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FEKETE-SZEGÖ PROBLEM FOR SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH QUASI-SUBORDINATION

Hava ARIKAN[1], Murat ÇAĞLAR[2] and Halit ORHAN[*,3]

Abstract

In this paper, we define certain subclasses of analytic and univalent functions associated with quasi-subordination and we derive the bounds for the Fekete-Szegö functional $\left|a_3 - va_2^2\right|$ for functions belonging to these subclasses.

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1 Introduction and definitions

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U) \tag{1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$, let S denote the subclass of A consisting of analytic and univalent functions f in U. Let g and f be two analytic functions in U then function g is said to be subordinate to f if there exists an analytic function w in the unit disk U with $w(0) = 0$ and $|w(z)| < 1$ such that $g(z) = f(w(z))$ in U.

We denote this subordination by $g \prec f$. In particular, if the f is univalent in U, the above subordination is equivalent to $g(0) = f(0)$ and $g(U) \subset f(U)$. In the year 1970, Robertson [18] introduced the concept of quasi-subordination. The function g is said to be quasi-subordinate to f in the unit disk U if there

[1]Department of Mathematics, Faculty of Science, *Atatürk* University, Erzurum, Turkey, e-mail: hw.arikan@gmail.com
[2]Department of Mathematics, Faculty of Science and Letters, *Kafkas* University, Kars, Turkey, e-mail: mcaglar25@gmail.com
[3*] *Corresponding author*, Department of Mathematics, Faculty of Science, *Atatürk* University, Erzurum, Turkey, e-mail: horhan@atauni.edu.tr

exist the functions w (with constant coefficient zero) and Φ which are analytic and bounded by one in the unit disk U such that $g(z)=\Phi(z)f(w(z))$ and this is equivalent to $\frac{g(z)}{\Phi(z)} \prec f(z)$ in U.

We denote this quasi-subordination by $g \prec_q f$. Observe that when $\Phi(z)=1$, then $g(z)=f(w(z))$, so that $g(z) \prec f(z)$ in U. Also notice that if $w(z)=z$, then $g(z)=\Phi(z)f(z)$ and it is said that g is majorized by f and written $g(z) \ll f(z)$ in U.

Some typical problems in geometric function theory are to study functionals made up of combinations of the coefficients of f. In 1933, Fekete and Szegö [4] obtained a sharp bound of the functional $\lambda a_2^2 - a_3$, with real $\lambda (0 \leq \lambda \leq 1)$ for a univalent function f. Since then, the problem of finding the sharp bounds for this functional of any compact family of functions or $f \in A$ with any complex λ is known as the classical Fekete-Szegö problem or inequality. Lawrence Zalcman posed a conjecture in 1960 that the coefficients of S satisfy the sharp inequality $\left|a_n^2 - a_{2n-1}\right| \leq (n-1)^2$, $n \geq 2$.

More general versions of Zalcman conjecture have also been considered ([3], [12]-[14]) for the functional such as $\lambda a_n^2 - a_{2n-1}$ and $\lambda a_m a_n - a_{m+n-1}$ for certain positive value of λ. These functionals can be seen as generalizations of the Fekete-Szegö functional $\lambda a_2^2 - a_3$. Several authors including [1]-[3], [9]-[15], [17], [20] have investigated the Fekete-Szegö and Zalcman functionals for various subclasses of univalent functions Φ and h analytic in U. Also let

$$\Phi(z) = A_0 + A_1 z + A_2 z^2 + ... \quad (|\Phi(z)| \leq 1, \ z \in U) \tag{2}$$

and

$$h(z) = 1 + B_1 z + B_2 z^2 + ... \quad (B_1 > 0). \tag{3}$$

Motivated by earlier works in ([5]-[7], [15], [17], [19]) on quasi-subordination, we introduce here the following subclass of analytic functions:

Definition 1. *For $0 \leq \beta \leq \lambda \leq 1$ and $b \in \mathbb{C} \setminus \{0\}$, a function $f \in A$ given by (1) is said to be in the class $K_q(\lambda, \beta, b, h)$ if the folowing condition are satisfied:*

$$\frac{1}{b}\left(\frac{\lambda\beta z^3 f'''(z) + (2\lambda\beta + \lambda - \beta) z^2 f''(z) + z f'(z)}{\lambda\beta z^2 f''(z) + (\lambda - \beta) z f'(z) + (1 - \lambda + \beta) f(z)} - 1\right) \prec_q h(z) - 1 \quad (z \in U) \tag{4}$$

where h is given by (3).

It follows that a function f is in the class $K_q(\lambda, \beta, b, h)$ if and only if there exists an analytic function Φ with $|\Phi(z)| \leq 1$, in U such that

$$\frac{\frac{1}{b}\left(\frac{\lambda\beta z^3 f'''(z) + (2\lambda\beta + \lambda - \beta) z^2 f''(z) + z f'(z)}{\lambda\beta z^2 f''(z) + (\lambda - \beta) z f'(z) + (1 - \lambda + \beta) f(z)} - 1\right)}{\Phi(z)} \prec h(z) - 1 \quad (z \in U)$$

where h is given by (3).

If we set $\Phi(z) \equiv 1$, then the class $K_q(\lambda, \beta, b, h)$ is denoted by $K(\lambda, \beta, b, h)$ satisfying the condition that

$$1 + \frac{1}{b}\left(\frac{\lambda\beta z^3 f'''(z) + (2\lambda\beta + \lambda - \beta) z^2 f''(z) + z f'(z)}{\lambda\beta z^2 f''(z) + (\lambda - \beta) z f'(z) + (1 - \lambda + \beta) f(z)} - 1\right) \prec h(z) \quad (z \in U).$$

In the present paper, we find sharp bounds on the Fekete-Szegö functional for functions belonging in the class $K_q(\lambda, \beta, b, h)$. Several known and new consequences of these results are also pointed out. In order to prove our results, we have to recall here the following well-known lemma:

Let Ω be class of analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + ... \tag{5}$$

in the unit disk U satisfying the condition $|w(z)| < 1$.

Lemma 1. *([8], p.10) If $w \in \Omega$, then for any complex number v:*

$$|w_1| \leq 1, \ \left|w_2 - v w_1^2\right| \leq 1 + (|v| - 1)\left|w_1^2\right| \leq \max\{1, |v|\}.$$

The result is sharp for the functions $w(z) = z$ or $w(z) = z^2$.

2 Main results

In this section, we shall obtain Fekete-Szegö inequality for functions in the class $K_q(\lambda, \beta, b, h)$.

Theorem 1. *Let $0 \leq \beta \leq \lambda \leq 1$ and $b \in \mathbb{C} \setminus \{0\}$. If $f \in A$ of the from (1) belong to the class $K_q(\lambda, \beta, b, h)$, then*

$$|a_2| \leq \frac{|b| B_1}{2\lambda\beta + \lambda - \beta + 1} \tag{6}$$

and for any $v \in \mathbb{C}$

$$\left|a_3 - v a_2^2\right| \leq \frac{|b| B_1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)} \max\left\{1, \left|\frac{B_2}{B_1} - Q B_1\right|\right\}, \tag{7}$$

where

$$Q = b\left(\frac{2v(6\lambda\beta + 2\lambda - 2\beta + 1)}{(2\lambda\beta + \lambda - \beta + 1)^2} - 1\right). \tag{8}$$

The results are sharp.

Proof. Let $f \in K_q(\lambda, \beta, b, h)$. In view of Definition 1, there exist a Schwarz function w and an analytic function Φ such that, for $z \in U$:

$$\frac{1}{b}\left(\frac{\lambda\beta z^3 f'''(z) + (2\lambda\beta + \lambda - \beta) z^2 f''(z) + z f'(z)}{\lambda\beta z^2 f''(z) + (\lambda - \beta) z f'(z) + (1 - \lambda + \beta) f(z)} - 1\right) = \Phi(z)(h(w(z)) - 1). \tag{9}$$

Series expansions for f given by (1) and its successive derivatives give us

$$\begin{aligned} & \frac{1}{b}\left(\frac{\lambda\beta z^3 f'''(z) + (2\lambda\beta + \lambda - \beta) z^2 f''(z) + z f'(z)}{\lambda\beta z^2 f''(z) + (\lambda - \beta) z f'(z) + (1 - \lambda + \beta) f(z)} - 1\right) \\ = & \frac{1}{b}(2\lambda\beta + \lambda - \beta + 1) a_2 z \\ & + \frac{1}{b}\left(2(6\lambda\beta + 2\lambda - 2\beta + 1) a_3 - (2\lambda\beta + \lambda - \beta + 1)^2 a_2^2\right) z^2 + ... \end{aligned} \tag{10}$$

Similarly from (2), (3) and (5), we obtain

$$h(w(z)) - 1 = B_1 w_1 z + (B_1 w_2 + B_2 w_1^2) z^2 + ...$$

and

$$\Phi(z)(h(w(z)) - 1) = A_0 B_1 w_1 z + \left[A_1 B_1 w_1 + A_0 (B_1 w_2 + B_2 w_1^2)\right] z^2 + ... \quad (11)$$

Equating (10) and (11) in view of (9) and comparing the coefficients of z and z^2, we get

$$a_2 = \frac{b A_0 B_1 w_1}{2\lambda\beta + \lambda - \beta + 1} \quad (12)$$

and

$$a_3 = \frac{b B_1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)} \left[A_1 w_1 + A_0 \left\{ w_2 + \left(\frac{B_2}{B_1} + b A_0 B_1 \right) w_1^2 \right\}\right]. \quad (13)$$

Thus, for any $v \in \mathbb{C}$, we have

$$\begin{aligned} & a_3 - v a_2^2 \\ = \; & \frac{b B_1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)} \\ & \times \left[A_1 w_1 + A_0 \left\{ w_2 + \left(\frac{B_2}{B_1} + b A_0 B_1 \right) w_1^2 \right\}\right] - v \frac{b^2 A_0^2 B_1^2 w_1^2}{(2\lambda\beta + \lambda - \beta + 1)^2} \\ = \; & \frac{b B_1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)} \\ & \times \left[A_1 w_1 + \left(w_2 + \frac{B_2}{B_1} w_1^2 \right) A_0 - \left(\frac{2(6\lambda\beta + 2\lambda - 2\beta + 1) b}{(2\lambda\beta + \lambda - \beta + 1)^2} v - b \right) A_0^2 B_1 w_1^2 \right] \\ = \; & \frac{b B_1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)} \left[A_1 w_1 + \left(w_2 + \frac{B_2}{B_1} w_1^2 \right) A_0 - Q B_1 A_0^2 w_1^2 \right] \end{aligned} \quad (14)$$

where Q is given by (8).

Since $\Phi(z) = A_0 + A_1 z + A_2 z^2 + ...$ is analytic and bounded by one in U, therefore we have (see [16], p. 172)

$$|A_0| \leq 1 \text{ and } A_1 = (1 - A_0^2) y \quad (y \leq 1). \quad (15)$$

Since $|w_1| \leq 1$ and $|A_0| \leq 1$, from (12) we have

$$|a_2| \leq \frac{|b| \, B_1}{2\lambda\beta + \lambda - \beta + 1}.$$

From (14) and (15), we obtain

$$a_3 - v a_2^2 = \frac{b B_1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)} \left[y w_1 + \left(w_2 + \frac{B_2}{B_1} w_1^2 \right) A_0 - \left(Q B_1 w_1^2 + y w_1 \right) A_0^2 \right]. \quad (16)$$

If $A_0 = 0$ in (16), we at once get

$$\left|a_3 - va_2^2\right| \leq \frac{|b|\, B_1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)}. \tag{17}$$

But if $A_0 \neq 0$, let us then suppose that

$$G(A_0) = yw_1 + \left(w_2 + \frac{B_2}{B_1}w_1^2\right) A_0 - \left(QB_1 w_1^2 + yw_1\right) A_0^2$$

which is a quadratic polynomial in A_0 and hence analytic in $|A_0| \leq 1$ and maximum value of $|G(A_0)|$ is attained at $A_0 = e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$, we find that

$$\begin{aligned} \max |G(A_0)| &= \max_{0\leq\theta\leq 2\pi} \left|G(e^{i\theta})\right| = |G(1)| \\ &= \left|w_2 - \left(QB_1 - \frac{B_2}{B_1}\right) w_1^2\right|. \end{aligned}$$

Therefore, it follows from (16) that

$$\left|a_3 - va_2^2\right| \leq \frac{|b|\, B_1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)} \left|w_2 - \left(QB_1 - \frac{B_2}{B_1}\right) w_1^2\right|, \tag{18}$$

which on using Lemma 1, shows that

$$\left|a_3 - va_2^2\right| \leq \frac{|b|\, B_1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)} \max\left\{1, \left|\frac{B_2}{B_1} - QB_1\right|\right\},$$

and this last above inequality together with (17) establish the results. The results are sharps for the function f given by

$$\begin{aligned} 1 + \frac{1}{b}\left(\frac{\lambda\beta z^3 f'''(z) + (2\lambda\beta + \lambda - \beta)z^2 f''(z) + zf'(z)}{\lambda\beta z^2 f''(z) + (\lambda - \beta)zf'(z) + (1 - \lambda + \beta)f(z)} - 1\right) &= h(z), \\ 1 + \frac{1}{b}\left(\frac{\lambda\beta z^3 f'''(z) + (2\lambda\beta + \lambda - \beta)z^2 f''(z) + zf'(z)}{\lambda\beta z^2 f''(z) + (\lambda - \beta)zf'(z) + (1 - \lambda + \beta)f(z)} - 1\right) &= h(z^2), \\ 1 + \frac{1}{b}\left(\frac{\lambda\beta z^3 f'''(z) + (2\lambda\beta + \lambda - \beta)z^2 f''(z) + zf'(z)}{\lambda\beta z^2 f''(z) + (\lambda - \beta)zf'(z) + (1 - \lambda + \beta)f(z)} - 1\right) &= z(h(z) - 1). \end{aligned}$$

This completes the proof of Theorem 1. □

Remark 1. *i) For $\lambda = 0$, $\beta = 0$ and $b = 1$ in Theorem 1, we obtain result of [15, Theorem 2.1].*

ii) For $\lambda = 1$, $\beta = 0$ and $b = 1$ in Theorem 1, we obtain result of [15, Theorem 2.4].

For $\beta = 0$, the Theorem 1 reduces to following corollary:

Corollary 1. *Let $0 \leq \lambda \leq 1$ and $b \in \mathbb{C} \setminus \{0\}$. If $f \in A$ of the from (1) belong to the class $K_q(\lambda, 0, b, h)$, then*

$$|a_2| \leq \frac{|b| \, B_1}{\lambda + 1},$$

and for some $v \in \mathbb{C}$

$$\left|a_3 - va_2^2\right| \leq \frac{|b| \, B_1}{2(1+2\lambda)} \max\left\{1, \left|\frac{B_2}{B_1} + bB_1\left(\frac{2v(1+2\lambda)}{(\lambda+1)^2} - 1\right)\right|\right\}.$$

The results are sharp.

If we choose $\lambda = 1$ and $b = 1$ in Corollary 1, we obtain the following corollary.

Corollary 2. *If $f \in A$ of the from (1) belong to the class $K_q(1, 0, 1, h)$, then*

$$|a_2| \leq \frac{B_1}{2},$$

and for some $v \in \mathbb{C}$

$$\left|a_3 - va_2^2\right| \leq \frac{B_1}{6} \max\left\{1, \left|\frac{B_2}{B_1} + B_1\left(\frac{3}{2}v - 1\right)\right|\right\}.$$

The results are sharp.

The next theorem gives the result based on majorization.

Theorem 2. *Let $0 \leq \beta \leq \lambda \leq 1$ and $b \in \mathbb{C} \setminus \{0\}$. If $f \in A$ of the form (1) satisfies*

$$\frac{1}{b}\left(\frac{\lambda\beta z^3 f'''(z) + (2\lambda\beta + \lambda - \beta)z^2 f''(z) + zf'(z)}{\lambda\beta z^2 f''(z) + (\lambda - \beta)zf'(z) + (1 - \lambda + \beta)f(z)} - 1\right) \ll h(z) - 1 \quad (z \in U), \tag{19}$$

then

$$|a_2| \leq \frac{|b| \, B_1}{2\lambda\beta + \lambda - \beta + 1}$$

and for any $v \in \mathbb{C}$

$$\left|a_3 - va_2^2\right| \leq \frac{|b| \, B_1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)} \left|QB_1 - \frac{B_2}{B_1}\right|,$$

where Q is given by (8). The results are sharp.

Proof. Assume that (19) holds. From the definition of majorization, there exist an analytic function Φ such that

$$\frac{1}{b}\left(\frac{\lambda\beta z^3 f'''(z) + (2\lambda\beta + \lambda - \beta)z^2 f''(z) + zf'(z)}{\lambda\beta z^2 f''(z) + (\lambda - \beta)zf'(z) + (1 - \lambda + \beta)f(z)} - 1\right) = \Phi(z)(h(z) - 1) \quad (z \in U).$$

Following similar steps as in the proof of Teorem 1, and by setting $w(z) \equiv z$, so that $w_1 = 1$, $w_n = 0$, $n \geq 2$, we obtain

$$a_2 = \frac{bA_0B_1}{2\lambda\beta + \lambda - \beta + 1}$$

and also we obtain that

$$a_3 - va_2^2 = \frac{bB_1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)}\left[A_1 + \frac{B_2}{B_1}A_0 - QB_1A_0^2\right].$$

Since $|A_0| \leq 1$ we easily obtain

$$|a_2| \leq \frac{|b|\,B_1}{2\lambda\beta + \lambda - \beta + 1}.$$

Substituting the value of A_1 from (15), we obtain

$$a_3 - va_2^2 = \frac{bB_1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)}\left[y + \frac{B_2}{B_1}A_0 - (QB_1 + y)A_0^2\right]. \tag{20}$$

If $A_0 = 0$ in (20), we at once get

$$\left|a_3 - va_2^2\right| \leq \frac{|b|\,B_1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)}. \tag{21}$$

But, if $A_0 \neq 0$, let us then suppose that

$$T(A_0) = y + \frac{B_2}{B_1}A_0 - (QB_1 + y)A_0^2,$$

which is a quadratic polynomial in A_0, hence analytic in $|A_0| \leq 1$ and maximum value of $|T(A_0)|$ is attained at $A_0 = e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$, we find that

$$\max|T(A_0)| = \max_{0\leq\theta\leq 2\pi}\left|T(e^{i\theta})\right| = |T(1)|\,.$$

Hence, from (20), we obtain

$$\left|a_3 - va_2^2\right| \leq \frac{|b|\,B_1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)}\left|QB_1 - \frac{B_2}{B_1}\right|.$$

Thus, the assertion of Theorem 2 follows from this last above inequality together with (21). The result are sharp for the function given by

$$1 + \frac{1}{b}\left(\frac{\lambda\beta z^3 f'''(z) + (2\lambda\beta + \lambda - \beta)z^2 f''(z) + zf'(z)}{\lambda\beta z^2 f''(z) + (\lambda - \beta)zf'(z) + (1 - \lambda + \beta)f(z)} - 1\right) = h(z),$$

which completes the proof of Theorem 2. □

Remark 2. *i) For $\lambda = 0$, $\beta = 0$ and $b = 1$ in Theorem 2, we obtain result of [15, Theorem 2.3].*

ii) For $\lambda = 1$, $\beta = 0$ and $b = 1$ in Theorem 2, we obtain result of [15, Theorem 2.5].

Theorem 3. *Let $0 \leq \beta \leq \lambda \leq 1$ and $b \in \mathbb{C} \setminus \{0\}$. If $f \in A$ of the form (1) belong to the class $K(\lambda, \beta, b, h)$, then*

$$|a_2| \leq \frac{|b|\, B_1}{2\lambda\beta + \lambda - \beta + 1}$$

and for any $v \in \mathbb{C}$

$$\left|a_3 - va_2^2\right| \leq \frac{|b|\, B_1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)} \max\left\{1, \left|\frac{B_2}{B_1} - QB_1\right|\right\},$$

where Q is given by (8), the results are sharp.

Proof. The proof is similar to Theorem 1. Let $f \in K(\lambda, \beta, b, h)$. If $\Phi(z) = 1$, then $A_0 = 1$, $A_n = 0$ $(n \in \mathbb{N})$. Therefore, in view of (12) and (14) and by application of Lemma 1, we obtain the desired assertion. The results are sharp for the function f given by

$$\begin{aligned} 1 + \frac{1}{b}\left(\frac{\lambda\beta z^3 f'''(z) + (2\lambda\beta + \lambda - \beta)z^2 f''(z) + zf'(z)}{\lambda\beta z^2 f''(z) + (\lambda - \beta)zf'(z) + (1 - \lambda + \beta)f(z)} - 1\right) &= h(z), \\ 1 + \frac{1}{b}\left(\frac{\lambda\beta z^3 f'''(z) + (2\lambda\beta + \lambda - \beta)z^2 f''(z) + zf'(z)}{\lambda\beta z^2 f''(z) + (\lambda - \beta)zf'(z) + (1 - \lambda + \beta)f(z)} - 1\right) &= h(z^2). \end{aligned}$$

Thus, the proof of Theorem 3 is completed. □

Theorem 4. *Let $0 \leq \beta \leq \lambda \leq 1$. If $f \in A$ of the form (1) belong to the class $K_q(\lambda, \beta, b, h)$, then for real v and b, we have*

$$\begin{aligned} &\left|a_3 - va_2^2\right| \\ &\leq \begin{cases} \frac{|b|B_1}{2(6\lambda\beta+2\lambda-2\beta+1)}\left[B_1 b\left(1 - \frac{2(6\lambda\beta+2\lambda-2\beta+1)}{(2\lambda\beta+\lambda-\beta+1)^2}v\right) + \frac{B_2}{B_1}\right], & v \leq \sigma_1, \\ \frac{|b|B_1}{2(6\lambda\beta+2\lambda-2\beta+1)}, & \sigma_1 \leq v \leq \sigma_1 + 2\rho, \\ -\frac{|b|B_1}{2(6\lambda\beta+2\lambda-2\beta+1)}\left[B_1 b\left(1 - \frac{2(6\lambda\beta+2\lambda-2\beta+1)}{(2\lambda\beta+\lambda-\beta+1)^2}v\right) + \frac{B_2}{B_1}\right], & v \geq \sigma_1 + 2\rho, \end{cases} \end{aligned} \tag{22}$$

where

$$\sigma_1 = \frac{(2\lambda\beta + \lambda - \beta + 1)^2}{2(6\lambda\beta + 2\lambda - 2\beta + 1)} - \frac{(2\lambda\beta + \lambda - \beta + 1)^2}{2b(6\lambda\beta + 2\lambda - 2\beta + 1)}\left(\frac{1}{B_1} - \frac{B_2}{B_1^2}\right), \tag{23}$$

$$\rho = \frac{(2\lambda\beta + \lambda - \beta + 1)^2}{2b(6\lambda\beta + 2\lambda - 2\beta + 1)B_1}. \tag{24}$$

Each of the estimates in (22) are sharp.

Proof. For real values of v and b the above bounds can be obtained from (7), respectively, under the following cases:

$$QB_1 - \frac{B_2}{B_1} \leq -1, \ -1 \ \leq QB_1 - \frac{B_2}{B_1} \leq 1 \text{ and } QB_1 - \frac{B_2}{B_1} \geq 1,$$

where Q is given by (8). We also note the following: □

(i) When $v < \sigma_1$ or $v > \sigma_1 + 2\rho$, then the equality holds if and only if $\Phi(z) \equiv 1$ and $w(z) = z$ or one of its rotations.

(ii) When $\sigma_1 < v < \sigma_1 + 2\rho$, then the equality holds if and only if $\Phi(z) \equiv 1$ and $w(z) = z^2$ or one of its rotations.

(iii) Equality holds for $v = \sigma_1$ if and only if $\Phi(z) \equiv 1$ and $w(z) = \frac{z(z+\varepsilon)}{1+\varepsilon z}$ $(0 \leq \varepsilon \leq 1)$, or one of its rotations, while for $v = \sigma_1 + 2\rho$, the equality holds if and only if $\Phi(z) \equiv 1$ and $w(z) = -\frac{z(z+\varepsilon)}{1+\varepsilon z}$ $(0 \leq \varepsilon \leq 1)$, or one of its rotations.

The bounds of the functional $a_3 - va_2^2$ for real values of v and b for the middle range of the parameter v can be improved further as follows:

Theorem 5. *Let $0 \leq \beta \leq \lambda \leq 1$. If $f \in A$ of the form (1) belong to the class $K_q(\lambda, \beta, b, h)$, then for real v and b, we arrive*

$$\left|a_3 - va_2^2\right| + (v - \sigma_1)\left|a_2\right|^2 \leq \frac{|b|\, B_1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)} \quad (\sigma_1 \leq v \leq \sigma_1 + \rho) \tag{25}$$

and

$$\left|a_3 - va_2^2\right| + (\sigma_1 + 2\rho - v)\left|a_2\right|^2 \leq \frac{|b|\, B_1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)} \quad (\sigma_1 + \rho \leq v \leq \sigma_1 + 2\rho) \tag{26}$$

where σ_1 and ρ are given by (23) and (24), respectively.

Proof. Let $f \in K_q(\lambda, \beta, b, h)$. For real v satisfying $\sigma_1 + \rho \leq v \leq \sigma_1 + 2\rho$ and using (12) and (18) we get

$$\begin{aligned} &\left|a_3 - va_2^2\right| + (v - \sigma_1)\left|a_2\right|^2 \\ \leq\ & \frac{|b|\, B_1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)} \left[|w_2| - \frac{2\,|b|\, B_1(6\lambda\beta + 2\lambda - 2\beta + 1)}{(2\lambda\beta + \lambda - \beta + 1)^2}(v - \sigma_1 - \rho)\left|w_1\right|^2 \right. \\ & \left. + \frac{2\,|b|\, B_1(6\lambda\beta + 2\lambda - 2\beta + 1)}{(2\lambda\beta + \lambda - \beta + 1)^2}(v - \sigma_1)\left|w_1\right|^2 \right]. \end{aligned}$$

Therefore, by virtue of Lemma 1, we get

$$\left|a_3 - va_2^2\right| + (v - \sigma_1)\left|a_2\right|^2 \leq \frac{|b|\, B_1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)},$$

which yields the assertion (25).

If $\sigma_1 + \rho \leq v \leq \sigma_1 + 2\rho$, then again from (12), (18) and the application of Lemma 1, we have

$$\begin{aligned} &\left|a_3 - va_2^2\right| + (\sigma_1 + 2\rho - v)\left|a_2\right|^2 \\ \leq\ & \frac{|b|\, B_1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)} \left[|w_2| + \frac{2\,|b|\, B_1(6\lambda\beta + 2\lambda - 2\beta + 1)}{(2\lambda\beta + \lambda - \beta + 1)^2}(v - \sigma_1 - \rho)\left|w_1\right|^2 \right. \\ & \left. + \frac{2\,|b|\, B_1(6\lambda\beta + 2\lambda - 2\beta + 1)}{(2\lambda\beta + \lambda - \beta + 1)^2}(\sigma_1 + 2\rho - v)\left|w_1\right|^2 \right], \end{aligned}$$

$$\left|a_3 - va_2^2\right| + (\sigma_1 + 2\rho - v)\left|a_2\right|^2 \leq \frac{|b|\, B_1}{2(6\lambda\beta + 2\lambda - 2\beta + 1)},$$

which estimates (26). □

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