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STUDY ON GENERALIZED PSEUDO (RICCI) SYMMETRIC SASAKIAN MANIFOLD ADMITTING GENERAL CONNECTION

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Abstract

The object of the present paper is to study the generalized pseudo (Ricci) symmetric Sasakian manifold with respect to a new connection named general connection. The general connection has the flavour of quarter-symmetric connection, generalized Tanaka-Webster connection, Zamkovoy and Schouten-van Kampen connection. The existence of generalized pseudo (Ricci) symmetric Sasakian manifold with respect to general connection is ensured by an example.

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1 Introduction

Let the symbols ∇, ∇^q, ∇^T, ∇^z, ∇^s and ∇^* stand for Levi-civita connection, quarter-symmetric metric connection, Generalized Tanaka-Webster connection, Zamkovoy connection, Schouten-Van Kampen connection and general connections respectively. Also we denote $(GPS)_n$ and $(GPRS)_n$ for generalized pseudo symmetric Sasakian manifold and generalized pseudo Ricci symmetric Sasakian manifold. In 1924, A. Friedmann and J. A. Schouten[1] founded the idea of semi-symmetric connection on a differentiable manifold. In 1932, H. A. Hayden[8] further studied the thought of semi-symmetric connection with torsion on a Riemannian manifold. A comprehensive study of Semi-symmetric metric connection was carriedout by Yano[21]. Semi-symmetric metric connection plays a vital role in Riemannian geometry. Generalising the concept of semi-symmetric connection

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Golab[7] defined and studied the quarter-symmetric connection in 1975. There after, many geometers like as Rastogi ([13], [14]), Mishra et al. [11], Yano et al.[20] and others studied quarter symmetric connection. An affine connection ∇^q on an n-dimensional Riemannian manifold (M, g) is called a quarter-symmetric connection[7] if its torsion tensor T of the connection ∇^q satisfies

$$\begin{aligned} T(X,Y) &= \nabla^q_X Y - \nabla^q_Y X - [X,Y] \\ &= \eta(Y)\phi X - \eta(X)\phi Y, \end{aligned}$$

where η is a 1-form and ϕ is a $(1,1)$ tensor field. In particular, if $\phi X = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection[1]. Furthermore, if a quarter-symmetric connection ∇^q admits the condition

$$(\nabla^q_X g)(Y,Z) = 0,$$

then ∇^q is said to be a quarter-symmetric metric connection, otherwise it is said to be a quarter-symmetric non-metric connection. If ∇ and ∇^q are the Levi-civita connection and quarter-symmetric connection respectively, then

$$\nabla^q_X Y = \nabla_X Y - \eta\,(X)\,\phi Y. \tag{1}$$

In 1970 Tanno[18] characterized and considered the idea of summed up Tanaka-Webster connection by summing up the connection by Tanaka [12] and Webster[17] , which harmonizes with the Tanaka-Webster connection if the related contact Riemann structure is integrable. Likewise J. T. Cho([9],[10]) has considered the Generalized Tanaka-Webster connection with regards to Kahler manifod. Let ∇ and ∇^T be the Levi-civita connection and generalized Tanaka-Webster connection respectively.Then

$$\nabla^T_X Y = \nabla_X Y + (\nabla_X \eta)\,(Y)\,\xi - \eta\,(Y)\,\nabla_X \xi - \eta\,(X)\,\phi Y. \tag{2}$$

Also Z. Zamkovoy[22] has introduced a new connection known as Zamkovoy connection ∇^z , which is related with Levi-civita connection as

$$\nabla^z_X Y = \nabla_X Y + (\nabla_X \eta)\,(Y)\,\xi - \eta\,(Y)\,\nabla_X \xi + \eta\,(X)\,\phi Y. \tag{3}$$

Schouten, J. A. and Van Kampen[16] introduced the Schouten-van Kampen connection in the third decade of last century. The relation[16] between Schouten-van Kampen connection ∇^s and Levi-Civita connection ∇ is

$$\nabla^s_X Y = \nabla_X Y + (\nabla_X \eta)\,(Y)\,\xi - \eta\,(Y)\,\nabla_X \xi. \tag{4}$$

In the setting of Sasakian geometry, we like to define a new connection named general connection ∇^*, which is related with ∇

$$\nabla^*_X Y = \nabla_X Y + \lambda\,[(\nabla_X \eta)\,(Y)\,\xi - \eta\,(Y)\,\nabla_X \xi] + \mu\eta\,(X)\,\phi Y, \tag{5}$$

for all $X, Y \in \chi\,(M)$ and the pair (λ, μ) being real constants.Connection defined in (5) has an important characteristic because it has a flavour of quarter symmetric metric connection for $(\lambda, \mu) \equiv (0, -1)$; Tanaka Webster connection for

$(\lambda, \mu) \equiv (1, -1)$; Zamkovoy connection for $(\lambda, \mu) \equiv (1, 1)$; Schouten Kampen-Van connection $(\lambda, \mu) \equiv (1, 0)$.

This paper is structured as follows: After the introduction, section 2, we give a brief account of the Sasakian manifolds. Section 3 is dedicated to establishing the relation between the curvature tensor of the Sasakian manifolds with respect to the general connection and the levi-civita connection. Section 4 is concerned with generalized pseudo symmetric Sasakian manifolds admitting general connection and we prove that there exists no generalized pseudo symetric Sasakian manifold with respect to general connection unless $\beta - \overline{C}\alpha$ vanishes everywhere, provided $\overline{C} \neq 0$. It is worthy to note that there exists no $[(GPS)_n, \nabla^T]$, $[(GPS)_n, \nabla^s]$, $[(GPS)_n, \nabla^z]$ unless β vanishes everywhere. In the next section, we investigate the generalized pseudo Ricci-symmetric Sasakian manifolds admitting general connection. Finally, we construct an example for the existence of generalized pseudo symmetric and generalized pseudo Ricci-symmetric Sasakian manifolds with respect to the general connection.

2 Preliminaries

Let M be an n-dimensional almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g. Then

$$\phi^2 Y = -Y + \eta(Y)\xi, \eta(\xi) = 1, \eta(\phi X) = 0, \ \phi\xi = 0, \tag{6}$$

$$g(X, Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y), \tag{7}$$

$$g(X, \phi Y) = -g(\phi X, Y), \eta(Y) = g(Y, \xi), \forall X, Y \in TM. \tag{8}$$

An almost contact metric manifold M is said to be (a) a contact metric manifold if

$$g(X, \phi Y) = d\eta(X, Y), \forall X, Y \in TM; \tag{9}$$

(b) a K-contact manifold if the vector field ξ is Killing equivalently

$$\nabla_Y \xi = -\phi Y, \tag{10}$$

where ∇ is Riemannian connection and (c) a Sasakian manifold if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \forall X, \ Y \in TM. \tag{11}$$

A K-contact manifold is a contact metric manifold, while the converse is true if the Lie derivative of ϕ in the characteristic direction ξ vanishes identically. A Sasakian manifold is always a K-contact manifold. A 3-dimensional K-contact manifold is a Sasakian manifold.

It is well known that a contact metric manifold is Sasakian if and only if

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \forall X, Y \in TM. \tag{12}$$

In a Sasakian manifold equipped with the structure $(\phi,\ \xi,\ \eta,\ g)$, the following relations also hold ([2], [19], [15]):

$$\begin{aligned} (\nabla_X \eta) Y &= g(X, \phi Y), && (13)\\ R(\xi, X)Y &= g(X,Y)\xi - \eta(Y)X, && (14)\\ S(X,\xi) &= (n-1)\eta(X), && (15)\\ R(X,\xi)Y &= \eta(Y)X - g(X,Y)\xi, && (16) \end{aligned}$$

$$Q\xi = (n-1)\xi \qquad (17)$$

3 Some properties of Sasakian manifold admitting general connection

In Sasakian manifold, the relation (5) reduces to

$$\nabla^{*}_X Y = \nabla_X Y + \lambda\left[g(X,\phi Y)\xi + \eta(Y)\phi X\right] + \mu\eta(X)\phi Y. \qquad (18)$$

Putting $Y=\xi$ in (5)

$$\nabla^{*}_X \xi = -\phi X + \lambda\phi X. \qquad (19)$$

Now with the help of (18), (10) and (11) we get the followings

$$\begin{aligned} &\nabla^{*}_X \eta(Y)\\ =\ &\nabla^{*}_X g(Y,\xi)\\ =\ &\eta(\nabla_X Y) + \lambda g(X,\phi Y) - g(Y,\phi X) + \lambda g(Y,\phi X), \end{aligned} \qquad (20)$$

$$\begin{aligned} &\nabla^{*}_X (\phi Y)\\ =\ &\nabla_X (\phi Y) - \lambda g(\phi X,\phi Y)\xi - \mu\eta(X)Y + \mu\eta(X)\eta(Y)\xi, \end{aligned} \qquad (21)$$

$$\begin{aligned} &\nabla^{*}_X g(Y,\phi Z)\\ =\ &g(\nabla_X Y,\phi Z) + \mu\eta(X)g(\phi Y,\phi Z) + g(Y,\nabla_X(\phi Z))\\ &-\mu\eta(X)g(Y,Z) + \mu\eta(X)\eta(Y)\eta(Z). \end{aligned} \qquad (22)$$

Now we know that

$$R^{*}(X,Y)Z = \nabla^{*}_X\nabla^{*}_Y Z - \nabla^{*}_Y\nabla^{*}_X Z - \nabla^{*}_{[X,Y]}Z. \qquad (23)$$

By using (18), (19), (20), (21) and (22) we obtain the following

$$\begin{aligned}
& \nabla^{*}_{X}\nabla^{*}_{Y}Z \\
= \; & \nabla_X\nabla_Y Z + \lambda g(X,\phi\nabla_Y Z)\xi + \lambda\eta(\nabla_Y Z)\phi X + \mu\eta(X)\phi\nabla_Y Z \\
& +\lambda g(\nabla_X Y,\phi Z)\xi + \lambda\mu\eta(X)g(\phi Y,\phi Z)\xi + \lambda g(Y,\nabla_X(\phi Z))\xi \\
& -\lambda\mu\eta(X)g(Y,Z)\xi + \lambda\mu\eta(X)\eta(Y)\eta(Z)\xi - \lambda g(Y,\phi Z)\phi X \\
& +\lambda^2 g(Y,\phi Z)\phi X + \lambda\eta(\nabla_X Z)\phi Y + \lambda^2 g(X,\phi Z)\phi Y \\
& +\lambda^2 g(Z,\phi X)\phi Y + \lambda\eta(Z)\nabla_X(\phi Y) - \lambda^2\eta(Z)g(\phi X,\phi Y)\xi \\
& -\lambda\mu\eta(Z)\eta(X)Y + \lambda\mu\eta(Z)\eta(X)\eta(Y)\xi + \mu\eta(\nabla_X Y)\phi Z \\
& +\lambda\mu g(X,\phi Y)\phi Z - \mu g(Y,\phi X)\phi Z + \lambda\mu g(Y,\phi X)\phi Z \\
& +\mu\eta(Y)\nabla_X(\phi Z) - \lambda\mu\eta(Y)g(\phi X,\phi Z)\xi - \lambda g(Z,\phi X)\phi Y \\
& -\mu^2\eta(Y)\eta(X)Z + \mu^2\eta(Y)\eta(X)\eta(Z)\xi, \qquad (24)
\end{aligned}$$

$$\begin{aligned}
& \nabla^{*}_{Y}\nabla^{*}_{X}Z \\
= \; & \nabla_Y\nabla_X Z + \lambda g(Y,\phi\nabla_X Z)\xi + \lambda\eta(\nabla_X Z)\phi Y + \mu\eta(Y)\phi\nabla_X Z \\
& +\lambda g(\nabla_Y X,\phi Z)\xi + AB\eta(Y)g(\phi X,\phi Z)\xi + \lambda g(X,\nabla_Y(\phi Z))\xi \\
& -\lambda\mu\eta(Y)g(X,Z)\xi + \lambda\mu\eta(Y)\eta(X)\eta(Z)\xi - \lambda g(X,\phi Z)\phi Y \\
& +\lambda^2 g(X,\phi Z)\phi Y + \lambda\eta(\nabla_Y Z)\phi X + \lambda^2 g(Y,\phi Z)\phi X \\
& -\lambda g(Z,\phi Y)\phi X + \lambda^2 g(Z,\phi Y)\phi X + \lambda\eta(Z)\nabla_Y(\phi X) \\
& -\lambda^2\eta(Z)g(\phi Y,\phi X)\xi - \lambda\mu\eta(Z)\eta(Y)X + \lambda\mu\eta(Z)\eta(Y)\eta(X)\xi \\
& +\mu\eta(\nabla_Y X)\phi Z + \lambda\mu g(Y,\phi X)\phi Z - \mu g(X,\phi Y)\phi Z \\
& +\lambda\mu g(X,\phi Y)\phi Z + \mu\eta(X)\nabla_Y(\phi Z) - \lambda\mu\eta(X)g(\phi Y,\phi Z)\xi \\
& -\mu^2\eta(X)\eta(Y)Z + \mu^2\eta(X)\eta(Y)\eta(Z)\xi, \qquad (25)
\end{aligned}$$

and

$$\begin{aligned}
& \nabla^{*}_{[X,Y]}Z \\
= \; & \nabla_{[X,Y]}Z + \lambda g(\nabla_X Y,\phi Z)\xi - \lambda g(\nabla_Y X,\phi Z)\xi + \lambda\eta(Z)\phi\nabla_X Y \\
& -\lambda\eta(Z)\phi\nabla_Y X + \mu\eta(\nabla_X Y)\phi Z - \mu\eta(\nabla_Y X)\phi Z. \qquad (26)
\end{aligned}$$

Now in reference of (24), (25) and (26) we get from (23)

$$\begin{aligned}
& R^{*}(X,Y)Z \\
= \; & R(X,Y)Z + \left(\lambda^2 - 2\lambda\right)[g(Z,\phi X)\phi Y + g(Y,\phi Z)\phi X] \\
& -2\mu g(Y,\phi X)\phi Z \\
& +(\lambda - \lambda\mu + \mu)[g(X,Z)\eta(Y)\xi - \eta(X)g(Y,Z)\xi] \\
& +(\lambda - \lambda\mu + \mu)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X] \qquad (27)
\end{aligned}$$

Consequently one can easily bring out the following

$$S^{*}(Y,Z) = S(Y,Z) - \overline{A}g(Y,Z) + \overline{B}\eta(Y)\eta(Z), \tag{28}$$
$$S^{*}(Y,\xi) = -(n-1)\overline{C}\eta(Y), \tag{29}$$
$$S^{*}(\xi,Z) = -(n-1)\overline{C}\eta(Z), \tag{30}$$
$$Q^{*}Y = QY - \overline{A}Y + \overline{B}\eta(Y)\xi, \tag{31}$$
$$Q^{*}\xi = -(n-1)\overline{C}\xi, \tag{32}$$
$$r^{*} = r - \overline{A}n + \overline{B}, \tag{33}$$

$$R^{*}(X,Y)\xi = \overline{C}[\eta(X)Y - \eta(Y)X], \tag{34}$$
$$R^{*}(\xi,Y)Z = \overline{C}[\eta(Z)Y - g(Y,Z)\xi], \tag{35}$$
$$R^{*}(X,\xi)Z = \overline{C}[g(X,Z)\xi - \eta(Z)X]. \tag{36}$$

where

$$\overline{A} = \left(\lambda^2 - \lambda - \mu - \lambda\mu\right), \tag{37}$$
$$\overline{B} = \left[\lambda^2 + (n-2)\lambda\mu - n(\lambda+\mu)\right], \tag{38}$$
$$\overline{C} = (\lambda - \lambda\mu + \mu - 1). \tag{39}$$

Therefore, for quarter-symmetric metric connection

$$\overline{A} = 1; \overline{B} = n; \overline{C} = -2, \tag{40}$$

Tanaka Webster connection

$$\overline{A} = 2; \overline{B} = 3 - n; \overline{C} = 0, \tag{41}$$

Zamkovoy connection

$$\overline{A} = -2; \overline{B} = -1 - n; \overline{C} = 0, \tag{42}$$

and for Schouten Kampen-Van connection

$$\overline{A} = 0; \overline{B} = 1 - n; \overline{C} = 0. \tag{43}$$

Thus, we can state the following

Proposition 1. *Let M be an n-dimensional Sasakian manifold admitting general connection ∇^{*}, Then*

(i) The curvature tensor R^{*} of ∇^{*} is given by (27),
(ii) The Ricci tensor S^{*} of ∇^{*} is given by (28),
(iii) The scalar curvature r^{*} of ∇^{*} is given by (33),
(iv) The Ricci tensor S^{*} of ∇^{*} is symmetric.

Now if we suppose that the Sasakian manifold is Ricci flat with respect to the general connection then from (28) we get

$$S(Y,Z) = \overline{A}g(Y,Z) - \overline{B}\eta(Y)\eta(Z).$$

This leads to the following

Theorem 1. *If manifold M^n is Ricci flat with respect to the general connection if and only if M^n is an η-Einstein manifold.*

4 Generalised pseudo symmetric Sasakian manifold with respect to general connection

A non-flat Sasakian manifold (M^n, g) $(n \geq 3)$ is said to be pseudo symmetric[3], if the curvature tensor R satisfies the condition.

$$\begin{aligned} & (\nabla_X R)(U,V,W,Y) \\ = \; & 2\alpha(X) R(U,V,W,Y) + \alpha(U) R(X,V,W,Y) \\ & +\alpha(V) R(U,X,W,Y) + \alpha(W) R(U,V,X,Y) \\ & +\alpha(Y) R(U,V,W,X). \end{aligned}$$

Keeping in tune with Dubey[6], a non-flat Sasakian manifold (M^n, g) $(n \geq 3)$ is said to be generalized pseudo symmetric if the curvature tensor R satisfies the condition

$$\begin{aligned} & (\nabla_X R)(U,V,W,Y) \\ = \; & 2\alpha(X) R(U,V,W,Y) + \alpha(U) R(X,V,W,Y) \\ & +\alpha(V) R(U,X,W,Y) + \alpha(W) R(U,V,X,Y) \\ & +\alpha(Y) R(U,V,W,X) + 2\beta(X) G(U,V,W,Y) \\ & +\beta(U) G(X,V,W,Y) + \beta(V) G(U,X,W,Y) \\ & +\beta(W) G(U,V,X,Y) + \beta(Y) G(U,V,W,X), \end{aligned} \tag{44}$$

where

$$G(Y,U,V,\ W) = [g(U,V)g(Y,\ W) - g(Y,\ V)g(U,\ W)]. \tag{45}$$

for all vectors fields $X, U, V \in \chi(M)$ and where non zero one forms α, β are defined by

$$\alpha(X) = g(X, \rho_1);\beta(X) = g(X, \rho_2). \tag{46}$$

Now we denote the generalised pseudo symmetric-Sasakian manifold with respect to the general connection by $([(GPS)_n, \overset{*}{\nabla}])$, whose curvature tensor satisfies the following condition

$$\begin{aligned} & \left(\overset{*}{\nabla}_X \overset{*}{R}\right)(U,V,W,Y) \\ = \; & 2\alpha(X) \overset{*}{R}(U,V,W,Y) + \alpha(U) \overset{*}{R}(X,V,W,Y) \\ & +\alpha(V) \overset{*}{R}(U,X,W,Y) + \alpha(W) \overset{*}{R}(U,V,X,Y) \\ & +\alpha(Y) \overset{*}{R}(U,V,W,X) + 2\beta(X) G(U,V,W,Y) \\ & +\beta(U) G(X,V,W,Y) + \beta(V) G(U,X,W,Y) \\ & +\beta(W) G(U,V,X,Y) + \beta(Y) G(U,V,W,X) \end{aligned} \tag{47}$$

which yields after contraction,

$$\begin{aligned} & \left(\nabla^{*}_{X}S^{*}\right)(V,W) \\ = \; & 2\alpha(X)\,S^{*}(V,W)+\alpha\left(R^{*}(X,V)\,W\right) \\ & +\alpha(V)\,S^{*}(X,W)+\alpha(W)\,S^{*}(V,X) \\ & -\alpha\left(R^{*}(W,X)\,V\right)+2\beta(X)\,ng(V,W) \\ & -2\beta(X)\,g(V,W)+\beta(X)\,g(V,W) \\ & -g(X,W)\,\beta(V)+\beta(V)\,ng(X,W) \\ & -\beta(V)\,g(X,W)+\beta(W)\,ng(V,X) \\ & -\beta(W)\,g(X,V)+\beta(X)\,g(V,W) \\ & -\beta(W)\,g(X,V). \end{aligned} \tag{48}$$

Replacing W by ξ in (48), we get

$$\begin{aligned} & \left(\nabla^{*}_{X}S^{*}\right)(V,\xi) \\ = \; & -2(n-1)\,\overline{C}\alpha(X)\,\eta(V)+\overline{C}\eta(X)\,\alpha(V)-\overline{C}\eta(V)\,\alpha(X) \\ & -(n-1)\,\overline{C}\alpha(V)\,\eta(X)+\alpha(\xi)\,S^{*}(V,X) \\ & -\overline{C}\eta(V)\,\alpha(X)+\overline{C}g(X,V)\,\alpha(\xi)+2n\beta(X)\,\eta(V) \\ & -2\beta(X)\,\eta(V)+\beta(X)\,\eta(V)-\eta(X)\,\beta(V) \\ & +\beta(V)\,n\eta(X)-\beta(V)\,\eta(X) \\ & +\beta(\xi)\,ng(V,X)-\beta(\xi)\,g(X,V) \\ & +\beta(X)\,\eta(V)-\beta(\xi)\,g(X,V). \end{aligned} \tag{49}$$

Now, by using the definition covariant derivative, (8) and (29) we get the following

$$\left(\nabla^{*}_{X}S^{*}\right)(V,\xi)=(1-\lambda)\left[S(V,\phi X)+\overline{C}(n-1)\,g(V,\phi X)-\overline{A}g(V,\phi X)\right]. \tag{50}$$

Taking account of (29), (34), (36), (49) and (50) we get

$$\begin{aligned} & (1-\lambda)\left[S(V,\phi X)+\overline{C}(n-1)\,g(V,\phi X)-\overline{A}g(V,\phi X)\right] \\ = \; & -2(n-1)\,\overline{C}\alpha(X)\,\eta(V)+\overline{C}\eta(X)\,\alpha(V)-\overline{C}\eta(V)\,\alpha(X) \\ & -(n-1)\,\overline{C}\alpha(V)\,\eta(X)+\alpha(\xi)\,S^{*}(V,X) \\ & -\overline{C}\eta(V)\,\alpha(X)+\overline{C}g(X,V)\,\alpha(\xi)+2n\beta(X)\,\eta(V) \\ & -2\beta(X)\,\eta(V)+\beta(X)\,\eta(V)-\eta(X)\,\beta(V) \\ & +\beta(V)\,n\eta(X)-\beta(V)\,\eta(X) \\ & +\beta(\xi)\,ng(V,X)-\beta(\xi)\,g(X,V) \\ & +\beta(X)\,\eta(V)-\beta(\xi)\,g(X,V). \end{aligned} \tag{51}$$

Replacing again, X and V by ξ in the foregoing equation and using (29), (6) we obtain

$$\left[\beta(\xi)-\overline{C}\alpha(\xi)\right]=0. \tag{52}$$

Putting V by ξ in (48) and then using (29), (34), (36) and (50) we get

$$\begin{aligned} & (1-\lambda)\left[S(\phi X,W)+\overline{C}(n-1)g(\phi X,W)-\overline{A}g(\phi X,W)\right] \\ = & -2(n-1)\overline{C}\alpha(X)\eta(W)-2\overline{C}\eta(W)\alpha(X) \\ & +\overline{C}g(X,W)\alpha(\xi)+\alpha(\xi)S^{*}(X,W) \\ & -(n-1)\overline{C}\alpha(W)\eta(X)+\overline{C}\eta(X)\alpha(W) \\ & +(n-2)\beta(\xi)g(X,W)+(n-1)\beta(W)\eta(X) \\ & -\beta(W)\eta(X)+2n\beta(X)\eta(W). \end{aligned} \tag{53}$$

Setting $X=\xi$ in (53) and using (6), and (36) we obtain

$$\begin{aligned} & 0 \\ = & -3(n-1)\overline{C}\alpha(\xi)\eta(W)-\overline{C}\eta(W)\alpha(\xi) \\ & -(n-1)\overline{C}\alpha(W)+\overline{C}\alpha(W) \\ & +3n\beta(\xi)\eta(W)-2\beta(\xi)\eta(W) \\ & +(n-2)\beta(W). \end{aligned} \tag{54}$$

Taking $W=\xi$ in (53) and using (6), and (36) we obtain

$$\begin{aligned} & 0 \\ = & -2(n-1)\overline{C}\alpha(X)-2\overline{C}\alpha(X)+2\overline{C}\eta(X)\alpha(\xi) \\ & -2(n-1)\overline{C}\eta(X)\alpha(\xi)+2(n-2)\beta(\xi)\eta(X) \\ & +2n\beta(X). \end{aligned} \tag{55}$$

Replacing W by X in (54)

$$\begin{aligned} & 0 \\ = & -3(n-1)\overline{C}\alpha(\xi)\eta(X)-\overline{C}\eta(X)\alpha(\xi) \\ & -(n-1)\overline{C}\alpha(X)+\overline{C}\alpha(X) \\ & +3n\beta(\xi)\eta(X)-2\beta(\xi)\eta(X) \\ & +(n-2)\beta(X). \end{aligned} \tag{56}$$

Finally, in view of (52), (54) and (55) we get

$$\left[\beta(X)-\overline{C}\alpha(X)\right]=0. \tag{57}$$

Thus, we can state following

Theorem 2. *There exists no generalised pseudo symmetric Sasakian manifold admitting a general connection, unless $\beta=\overline{C}\alpha$ everywhere, provided $\overline{C}\neq 0$.*

Corollary 1. *There do not exist $[(GPS)_n,\nabla^T]$, $[(GPS)_n,\nabla^s]$, $[(GPS)_n,\nabla^z]$ unless $\beta=0$, everywhere.*

5 Generalised pseudo Ricci symmetric Sasakian manifold admitting general connection

A non flat n-dimensional Sasakian manifold M^n $(n \geq 3)$ is said to be generalised pseudo Ricci symmetric with respect to general connection briefly if the Ricci tensor S^* satisfies the following condition

$$\begin{aligned} & \left(\nabla^*_X S^*\right)(U,V) \\ = \; & 2\alpha^*(X) S^*(U,V) + \alpha^*(U) S^*(X,V) + \alpha^*(V) S^*(U,X) \\ & +2\beta^*(X) g(U,V) + \beta^*(U) g(X,V) + \beta^*(V) g(U,X). \end{aligned} \tag{58}$$

for all vectors fields $U, V \in \chi(M)$ and where α, β being non zero one forms defined by

$$\alpha^*(X) = g(X,\rho); \beta^*(X) = g(X,\sigma). \tag{59}$$

Replacing V by ξ in (58) and using (36) and (50)

$$\begin{aligned} & \left(\nabla^*_X S^*\right)(U,\xi) \\ = \; & -2(n-1)\overline{C}\alpha^*(X)\eta(U) - (n-1)\overline{C}\alpha^*(U)\eta(X) + \alpha^*(\xi) S^*(U,X) \\ & +2\beta^*(X)\eta(U) + \beta^*(U)\eta(X) + \beta^*(\xi) g(U,X), \end{aligned} \tag{60}$$

Putting $X = U = \xi$ in foregoing equation and using (36) we get

$$-(n-1)\overline{C}\alpha^*(\xi) + \beta^*(\xi) = 0. \tag{61}$$

By replacing X by ξ in (60) and using (36), (6) we obtain

$$\begin{aligned} & 0 \\ = \; & -3(n-1)\overline{C}\alpha^*(\xi)\eta(U) - (n-1)\overline{C}\alpha^*(U) \\ & +3\beta^*(\xi)\eta(U) + \beta^*(U). \end{aligned} \tag{62}$$

Putting $U = \xi$ in (60) and using (36), (6) we get

$$\begin{aligned} & 0 \\ = \; & -2(n-1)\overline{C}\alpha^*(X) - 2(n-1)\overline{C}\alpha^*(\xi)\eta(X) \\ & +2\beta^*(X) + 2\beta^*(\xi)\eta(X). \end{aligned} \tag{63}$$

Finally, in view of (61), (62) and (63) we obtain

$$0 = -(n-1)\overline{C}\alpha^*(X) + \beta^*(X). \tag{64}$$

Thus, we can state followings

Theorem 3. *There exists no generalised pseudo ricci symmetric Sasakian manifold admitting a general connection, unless* $(n-1)\overline{C}\alpha^* = \beta^*$ *everywhere, provided* $\overline{C} \neq 0$.

Corollary 2. *There do not exist* $[(GPRS)_n, \nabla^T]$, $[(GPRS)_n, \nabla^s]$ *and* $[(GPRS)_n, \nabla^z]$ *unless* $\beta^* = 0$.

6 Example of a Sasakian manifold with $\left(\left[(GPS)_n, \nabla^*\right]\right)$

With the help of [5](See, Ex. 3.1, P.P. 21-22) we introduce an example of 3-dimensional Sasakian manifold spanned by a set of vector fields $\{e_1, e_2, e_3\}$ defined by

$$e_1 = x(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}) - 2y\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \xi = \frac{\partial}{\partial z},$$

where $\{x; y; z\}$ is a standard coordinate in R^3. Define 1-form η , characteristic vector field ξ, Riemannian metric g and (1-1) tensor ϕ by $\eta(Z) = g(Z, e_3)$, $\xi = \frac{\partial}{\partial z}$, $g(e_i, e_j) = \delta_{ij}$ and $\phi e_1 = -e_2$, $\phi e_2 = e_1$and $\phi e_3 = 0$. Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g. Then we have $[e_1, e_2] = 2e_3$, $[e_1, e_3] = 0$, $[e_2, e_3] = 0$. Thus, $M(\phi, \xi, \eta\ , g)$ defines a Sasakian manifold and the Levi-Civita connection ∇ of the metric tensor g can be obtained by using Koszul's formulas which are as follows:

$$\begin{aligned}
\nabla_{e_1} e_3 &= -e_2, \quad \nabla_{e_1} e_2 = e_3, \quad \nabla_{e_1} e_1 = 0, \\
\nabla_{e_2} e_3 &= e_1, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_1 = -e_3, \\
\nabla_{e_3} e_3 &= 0, \quad \nabla_{e_3} e_2 = e_1, \quad \nabla_{e_3} e_1 = -e_2,
\end{aligned} \tag{65}$$

$$\begin{aligned}
R(e_1, e_2)e_2 &= -3e_1, \quad R(e_1, e_3)e_3 = e_1, \quad R(e_2, e_3)e_3 = e_2, \\
R(e_1, e_2)e_3 &= 0, \quad R(e_3, e_2)e_2 = e_3, \quad R(e_2, e_1)e_1 = -3e_2, \\
R(e_1, e_3)e_2 &= 0, \quad R(e_3, e_1)e_1 = e_3, \quad R(e_2, e_1)e_3 = 0,
\end{aligned} \tag{66}$$

and

$$S\left(e_1, e_1\right) = -2, S\left(e_2, e_2\right) = -2, S\left(e_3, e_3\right) = 2. \tag{67}$$

Using (18), (27) (28), (65) and (66) we obatin the following

$$\begin{aligned}
\nabla^*_{e_1} e_2 &= e_3 + \lambda e_3, \ \nabla^*_{e_1} e_1 = 0; \ \nabla^*_{e_1} e_3 = -e_2 - \lambda e_2; \\
\nabla^*_{e_2} e_1 &= -e_3 - \lambda e_3; \ \nabla^*_{e_2} e_2 = 0; \ \nabla^*_{e_2} e_3 = e_1 + \lambda e_1; \\
\nabla^*_{e_3} e_1 &= -e_2 - \mu e_2; \ \nabla^*_{e_3} e_2 = e_1 + \mu e_1; \ \nabla^*_{e_3} e_3 = 0,
\end{aligned} \tag{68}$$

$$\begin{aligned}
R^*(e_1, e_2)e_2 &= -3e_1 - \left(\lambda^2 - 2\lambda\right) e_1 + 2\mu e_1, \\
R^*(e_2, e_3)e_3 &= e_2 - \left(\lambda - \lambda\mu + \mu\right) e_2, , \\
R^*(e_1, e_3)e_3 &= e_1 - \left(\lambda - \lambda\mu + \mu\right) e_1 \\
R^*(e_2, e_1)e_1 &= -3e_2 - \left(\lambda^2 - 2\lambda\right) e_2 + 2\mu e_2, \\
R^*(e_3, e_2)e_2 &= e_3 - \left(\lambda - \lambda\mu + \mu\right) e_3, \\
R^*(e_2, e_1)e_3 &= 0, \ R^*(e_1, e_2)e_3 = 0, \\
R^*(e_3, e_1)e_1 &= e_3 - \left(\lambda - \lambda\mu + \mu\right) e_3, \\
R^*(e_1, e_3)e_2 &= 0,
\end{aligned} \tag{69}$$

$$S^*\left(e_1, e_1\right) = -2 - \overline{A}, \tag{70}$$

$$S^{*}(e_2,e_2) = -2-\overline{A}, \tag{71}$$

$$S^{*}(e_3,e_3) = -2\overline{C}, \tag{72}$$

and

$$S^{*}(e_1,e_1) = -2-\overline{A}, S^{*}(e_2,e_2) = -2-\overline{A}, S^{*}(e_3,e_3) = -2\overline{C}. \tag{73}$$

Using the above results of curvature tensor and (47) and (58) we obtain

$$\left[\beta(e_i) - \overline{C}\alpha(e_i)\right] = 0; \forall i = 1,2,3; \text{ provided } \overline{C} \neq 0.$$

and

$$-(n-1)\overline{C}\alpha(e_i) + \beta(e_i) = 0, \forall i = 1,2,3; \text{ provided } \overline{C} \neq 0.$$

Hence, this example is the necessary condition for the existence of generalized pseudo symmetric and generalized pseudo Ricci-symmetric Sasakian manifolds admitting a general connection, that is, this example supports **Theorem4.1** and **Theorem5.1**.

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