X^H,Y^V)_{(x,u)} &= 0 \\ G(X^V,Y^V)_{(x,u)} &= \omega^p(g_x(X,Y)+qg_x(X,u)g_x(Y,u)) \end{aligned} \tag{5}$$

for all vectors fields $X, Y \in \Gamma(TM)$, $r^2 = \|u\| = \sqrt{g(u,u)}$, where $\omega = (1 + \|u\|^2)^{-1}$, $p, q \in \mathbb{R}$ and q positive ensure non-degeneracy.

Dida H.M, Hathout F in [8], we define a new class of naturally metric on TM given by

$$\begin{aligned} G^f_{(p,u)}(X^H, Y^H) &= g_p(X,Y) \\ G^f_{(p,u)}(X^H, Y^V) &= 0 \\ G^f_{(p,u)}(X^V, Y^V) &= f(p) g_p(X,Y) \end{aligned} \tag{6}$$

for some strictly positive smooth function f in (M, g) and any vector fields X and Y on M. We call G^f vertical rescaled metric.
Motivated by the above studies, we define a new class of naturally metric on TM given by

$$\begin{aligned} G^f(X^H, Y^H)_{(x,u)} &= g_x(X,Y) \\ G^f(X^H, Y^V)_{(x,u)} &= 0 \\ G^f(X^V, Y^V)_{(x,u)} &= f(x)\omega^p(g_x(X,Y) + q g_x(X,u) g_x(Y,u)) \end{aligned}$$

where f be strictly positive smooth function on M and any vector fields X and Y on M. For $f = 1$ the metric G^f is exactly the generalized Cheeger-Gromoll metric. We call G^f the vertical rescaled generalized Cheeger-Gromoll metric.

In this paper, we introduce the vertical rescaled generalized Cheeger-Gromoll metric on the tangent bundle TM as a new natural metric non-rigid on TM. First we investigate the geometry of the vertical rescaled generalized Cheeger-Gromoll metric and we characterize the sectional curvature (Proposition 3.1) and the scalar curvature (Proposition 3.3).

2 Vertical rescaled generalized Cheeger-Gromoll metric

Definition 2.1. *Let (M, g) be a Riemannian manifold and $f : M \to]0, +\infty[$ be a strictly positive smooth function. We define the vertical rescaled generalized Cheeger-Gromoll metric G^f on the tangent bundle TM by*

$$\begin{aligned} G^f(X^H, Y^H)_{(x,u)} &= g_x(X,Y) \\ G^f(X^H, Y^V)_{(x,u)} &= 0 \\ G^f(X^V, Y^V)_{(x,u)} &= f(x)\omega^p(g_x(X,Y) + q g_x(X,u) g_x(Y,u)) \end{aligned}$$

for all vector fields $X, Y \in \Gamma(TM)$, and $r^2 = \|u\| = \sqrt{g(u,u)}$, where $\omega = (1 + \|u\|^2)^{-1}$, $p, q \in \mathbb{R}$ and q positive ensure non-degeneracy.

Remark 2.1.

1. *If* $f = 1, p = q = 1$, *then* G^f *is the Cheeger-Gromoll metric.*
2. *If* $f = 1$, *then* G^f *is the generalized Cheeger-Gromoll metric.*
3. *If* $f = 1, p = q = 0$, *then* G^f *is the Sasaki metric.*
4. $G^f(X^V, U^V) = f\omega^p g(X, u)(1 + qr^2)$.
5. $G^f(U^V, U^V) = f\omega^p r^2(1 + qr^2)$.

where $X, U \in \Gamma(TM)$ *and* $U_x = u = u^i \dfrac{\partial}{\partial x_i} \in T_xM$ *and* $(x, u) \in TM$.

Lemma 2.1. *([3]) . Let* (M, g) *be a Riemannian manifold and* TM *be the tangent bundle of* M*. Then, for each* $(x, u) \in TM$ *and every real valued function* h *on* M*, we have the following:*

1. $X^H_{(x,u)}(h(r^2)) = 0$,
2. $X^V_{(x,u)}(h(r^2)) = 2h'(r^2)g(X, u)$,
3. $X^V(g(X, u) = g(X, Y)$,
4. $X^H(g(Y, u) = g(\nabla_X Y, u)$,
5. $X^H(g(Y, Z) = X(g(Y, Z))$.

for all vector fields $X, Y, Z \in \Gamma(TM)$.

Lemma 2.2. *Let* (M, g) *be a Riemannian manifold and* (TM, G^f) *its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric, then we have*

$$\begin{aligned} X^H(G^f(Y^V, Z^V)) &= \frac{X(f)}{f} G^f(Y^V, Z^V) \\ &\quad + G^f((\nabla_X Y)^V, Z^V) + G^f((\nabla_X Z)^V, Y^V) \\ X^V(G^f(Y^V, Z^V)) &= -2pf\omega^{p+1} g(X, u)\Big[g(Y, Z) + g(Y, u)g(Z, u)\Big] \\ &\quad + qf\omega^p \Big[g(X, Y)g(Z, u) + g(X, Z)g(Y, u)\Big] \end{aligned}$$

for all $X, Y, Z \in \Gamma(TM)$.

Proof. Using Lemma 2.1, we have

- $$\begin{aligned} X^H(G^f(Y^V,Z^V)) &= X^H(f)\omega^p\Big[g(Y,Z)+qg(Y,u)g(Z,u)\Big] \\ &\quad +f\omega^p X^H\Big[g(Y,Z)+qg(Y,u)g(Z,u)\Big] \\ &= X(f)\omega^p\Big[g(Y,Z)+qg(Y,u)g(Z,u)\Big] \\ &\quad +f\omega^p\Big[g(\nabla_X Y,Z)+g(Y,\nabla_X Z) \\ &\qquad +qg(\nabla_X Y,u)g(Z,u)+qg(\nabla_X Z,u)g(Y,u)\Big] \\ &= \frac{X(f)}{f}G^f(Y^V,Z^V)+G^f((\nabla_X Y)^V,Z^V) \\ &+ G^f((\nabla_X Z)^V,Y^V) \end{aligned}$$

- $$\begin{aligned} X^V(G^f(Y^V,Z^V)) &= X^V(\omega^p)f\Big[g(Y,Z)+qg(Y,u)g(Z,u)\Big] \\ &\quad ++f\omega^p X^V\Big[g(Y,Z)+qg(Y,u)g(Z,u)\Big] \\ &= -2pf\omega^{p+1}g(X,u)\Big[g(Y,Z)+g(Y,u)g(Z,u)\Big] \\ &\quad +qf\omega^p\Big[g(X,Y)g(Z,u)+g(X,Z)g(Y,u)\Big] \end{aligned}$$

□

2.1 Levi-Civita connection of G^f

Lemma 2.3. *Let (M,g) be a Riemannian manifold. If ∇ (resp. $\overline{\nabla}$) denote the Levi-Civita connection of (M,g) (resp. (TM,G^f)), then we have*

1. $G^f(\overline{\nabla}_{X^H}Y^H,Z^H)=G^f((\nabla_X Y)^H,Z^H)$
2. $G^f(\overline{\nabla}_{X^H}Y^H,Z^V)=-\dfrac{1}{2}G^f(Z^V,(R(X,Y)u)^V)$
3. $G^f(\overline{\nabla}_{X^H}Y^V,Z^H)=G^f(f\omega^p(R(u,Y)X)^H,Z^H)$
4. $G^f(\overline{\nabla}_{X^H}Y^V,Z^V)=G^f(\frac{X(f)}{f}Y^V+2(\nabla_X Y)^V,Z^V)$
5. $G^f(\overline{\nabla}_{X^V}Y^H,Z^H)=G^f(\dfrac{f}{2}\omega^p(R(u,X)Y)^H,Z^H)$
6. $G^f(\overline{\nabla}_{X^V}Y^H,Z^V)=\dfrac{Y(f)}{f}G^f\Big(Z^V,X^V)$
7. $G^f(\overline{\nabla}_{X^V}Y^V,Z^H)=G^f(-\dfrac{1}{f}G^f(X^V,Y^V)(gradf)^H,Z^H)$

8. $G^f(\overline{\nabla}_{X^V}Y^V,Z^V) = -2\dfrac{p\omega^{-p+1}}{f(1+qr^2)}G^f(\Big[G^f(X^V,U^V)Y^V + G^f(Y^V,U^V)X^V\Big],Z^V)$

$$+2\frac{(p\omega+q)\omega^{-p}}{f(1+qr^2)}G^f(X^V,Y^V)G^f(Z^V,U^V)$$

$$-2\frac{q^2\omega^{-2p}}{f^2(1+qr^2)^3}G^f(X^V,U^V)G^f(Y^V,U^V)G^f(Z^V,U^V)$$

for all vector fields $X,Y,Z\in\Gamma(TM)$.

Proof. We shall repeatedly make use of the Koszul formula for the Levi-Civita connection $\overline{\nabla}$ stating that

$$\begin{aligned} 2G^f(\overline{\nabla}_{X^i}Y^j,Z^k) &= X^i(G^f(Y^j,Z^k)) + Y^j(G^f(Z^k,X^i)) - Z^k(G^f(X^i,Y^j)) \\ &- G^f(X^i,[Y^j,Z^k]) + G^f(Y^j,[Z^k,X^i]) + G^f(Z^k,[X^i,Y^j]) \end{aligned}$$

for all vector fields $X,Y,Z\in\Gamma(TM)$ and $i,j,k\in\{H,V\}$.
The result is a direct consequence of the following calculations using Definition 2.1 and Lemma 2.2

$$\begin{aligned}
2G^f(\overline{\nabla}_{X^H}Y^H,Z^H) &= X(g(Y,Z)) + Y(g(Z,X)) - Zg(g(X,Y)) \\
&\quad -g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]) \\
&= 2g(\nabla_X Y\, Z) \\
&= 2G^f((\nabla_X Y)^H,Z^H) \\
2G^f(\overline{\nabla}_{X^H}Y^H,Z^V) &= G^f(Z^V,[X^H,Y^H]) \\
&= -G^f(Z^V,(R(X,Y)u)^V) \\
2G^f(\overline{\nabla}_{X^H}Y^V,Z^H) &= G^f(Y^V,[Z^H,X^H]) \\
&= -G^f(Y^V,(R(Z,X)u)^V) \\
&= G^f((R(X,Z)u)^V,Y^V) \\
&= f\omega^p(g(R(X,Z)u,Y) + qg(R(X,Z)u,u).g(Y,u)) \\
&= G^f(f\omega^p(R(u,Y)X)^H,Z^H) \\
2G^f(\overline{\nabla}_{X^H}Y^V,Z^V) &= X^H(G^f(Y^V,Z^V)) \\
&\quad -G^f(Y^V,(\nabla_X Z)^V) + G^f(Z^V,(\nabla_X Y)^V) \\
&= G^f(\frac{X(f)}{f}Y^V + 2(\nabla_X Y)^V,Z^V) \\
2G^f(\overline{\nabla}_{X^V}Y^H,Z^H) &= -G^f(X^V,[Y^H,Z^H]) \\
&= G^f(X^V,(R(Y,Z)u)^V) \\
&= f\omega^p\Big[g(R(u,X)Y,Z) + qg(R(Y,Z)u,u)g(X,u)\Big] \\
&= G^f(f\omega^p(R(u,X)Y)^H,Z^H) \\
2G^f(\overline{\nabla}_{X^V}Y^H,Z^V) &= Y^H(G^f(Z^V,X^V)) \\
&\quad - G^f(X^V,(\nabla_Y Z)^V) - G^f(Z^V,(\nabla_Y X)^V)
\end{aligned}$$

$$\begin{aligned}
&= Y^H(G^f(X^V,Z^V)) + G^f(X^V,(\nabla_Y Z)^V) + G^f(Z^V,(\nabla_Y X)^V) \\
&\quad - G^f(X^V,(\nabla_Y Z)^V) - G^f(Z^V,(\nabla_Y X)^V) \\
&= \frac{Y(f)}{f} G^f(Z^V,X^V)
\end{aligned}$$

$$\begin{aligned}
2G^f(\overline{\nabla}_{X^V}Y^V,Z^H) &= -Z^H(G^f\Big(X^V,Y^V)) \\
&+ G^f(X^V,(\nabla_Z Y)^V) + G^f(Y^V,(\nabla_Z X)^V) \\
&= -\frac{Z(f)}{f} G^f(X^V,Y^V) \\
&= G^f(-\frac{1}{f}G^f(X^V,Y^V)(gradf)^H,Z^H)
\end{aligned}$$

From the lemma 2.1, we have

$$\begin{aligned}
2G^f(\overline{\nabla}_{X^V}Y^V,Z^V) &= X^V(G^f(Y^V,Z^V) + Y^V(G^f(Z^V,X^V)) - Z^V(G^f(X^V,Y^V)) \\
&= -2fp\omega^{p+1}g(X,u)\left(g(Y,Z) + qg(Y,u)g(Z,u)\right) \\
&\quad + qf\omega^p(g(X,Y)g(Z,u) + g(X,Z)g(Y,u)) \\
&\quad - 2fp\omega^{p+1}g(Y,u)\left(g(X,Z) + qg(X,u)g(Z,u)\right) \\
&\quad + qf\omega^p(g(X,Y)g(Z,u) + g(Y,Z)g(X,u)) \\
&\quad + 2fp\omega^{p+1}g(Z,u)\left(g(Y,X) + qg(Y,u)g(X,u)\right) \\
&\quad - qf\omega^p(g(Y,Z)g(X,u) + g(X,Z)g(Y,u)) \\
&= -2fpq\omega^{p+1}g(X,u)g(Y,u)g(Z,u) \\
&\quad + 2f\omega^p(p\omega+q)g(X,Y)g(Z,u) \\
&\quad - 2fp\omega^{p+1}\Big[g(X,u)g(Y,Z) + g(Y,u)g(X,Z)\Big]
\end{aligned}$$

$$\begin{aligned}
2G^f(\nabla^f_{X^V}Y^V,Z^V) &= -\frac{2p\omega^{-p+1}}{f(1+qr^2)}\Big[G^f(X^V,U^V)Y^V + G^f(Y^V,U^V)X^V\Big] \\
&\quad + \frac{2(p\omega+q)\omega^{-p}}{f(1+qr^2)} G^f(X^V,Y^V) \\
&\quad - \frac{2q^2\omega^{-2p}}{f^2(1+qr^2)^3} G^f(X^V,U^V)G^f(Y^V,U^V)U^V
\end{aligned}$$

□

Using Lemma 2.3, we have

Theorem 2.1. *Let (M,g) be a Riemannian manifold and $\overline{\nabla}$ be the Levi-Civita connection of the tangent bundle (TM,G^f). Then we have*

$$(\overline{\nabla}_{X^H}Y^H)_{(x,u)} = \Big(\nabla_X Y\Big)^H - \frac{1}{2}\Big(R(X,Y)u\Big)^V$$

$$\begin{aligned}
(\overline{\nabla}_{X^H}Y^V)_{(x,u)} &= \frac{f}{2}\omega^p\Big(R(u,Y)X\Big)^H + \frac{X(f)}{2f}Y^V + \Big(\nabla_X Y\Big)^V \\
(\overline{\nabla}_{X^V}Y^H)_{(x,u)} &= \frac{f}{2}\omega^p\Big(R(u,X)Y\Big)^H + \frac{Y(f)}{2f}X^V \\
(\overline{\nabla}_{X^V}Y^V)_{(x,u)} &= -\frac{p\omega^{-p+1}}{f(1+qr^2)}\Big[G^f(X^V,U^V)Y^V + G^f(Y^V,U^V)X^V\Big] \\
&\quad +\frac{(p\omega+q)\omega^{-p}}{f(1+qr^2)}G^f(X^V,Y^V) \\
&\quad -\frac{q^2\omega^{-2p}}{f^2(1+qr^2)^3}G^f(X^V,U^V)G^f(Y^V,U^V)U^V \\
&\quad -\frac{1}{2f}G^f(X^V,Y^V)\Big(gradf\Big)^H
\end{aligned}$$

for all vector fields $X, Y \in \Gamma(TM)$ and $(x,u) \in TM$, where R denote the curvature tensor of (M,g).

Definition 2.2. *Let (M,g) be a Riemannian manifold and $K : TM \to TTM$ be a smooth endomorphism bundle of the tangent bundle TM. Then we define the vertical and horizontal lifts $K^V : TM \to TTM, K^H : TM \to TTM$ of K by*

$$K^V(\eta) = \sum_{i=1}^{m} \eta_i K(\partial i)^V$$

and

$$K^H(\eta) = \sum_{i=1}^{m} \eta_i K(\partial i)^H$$

where $\sum_{i=1}^{m} \eta_i \partial i \in \pi^{-1}(V)$ is a local representation of $\eta \in \mathcal{C}^\infty(TM)$.

From Definition 2.2 and Theorem 2.1, we have

Proposition 2.1. *Let (M,g) be a Riemannian manifold and $\overline{\nabla}$ be a the Levi-Civita connection of the tangent bundle (TM, G^f). If K is a tensor field of type $(1,1)$ on M, then*

$$\begin{aligned}
(\overline{\nabla}_{X^H}K^H)_{(x,u)} &= \Big(\nabla_X K\Big)^H - \frac{1}{2}\Big(R(X,K(u))u\Big)^V \\
(\overline{\nabla}_{X^H}K^V)_{(x,u)} &= \frac{f}{2}\omega^p\Big(R(u,K(u))X\Big)^H + \frac{X(f)}{2f}K(u)^V + \Big(\nabla_X K(u)\Big)^V \\
(\overline{\nabla}_{X^V}K^H)_{(x,u)} &= \Big(K(X)\Big)^H + \frac{f}{2}\omega^p\Big(R(u,X)K(u)\Big)^H + \frac{1}{2f}g(K(u), gradf)X^V \\
(\overline{\nabla}_{X^V}K^V)_{(x,u)} &= -\frac{p\omega^{-p+1}}{f(1+qr^2)}\Big[G^f(X^V,U^V)K(u)^V + G^f(K(u)^V,U^V)X^V\Big] \\
&\quad +\frac{(p\omega+q)\omega^{-p}}{f(1+qr^2)}G^f(X^V,K(u)^V) + \Big(K(X)\Big)^V
\end{aligned}$$

$$-\frac{q^2\omega^{-2p}}{f^2(1+qr^2)^3}G^f(X^V,U^V)G^f(K(u)^V,U^V)U^V$$
$$-\frac{1}{2f}G^f(X^V,K(u)^V)\Big(gradf\Big)^H$$

for any $X,Y\in\Gamma(TM)$ *and* $(p,u)\in TM$.

3 Curvature tensor of vertical rescaled generalized Cheeger-Gromoll metric

Using Theorem 2.1 and Proposition 2.1 and formula of curvature, we have

Theorem 3.1. *Let* (M,g) *be a Riemannian manifold and* (TM,G^f) *its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. If* R *(resp* $\overline{R}$*) denote the Riemann curvature tensor of* M *(resp*(TM,G^f)*), then for all* $X,Y,Z\in\Gamma(TM)$ *and* $p=(x,u)\in TM$*, we have the following formulas*

$$\begin{aligned}
\overline{R}(X^H,Y^H)Z^H &= \Big(R(X,Y)Z\Big)^H+\frac{f\omega^p}{2}\Big(R(u,R(X,Y)u)Z\Big)^H\\
&+\frac{f\omega^p}{4}\Big(R(u,R(X,Z)u)Y\Big)^H-\frac{f\omega^p}{4}\Big(R(u,R(Y,Z)u)X\Big)^H\\
&+\frac{1}{2}\Big(\nabla_Z R)(X,Y)u\Big)^V-\frac{X(f)}{4f}\Big(R(u,R(Y,Z)u)\Big)^V\\
&+\frac{Y(f)}{4f}\Big(R(u,R(X,Z)u)\Big)^V+\frac{Z(f)}{2f}\Big(R(X,Y)u\Big)^V\\
\overline{R}(X^H,Y^V)Z^V &= -\frac{\omega^p}{2}\Big[g(Y,Z)+qg(Y,u)g(Z,u)\Big]\Big(\nabla_X gradf\Big)^H\\
&+\frac{\omega^p}{4}\Big[g(Y,Z)+qg(Y,u)g(Z,u)\Big]\Big(R(X,gradf)u\Big)^V\\
&-\frac{fp\omega^{p+1}}{2}\Big[-g(Y,u)\Big(R(u,Z)X\Big)^H+g(Z,u)\Big(R(u,Y)X\Big)^H\Big]\\
&-\frac{f\omega^p}{2}\Big(R(Y,Z)X\Big)^H\\
&-\frac{f^2\omega^{2p}}{4}\Big(R(u,Y)R(u,Z)X\Big)^H-\frac{\omega^p}{4}g(R(u,Z)X,gradf)Y^V\\
&+X(f)\frac{\omega^p}{4f}\Big[g(Y,Z)+qg(Y,u)g(Z,u)\Big]\Big(grad_M f\Big)^H
\end{aligned}$$

$$\begin{aligned}
\overline{R}(X^V,Y^V)Z^H &= \frac{f\omega^p}{2}\Big(R(X,Y)Z\Big)^H\\
&-pf\omega^{p+1}\Big[g(X,u)\Big(R(u,Y)Z\Big)^H-g(Y,u)\Big(R(u,X)Z\Big)^H\Big]
\end{aligned}$$

$$+\frac{f^2\omega^{2p}}{4}\Big[\Big(R(u,X)R(u,Y)Z\Big)^H - \Big(R(u,Y)R(u,X)Z\Big)^H\Big]$$
$$+\frac{\omega^p}{4}\Big[g(R(u,Y)Z, gradf)X^V - g(R(u,X)Z, gradf)Y^V\Big]$$

$$\begin{aligned}
\overline{R}(X^H, Y^V)Z^H &= \frac{\omega^p}{2}X(f)\Big(R(u,Y)Z\Big)^H + \frac{\omega^p}{4}Z(f)\Big(R(u,Y)X\Big)^H \\
&+\frac{f\omega^p}{2}\Big((\nabla_X R)(u,Y)Z\Big)^H - \frac{\omega^p}{4}g(Y, R(X,Z)u)\Big(gradf\Big)^H \\
&+\frac{1}{2}\Big(R(X,Z)Y\Big)^V - \frac{f\omega^p}{4}\Big(R(X, R(u,Y)Z)u\Big)^V \\
&+\Big[\frac{1}{2f}X(Z(f)) - \frac{1}{4f^2}X(f)Z(f) - \frac{1}{2f}(\nabla_X Z)(f)\Big]Y^V \\
&-\frac{p\omega}{2}g(Y,u)\Big(R(X,Z)u\Big)^V \\
&+\frac{p\omega+q}{2(1+qr^2)}g(Y, R(X,Z)u)U^V
\end{aligned}$$

$$\begin{aligned}
\overline{R}(X^H Y^H)Z^V &= \frac{f\omega^p}{2}\Big((\nabla_X R)(u,Z)Y\Big)^H - \frac{f\omega^p}{2}\Big((\nabla_Y R)(u,Z)X\Big)^H \\
&+\frac{\omega^p}{4}X(f)\Big(R(u,Z)Y\Big)^H - \frac{\omega^p}{4}Y(f)\Big(R(u,Z)X\Big)^H \\
&-\frac{\omega^p}{2}g(R(X,Y)u, Z)\Big(gradf\Big)^H + \Big(R(X,Y)Z\Big)^V \\
&-\frac{f\omega^p}{4}\Big(R(X, R(u,Z)Y)u\Big)^V + \frac{f\omega^p}{4}\Big(R(Y, R(u,Z)X)u\Big)^V \\
&-p\omega g(Z,u)\Big(R(X,Y)u\Big)^V + (\frac{p\omega+q}{1+qr^2})g(R(X,Y)u, Z)U^V
\end{aligned}$$

$$\begin{aligned}
\overline{R}(X^V, Y^V)Z^V &= Ag(Z,u)\Big[g(Y,u)X^V - g(X,u)Y^V\Big] \\
&+ B\Big[g(Y,Z)X^V - g(X,Z)Y^V\Big] \\
&+ C\Big[g(X,u)g(Y,Z) - g(Y,u)g(X,Z)\Big]U^V \\
&- \frac{f\omega^{2p}}{4}\left[g(Z,Y)(R(u,X)gradf)^H - g(X,Z)(R(u,Y)gradf)^H\right] \\
&- \frac{qf\omega^{2p}}{4}g(Z,u)\left[g(Y,u)(R(u,X)gradf)^H\right. \\
&\left.-g(X,u)(R(u,Y)gradf)^H\right]
\end{aligned}$$

with

$$A = \frac{p\omega((p+2q-2)\omega - q)}{1+qr^2} - \frac{q\omega^p}{4f}\|gradf\|^2$$

$$B = \frac{p^2\omega^2 - p(p-2)\omega + q}{1+qr^2} - \frac{\omega^p}{4f}\|gradf\|^2$$
$$C = \frac{p(p-2)(1-q)\omega^2 + pq(p-3)\omega - q^2}{(1+qr^2)^2}$$

Notice that A, B and C are related by: $A - qB = C(1+qr^2)$.

3.1 Sectional curvature of vertical rescaled generalized Cheeger-Gromoll metric

In the following, we consider for all $V, W \in \Gamma(TTM), V \neq W$

$$\overline{Q}(V,W) = G(V,V)G(W,W) - |G(V,W)|^2$$
$$\overline{G}(V,W) = G(\overline{R}(V,W)W,V)$$
$$\overline{K}(V,W) = \frac{\overline{G}(V,W)}{\overline{Q}(V,W)}$$

Lemma 3.1. *Let (M,g) be a Riemannian manifold and (TM, G^f) its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric, then for any orthonormal vector fields $X, Y \in \Gamma(TM)$, we have*

1. $\overline{Q}(X^H, Y^H) = 1$,
2. $\overline{Q}(X^H, Y^V) = f\omega^p\Big[1 + q|g(Y,u)|^2\Big]$,
3. $\overline{Q}(X^V, Y^V) = f^2\omega^{2p}\Big(1 + q|g(X,u)|^2 + q|g(Y,u)|^2\Big)$.

Proof. The statement is a direct consequence of Definition 2.1. □

Lemma 3.2. *Let (M,g) be a Riemannian manifold and (TM, G^f) its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric, then for any orthonormal vector fields $X, Y \in \Gamma(TM)$, we have*

1. $\overline{G}(X^H, Y^H) = g(R(X,Y)Y,X) - \frac{3f\omega^p}{4}\|R(X,Y)u\|^2$,
2. $\overline{G}(X^H, Y^V) = \Big[\frac{|X(f)|^2\omega^p}{4f} - \frac{\omega^2}{2}g(\nabla_X gradf, X)\Big]\Big[1 + q|g(Y,u)|^2\Big] + \frac{f^2\omega^{2p}}{4}\|R(u,Y)X\|^2$,
3. $\overline{G}(X^V, Y^V) = f\omega^p\Big[A(|g(Y,u)|^2 + |g(X,u)|^2) + B\Big]$.

The constants A and B are as in (Theorem 3.1).

Proposition 3.1. *Let (M,g) be a Riemannian manifold and (TM, G^f) its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. If K (resp., $\overline{K}$ denote the sectional curvature tensor of (M,g)(resp., (TM, G^f)), then for any orthonormal vector fields $X, Y \in \Gamma(TM)$, we have*

1. $\overline{K}(X^H, Y^H) = K(X,Y) - \dfrac{3f\omega^p}{4}\|R(X,Y)u\|^2,$

2. $\overline{K}(X^H, Y^V) = \dfrac{f\omega^p}{4(1+q|g(Y,u)|^2)}\|R(u,Y)X)\|^2 + \frac{|X(f)|^2}{4f^2} - \dfrac{\omega^{-p+2}}{2f} g(\nabla_X gradf, X),$

3. $\overline{K}(X^V, Y^V) = \dfrac{\omega^{-p}}{fq(1+g(X,u)|^2+|g(Y,u)|^2)}\Big[A(|g(Y,u)|^2+|g(X,u)|^2)+B\Big].$

Proposition 3.2. *Let (M,g) be a Riemannian manifold of constant sectional curvature λ and (TM, G^f) its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. If $\overline{K}$ denotes the sectional curvature tensor of TM, then for any orthonormal vector fields $X, Y \in \Gamma(TM)$, we have*

1. $\overline{K}(X^H, Y^H) = \lambda - \dfrac{3\lambda^2 f\omega^p}{4}\Big[|g(X,u)|^2 + |g(Y,u)|^2\Big],$

2. $\overline{K}(X^H, Y^V) = \dfrac{f\omega^p\lambda^2}{4}\frac{|g(X,u)|^2}{(1+q|g(Y,u)|^2)} + \dfrac{|X(f)|^2}{4f^2} - \dfrac{\omega^{-p+2}}{2f} g(\nabla_X gradf, X),$

3. $\overline{K}(X^V, Y^V) = \dfrac{\omega^{-p}}{fq(1+g(X,u)|^2+|g(Y,u)|^2)}\Big[A(|g(Y,u)|^2+|g(X,u)|^2)+B\Big].$

Proof. The proof of Proposition 3.2 is deduced from Proposition 3.1 and the following equations

$$\begin{aligned}
R(X,Y)Z &= \lambda\Big[g(Y,Z)X - g(X,Z)Y\Big]\\
\|R(X,Y)u\|^2 &= \|\lambda\Big[g(Y,u)X - g(X,u)Y\Big]\|^2\\
&= \lambda^2[|g(X,u)|^2 + |g(y,u)|^2]\\
\|R(u,Y)X\|^2 &= \|\lambda\Big[g(Y,X)u - g(X,u)Y\Big]\|^2\\
&= \lambda^2|g(X,u)|^2
\end{aligned}$$

□

Lemma 3.3. *Let (x,u) be a point of TM with $u \neq 0$ and $(E_1, ..., E_m)$ be a local orthonormal on M such that $E_1 = \frac{u}{\|u\|}$. Then $(F_1, ..., F_{2m})$ is a local orthonormal frame on (TM, G^f).*
Where $F_i = E_i^H$, $F_{m+1} = \frac{1}{\sqrt{f\omega^p(1+qr^2)}}E_1^V$ and $F_{m+j} = \sqrt{\frac{\omega^{-p}}{f}}E_j^V$, $i = \overline{1,m}$, $j = \overline{2,m}$.

Lemma 3.4. *Let (M,g) be a Riemannian manifold and (TM, G^f) its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. If $(E_1, ..., E_m)$, (resp $(F_1, .., F_{2m})$) are local orthonormal on M (resp., TM), then for all $i, j = \overline{1,m}$ et $k, l = \overline{2,m}$, we have*

1. $\overline{K}(F_i, F_j) = K(E_i, E_j) - \dfrac{3f\omega^p}{4}\|R(E_i, E_j)\|^2,$

2. $\overline{K}(F_i, F_{m+1}) = \frac{|E_i(f)|^2}{4f^2} - \dfrac{\omega^{-p+2}}{2f} g(\nabla_{E_i} gradf, E_i),$

3. $\overline{K}(F_i, F_{m+l}) = \frac{1}{4}\|R(u, E_l)E_i)\|^2 + \dfrac{|E_i(f)|^2}{4f^2} - \dfrac{\omega^{-p+2}}{2f} g(\nabla_{E_i} gradf, E_i),$

4. $\overline{K}(F_{m+k}, F_{m+1}) = \frac{1+qr^2}{fq\omega^p(1+qr^2)+qr^2}\left[\frac{r^2}{f\omega^p(1+qr^2)}A + B\right],$

5. $\overline{K}(F_{m+k}, F_{m+l}) = \frac{\omega^{-p}}{fq}B.$

Lemma 3.5. *let $(E_1, ..., E_m)$ be local orthonormal frame on M, then for all $i, j = \overline{1, m}$, we have*

$$\sum_{i,j=1}^{m} \|R(u, E_i)E_j\|^2 = \sum_{i,j=1}^{m} \|R(E_i, E_j)u\|^2$$

Proof.

$$\begin{aligned}
\sum_{i,j=1}^{m} \|R(u, E_i)E_j\|^2 &= \sum_{i,j=1}^{m} g(R(u, E_i, E_j, R(u, E_i)E_j) \\
&= \sum_{i,j,k,l,s=1}^{m} u_k u_l g(R(E_k, E_i)E_j, E_s) g(R(E_k, E_i)E_j, E_s) \\
&= \sum_{i,j,k,l,s=1}^{m} u_k u_l g(R(E_j, E_s)E_k, E_i) g(R(E_j, E_s)E_l, E_i) \\
&= \sum_{i,j,s=1}^{m} g(R(E_j, E_s)u, g(E_j, E_s)u, E_i)E_i) \\
&= \sum_{i,j=1}^{m} \|R(E_i, E_j)u\|^2
\end{aligned}$$

□

Proposition 3.3. *Let (M, g) be a Riemannian manifold and (TM, G^f) its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. If σ (resp.,$\overline{\sigma}$) denote the scalar curvature of (M, g) (resp.,(TM, G^f)), then for any orthonormal frame $(E_1, ..., E_m)$, we have*

$$\begin{aligned}
\overline{\sigma} &= \sigma + \frac{2 - 3f\omega^p}{4} \sum_{i,j=1}^{m} \|R(E_i, E_j)\|^2 - \frac{m\omega^{-p+2}}{f}\Delta(f) + \frac{m\|gradf\|^2}{2f^2} \\
&\quad +2(m-1)\left[\frac{1 + qr^2}{fq\omega^p(1 + qr^2) + qr^2}(\frac{r^2}{f\omega^p(1 + qr^2)}A + B) + \frac{(m-2)\omega^{-p}}{2fq}B\right]
\end{aligned}$$

Proof. Using Lemma 3.4

$$\begin{aligned}
\overline{\sigma} &= \sum_{s,t=1}^{2m} \overline{K}(F_s, F_t) \\
&= \sum_{i,j=1,i\neq j}^{m} \overline{K}(F_i, F_j) + 2\sum_{i,j=1}^{m} \overline{K}(F_i, F_{m+j}) + \sum_{i,j=1,i\neq j}^{m} \overline{K}(F_{m+i}, F_{m+j}) \\
&= \sum_{i,j=1,i\neq j}^{m} \overline{K}(F_i, F_j) + 2\sum_{i=1}^{m} \overline{K}(F_i, F_{m+1}) + 2\sum_{i=1,j=2}^{m} \overline{K}(F_i, F_{m+j}) \\
&+ 2\sum_{i=1}^{m} \overline{K}(F_{m+i}, F_{m+1}) + \sum_{i,j=2,i\neq j}^{m} \overline{K}(F_{m+i}, F_{m+j})
\end{aligned}$$

$$\begin{aligned}
\overline{\sigma} &= \sum_{i,j=1,i\neq j}^{m} [K(E_i, E_j) - \frac{3f\omega^p}{4}\|R(E_i, E_j)\|^2] \\
&+2\sum_{i=1}^{m}[\frac{|E_i(f)|^2}{4f^2} - \frac{\omega^{-p+2}}{2f} g(\nabla_{E_i} gradf, E_i)] \\
&+2\sum_{i=1,j=2}^{m}[\frac{1}{4}\|R(u, E_j)E_i)\|^2 + \frac{|E_i(f)|^2}{4f^2} - \frac{\omega^{-p+2}}{2f} g(\nabla_{E_i} gradf, E_i)] \\
&+2\sum_{i=1}^{m}[\frac{1+\alpha r^2}{fq\omega^p(1+qr^2)+qr^2}[\frac{r^2}{f\omega^p(1+qr^2)}A + B]] \\
&+ \sum_{i,j=2,i\neq j}^{m}[\frac{\omega^{-p}}{fq}B] \\
&= \sigma - \sum_{i,j=1,i\neq j}^{m} \frac{3f\omega^p}{4}\|R(E_i, E_j)\|^2 + \frac{\|gradf\|^2}{2f^2} - \frac{\omega^{-p+2}}{f} trace_g(\nabla gradf) \\
&+\frac{2}{4}\sum_{i=1,j=2}^{m}\|R(u, E_j)E_i\|^2 + (m-1)\frac{\|gradf\|^2}{2f^2} \\
&- \frac{(m-1)\omega^{-p+2}}{f} trace_g(\nabla gradf) \\
&+2(m-1)[\frac{1+qr^2}{fq\omega^p(1+qr^2)+qr^2}[\frac{r^2}{f\omega^p(1+qr^2)}A + B] \\
&+(m-2)(m-1)[\frac{\omega^{-p}}{fq}B]
\end{aligned}$$

Hence,

$$\overline{\sigma} = \sigma + \frac{2-3f\omega^p}{4}\sum_{i,j=1}^{m}\|R(E_i, E_j)\|^2 - \frac{m\omega^{-p+2}}{f}\Delta(f) + \frac{m\|gradf\|^2}{2f^2}$$

$$+2(m-1)\left[\frac{1+qr^2}{fq\omega^p(1+qr^2)+qr^2}(\frac{r^2}{f\omega^p(1+qr^2)}A+B)+\frac{(m-2)\omega^{-p}}{2fq}B\right]$$

□

Corollary 3.1. *Let (M,g) be a Riemannian manifold of constant sectional curvature λ and (TM,G^f) its tangent bundle equipped with the vertical rescaled generalized Cheeger-Gromoll metric. If $\overline{\sigma}$ denotes the scalar curvature of (TM,G^f), then for any orthonormal frame $(E_1,...,E_m)$ on M, we have*

$$\begin{aligned}\overline{\sigma} &= +2(m-1)\left[\frac{1+qr^2}{fq\omega^p(1+qr^2)+qr^2}(\frac{r^2}{f\omega^p(1+qr^2)}A+B)\right.\\ &\qquad \left. +\frac{m\lambda}{2}+\lambda^2r^2\frac{2-3f\omega^p}{4}+\frac{(m-2)\omega^{-p}}{2fq}B\right]\\ &\quad +\frac{m}{f}\left[-\omega^{-p+2}\Delta(f)+\frac{\|gradf\|^2}{2f}\right]\end{aligned}$$

Proof. Taking account that $\sigma = m(m-1)\lambda$ and for any vector fields $X,Y,Z \in TM$

$$R(X,Y)Z = \lambda(g(Z,Y)X - g(X,Z)Y)$$

then we obtain

$$\begin{aligned}\sum_{i,j=1}^{m}\|R(E_i,E_j)u\|^2 &= \lambda^2\sum_{i,j=1}^{m}\|g(u,E_j)E_i - g(E_i,u)E_j\|^2\\ &= \lambda^2\sum_{i,j=1}^{m}[|g(u,E_j)|^2 - 2g(u,E_j)g(E_i,u)\delta_{ij} + |g(E_i,u)|^2]\\ &= \lambda^2[m\|u\|^2 - 2\|u\|^2 + m\|u\|^2]\\ &= 2\lambda^2(m-1)r^2\end{aligned}$$

we deduce that

$$\begin{aligned}\overline{\sigma} &= m(m-1)\lambda + 2(m-1)\lambda^2r^2\frac{2-3f\omega^p}{4} - \frac{m\omega^{-p+2}}{f}\Delta(f) + \frac{m\|gradf\|^2}{2f^2}\\ &\quad +2(m-1)\left[\frac{1+qr^2}{fq\omega^p(1+qr^2)+qr^2}(\frac{r^2}{f\omega^p(1+qr^2)}A+B)+\frac{(m-2)\omega^{-p}}{2fq}B\right]\end{aligned}$$

□

Acknowledgements

The authors would like to thank the reviewers for their valuable comments and suggestions to improve the paper.

References

[1] Belarbi, L. and El hendi, H., *Geometry of twisted Sasaki metric*, J. Geom. Symmetry Phys. **53** (2019), 1-19.

[2] Belarbi, L. and El hendi, H., *Harmonic and biharmonic maps between tangent bundles*, Acta Math. Univ. Comenian **88** (2019), 187-199.

[3] Benyounes, M., Loubeau, E. and Wood, C. M., *The geometry of generalised cheeger-gromoll metrics*, Tokyo J. Math. **32** (2009), 1-26.

[4] Cengiz, N. and Salimov, A. A., *Diagonal lift in the tensor bundle and its applications*, Appl. Math. Comput. **142** (2003), 309-319.

[5] Cheeger, J. and Gromoll, D., *On the structure of complete manifolds of non-negativ curvature*, Ann. of Math. **96** (1972), 413-443.

[6] Djaa, M., EL Hendi, H. and Ouakkas, S., *Biharmonic vector field*, Turkish J. Math. **36** (2012), 463-474.

[7] Djaa, M. and Gancarzewicz, J., *The geometry of tangent bundles of order r*, Boletin Academia. Galega de Ciencias. Espagne.**4** (1985), 147-165.

[8] Dida, H. M., Hathout, F. and Azzouz, A., *On the geometry of the tangent bundle with vertical rescaled metric*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **68** (2019), 222-235.

[9] Djaa,N. E. H., Ouakkas, S. and Djaa, M., *Harmonic sections on the tangent bundle of order two*, Ann. Math. Inform. **38**(2011), 15-25.

[10] El hendi, H. and Belarbi, L., *Deformed diagonal metrics on tangent bundle of order two and harmonicity*. Panamer. Math. J. **27**(2017), 90 - 106.

[11] El hendi, H. and Belarbi, L., *On paraquaternionic submersions of tangent bundle of order two*, Nonlinear Studies **25** (2018), 653-664.

[12] El hendi, H. Terbeche, M. and Djaa, M., *Tangent bundle of order two and biharmonicity*, Acta Math. Univ. Comenianae **83**(2014), 165-179.

[13] Eells, J. and Sampson, J. H., *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109-60.

[14] Gudmundsson, S. and Kappos, E., *On the geometry of the tangent bundles*, Expo. Math. **20** (2002). 1-41.

[15] Gudmundsson, S. and Kappos, E., *On the geometry of the tangent bundle with the Cheeger-Gromoll metric*, Tokyo J. Math. **25**(2002), 75-83.

[16] Ishihara, T., *Harmonic sections of tangent bundles*, J. Math. Univ. Tokushima **13** (1979), 23-27.

[17] Jiang, G. Y., *Harmonic maps and their first and second variational formulas*, Chinese Ann. Math. Ser. A. **7** (1986), 389-402.

[18] Konderak, J. J., *On harmonic vector fields. Publications Matmatiques* **36** (1992), 217-288.

[19] Mazouzi, H., El hendi, H, and Belarbi, L., *On the Generalized bi-f-harmonic map equations on singly warped product manifolds*, Comm. Appl. Nonlinear Anal. **25** (2018), 52-76.

[20] Oproiu, V., *On harmonic maps between tangent bundles*, Rend. Sem. Mat. **47** (1989) 47-55.

[21] Salimov, A.A., Gezer, A. and Akbulut, K., *Geodesics of Sasakian metrics on tensor bundles*, Mediterr. J. Math. **6** (2009), 135-147.

[22] Wang, J. and Wang, Y., *On the geometry of tangent bundles with the rescaled metric*, arXiv:1104.5584v1[math.DG] 29Apr 2011.

[23] Yano, K. and Ishihara, S., *Tangent and Cotangent Bundles*, Marcel Dekker. INC. New York 1973.