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ON UNIVALENCE OF A NEW INTEGRAL OPERATOR

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Abstract

In this paper we define a new integral operator and we obtain univalence criteria for this integral operator.

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1 Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of the functions $f \in \mathcal{A}$, wich are univalent in $\mathcal{U}$.

We consider the integral operator H_β for $f \in \mathcal{A}$ and some complex number $\beta\,(\beta \neq 0)$, which is given by

$$H_\beta(z) = \left\{ \frac{1}{\beta} \int_0^z u^{-1} \left(f(u)\right)^{\frac{1}{\beta}} du \right\}^\beta . \qquad (1)$$

Miller and Mocanu [3] have studied that the integral operator H_β is in the class $\mathcal{S}$ for $f \in \mathcal{S}^*$, $\beta \in \mathbb{R}$, $\beta > 0$, where $\mathcal{S}^*$ is the subclass of $\mathcal{S}$ consisting of all starlike functions in $\mathcal{U}$.

We define the integral operator

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$$J_{\gamma_1,\dots,\gamma_n,\alpha,n}(z) =$$

$$= \left\{ \left(\sum_{j=1}^{n} \frac{1}{\gamma_j} \right) \int_0^z u^{-\alpha} \left(f_1(u)\right)^{\frac{1}{\gamma_1}+\frac{\alpha-1}{n}} \dots \left(f_n(u)\right)^{\frac{1}{\gamma_n}+\frac{\alpha-1}{n}} du \right\}^{\frac{1}{\sum_{j=1}^{n}\frac{1}{\gamma_j}}}, \qquad (2)$$

for $f_j \in \mathcal{A}$ and complex numbers α, γ_j $(\gamma_j \neq 0)$, $j = \overline{1,n}$, which is a generalization of integral operator H_β.

For $n = 1$, $f_1 = f$, $\gamma_1 = \beta$, $\alpha = 1$, from (2) we obtain the integral operator H_β.

Properties of certain integral operators were studied by different authors in the following papers [10, 11, 12, 13, 14].

2 Preliminary results

We need the following lemmas.

Lemma 1. *[4]. Let α be a complex number, $Re\ \alpha > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \qquad (1)$$

for all $z \in \mathcal{U}$, then the integral operator F_α defined by

$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} f'(u) du \right]^{\frac{1}{\alpha}}, \qquad (2)$$

is in the class S.

Lemma 2. *(Schwarz [2]). Let f be the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m,\ (z \in \mathcal{U}_R), \qquad (3)$$

the equality (in the inequality (3) for $z \neq 0$) can hold if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

Lemma 3. *(Caratheodory [1]). Let f be analytic function in $\mathcal{U}$, with $f(0) = 0$. If f satisfies*

$$Re\ f(z) \leq M, \qquad (4)$$

for some $M > 0$, then

$$(1-|z|)|f(z)| \leq 2M|z|,\ \ (z \in \mathcal{U}). \qquad (5)$$

3 Main results

Theorem 1. *Let γ_j, α be complex numbers, $Re\,\gamma_j \neq 0$, M_j real positive numbers, $j=\overline{1,n}$, $p=\sum_{j=1}^{n} Re\,\frac{1}{\gamma_j}>0$ and $f_j \in \mathcal{A}$, $f_j(z)=z+a_{2j}z^2+a_{3j}z^3+\ldots$, $j=\overline{1,n}$.*

If

$$\left|\frac{zf_j'(z)}{f_j(z)}-1\right| \leq M_j, \quad \left(z\in\mathcal{U};\ j=\overline{1,n}\right) \tag{1}$$

and

$$\sum_{j=1}^{n} M_j\left[\frac{1}{|\gamma_j|}+\frac{|\alpha-1|}{n}\right] \leq \frac{3\sqrt{3}}{2}p, \quad (0<p<1) \tag{2}$$

or

$$\sum_{j=1}^{n} M_j\left[\frac{1}{|\gamma_j|}+\frac{|\alpha-1|}{n}\right] \leq \frac{3\sqrt{3}}{2}, \quad (p\geq 1), \tag{3}$$

then the integral operator $J_{\gamma_1,\gamma_2,\ldots,\gamma_n,\alpha,n}$ given by (2) is in the class $\mathcal{S}$.

Proof. We observe that

$$J_{\gamma_1,\gamma_2,\ldots,\gamma_n,\alpha,n}(z)=$$

$$=\left\{\left(\sum_{j=1}^{n}\frac{1}{\gamma_j}\right)\int_0^z u^{\sum_{j=1}^{n}\frac{1}{\gamma_j}-1}\left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}+\frac{\alpha-1}{n}}\ldots\left(\frac{f_n(u)}{u}\right)^{\frac{1}{\gamma_n}+\frac{\alpha-1}{n}}du\right\}^{\frac{1}{\sum_{j=1}^{n}\frac{1}{\gamma_j}}} \tag{4}$$

We consider the function

$$h(z)=\int_0^z\left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}+\frac{\alpha-1}{n}}\ldots\left(\frac{f_n(u)}{u}\right)^{\frac{1}{\gamma_n}+\frac{\alpha-1}{n}}du, \tag{5}$$

for $f_j\in\mathcal{A}$, $j=\overline{1,n}$. The function h is regular in $\mathcal{U}$.

We define the function g by

$$g(z)=\frac{zh''(z)}{h'(z)}, \quad (z\in\mathcal{U}). \tag{6}$$

We have $g(0)=0$ and from (5) and (6) we get

$$|g(z)|\leq\sum_{j=1}^{n}\left[\frac{1}{|\gamma_j|}+\frac{|\alpha-1|}{n}\right]\left|\frac{zf_j'(z)}{f_j(z)}-1\right|, \quad (z\in\mathcal{U}). \tag{7}$$

From (1) and (7) we obtain

$$|g(z)| \leq \sum_{j=1}^{n} M_j \left[\frac{1}{|\gamma_j|} + \frac{|\alpha - 1|}{n}\right], \tag{8}$$

for all $z \in \mathcal{U}$.

Applying Lemma 2 we get

$$|g(z)| \leq \sum_{j=1}^{n} M_j \left[\frac{1}{|\gamma_j|} + \frac{|\alpha - 1|}{n}\right] |z|, \quad (z \in \mathcal{U}). \tag{9}$$

From (6) and (9) we have

$$\frac{1-|z|^{2p}}{p}\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{1-|z|^{2p}}{p}|z| \sum_{j=1}^{n} M_j \left[\frac{1}{|\gamma_j|} + \frac{|\alpha - 1|}{n}\right], \tag{10}$$

for all $z \in \mathcal{U}$.

For $p \in (0,1)$ we have $1 - |z|^{2p} \leq 1 - |z|^2,\ z \in \mathcal{U}$ and from (10) we get

$$\frac{1-|z|^{2p}}{p}\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{(1-|z|^2)|z|}{p} \sum_{j=1}^{n} M_j \left[\frac{1}{|\gamma_j|} + \frac{|\alpha - 1|}{n}\right], \ (z \in \mathcal{U}). \tag{11}$$

Since

$$\max_{|z|\leq 1}\left\{(1-|z^2|)|z|\right\} = \frac{2}{3\sqrt{3}}, \tag{12}$$

from (2) and (11) we have

$$\frac{1-|z|^{2p}}{p}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1 \tag{13}$$

for all $z \in \mathcal{U}$ and $p \in (0,1)$.

For $p \in [1,\infty)$ we get $\frac{1-|z|^{2p}}{p} \leq 1 - |z|^2,\ z \in \mathcal{U}$ and using (10) we obtain

$$\frac{1-|z|^{2p}}{p}\left|\frac{zh''(z)}{h'(z)}\right| \leq (1-|z|^2)|z| \sum_{j=1}^{n} M_j \left[\frac{1}{|\gamma_j|} + \frac{|\alpha - 1|}{n}\right], \ (z \in \mathcal{U}). \tag{14}$$

From (3), (12) and (14) we have

$$\frac{1-|z|^{2p}}{p}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1, \tag{15}$$

for all $z \in \mathcal{U}$ and $p \in [1,\infty)$.

From (5) we obtain $h'(z) = \left(\frac{f_1(z)}{z}\right)^{\frac{1}{\gamma_1}+\frac{\alpha-1}{n}} \ldots \left(\frac{f_n(z)}{z}\right)^{\frac{1}{\gamma_n}+\frac{\alpha-1}{n}}$ and using (13), (15), by Lemma 1 it results that the integral operator $J_{\gamma_1,\gamma_2,\ldots,\gamma_n,\alpha,n}$ given by (2) is in the class $\mathcal{S}$. □

Corollary 1. *Let β be a complex number, $Re\ \beta \neq 0$, $Re\ \frac{1}{\beta} > 0$ and $f \in \mathcal{A}$,*

$$f(z) = z + a_{21}z^2 + a_{31}z^3 + \dots$$

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{3\sqrt{3}}{2}|\beta| Re\ \frac{1}{\beta}, \quad \left(z \in \mathcal{U};\ 0 < Re\ \frac{1}{\beta} < 1\right) \tag{16}$$

or

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{3\sqrt{3}}{2}|\beta|, \quad \left(z \in \mathcal{U};\ Re\ \frac{1}{\beta} \geq 1\right), \tag{17}$$

then the integral operator $H_\beta \in \mathcal{S}$.

Proof. For $n = 1,\ \alpha = 1,\ \gamma_1 = \beta,\ f_1 = f$, and $p = Re\ \frac{1}{\beta}$, from Theorem 1 we obtain Corollary 1. □

Corollary 2. *Let the function $f \in \mathcal{A},\ f(z) = z + a_{21}z^2 + a_{31}z^3 + \dots$*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{3\sqrt{3}}{2}, \quad (z \in \mathcal{U}), \tag{18}$$

then the integral operator Alexander, given by

$$G(z) = \int_0^z \frac{f(u)}{u} du$$

is in the class $\mathcal{S}$.

Proof. In Theorem 1 we take $n = 1,\ \alpha = 1,\ \gamma_1 = 1,\ f_1 = f$ and we obtain $G \in \mathcal{S}$. □

Theorem 2. *Let $\gamma_j,\ \alpha$ be complex numbers, $Re\ \gamma_j \neq 0,\ j = \overline{1,n}$,*

$$p = \sum_{j=1}^{n} Re\ \frac{1}{\gamma_j} > 0,\ f_j \in \mathcal{A},\ f_j(z) = z + a_{2j}z^2 + a_{3j}z^3 + \dots,\quad j = \overline{1,n}.$$

If

$$Re\left[e^{i\theta}\left(\frac{zf_j'(z)}{f_j(z)} - 1\right)\right] \leq \frac{Re\ \frac{1}{\gamma_j}}{4\left[\frac{1}{|\gamma_j|} + \frac{|\alpha-1|}{n}\right]},\ \left(j = \overline{1,n};\ p \in (0,1)\right) \tag{19}$$

or

$$Re\left[e^{i\theta}\left(\frac{zf_j'(z)}{f_j(z)} - 1\right)\right] \leq \frac{1}{4n\left[\frac{1}{|\gamma_j|} + \frac{|\alpha-1|}{n}\right]},\ \left(j = \overline{1,n};\ p \in [1,\infty)\right), \tag{20}$$

for all $z \in \mathcal{U}$ and $\theta \in [0, 2\pi)$, then the integral operator $J_{\gamma_1,\gamma_2,\dots,\gamma_n,\alpha,n}$, defined by (2), is in the class $\mathcal{S}$.

Proof. We take

$$g(z)=\int_0^z \left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}+\frac{\alpha-1}{n}} \dots \left(\frac{f_n(u)}{u}\right)^{\frac{1}{\gamma_n}+\frac{\alpha-1}{n}} du, \tag{21}$$

the function g is regular in $\mathcal{U}$. We have

$$\frac{zg''(z)}{g'(z)}=\sum_{j=1}^{n}\left[\frac{1}{\gamma_j}+\frac{\alpha-1}{n}\right]\left(\frac{zf_j'(z)}{f_j(z)}-1\right), \quad (z\in\mathcal{U}). \tag{22}$$

Let us consider the function

$$\varphi_j(z)=e^{i\theta}\left(\frac{zf_j'(z)}{f_j(z)}-1\right), \quad \left(j=\overline{1,n};\ \theta\in[0,2\pi)\right), \tag{23}$$

$z\in\mathcal{U}$ and we observe that $\varphi_j(0)=0,\ j=\overline{1,n}$.

By (19) and Lemma 3, for $p\in(0,1)$ we obtain

$$|\varphi_j(z)|\le\frac{|z|Re\,\frac{1}{\gamma_j}}{2(1-|z|)\left[\frac{1}{|\gamma_j|}+\frac{|\alpha-1|}{n}\right]} \ , \quad \left(j=\overline{1,n};\ z\in\mathcal{U}\right). \tag{24}$$

From (20) and Lemma 3, for $p\in[1,\infty)$ we have

$$|\varphi_j(z)|\le\frac{|z|}{2n(1-|z|)\left[\frac{1}{|\gamma_j|}+\frac{|\alpha-1|}{n}\right]} \ , \quad \left(j=\overline{1,n};\ z\in\mathcal{U}\right). \tag{25}$$

From (22) and (24) we get

$$\frac{1-|z|^{2p}}{p}\left|\frac{zg''(z)}{g'(z)}\right|\le\frac{1-|z|^{2p}|z|}{2(1-|z|)} \ , \quad (z\in\mathcal{U};\ \ p\in(0,1)). \tag{26}$$

Since $1-|z|^{2p}\le 1-|z|^2$ for $p\in(0,1),\ \ z\in\mathcal{U}$, from (26) we have

$$\frac{1-|z|^{2p}}{p}\left|\frac{zg''(z)}{g'(z)}\right|\le 1\ , \tag{27}$$

for all $z\in\mathcal{U},\ p\in(0,1)$.

For $p\in[1,\infty)$, we have $\frac{1-|z|^{2p}}{p}\le 1-|z|^2, \quad z\in\mathcal{U}$ and from (22), (25) it results that

$$\frac{1-|z|^{2p}}{p}\left|\frac{zg''(z)}{g'(z)}\right|\le 1\ , \tag{28}$$

for all $z\in\mathcal{U},\ p\in[1,\infty)$.

From (21) we have $g'(z)=\left(\frac{f_1(z)}{z}\right)^{\frac{1}{\gamma_1}+\frac{\alpha-1}{n}}\dots\left(\frac{f_n(z)}{z}\right)^{\frac{1}{\gamma_n}+\frac{\alpha-1}{n}}$ and using (27) and (28), by Lemma 1 it results that $J_{\gamma_1,\gamma_2\dots,\gamma_n,\alpha,n}$ given by (2) is in the class $\mathcal{S}$. □

Corollary 3. *Let β be a complex number, $Re\,\beta \neq 0,\ p = Re\,\frac{1}{\beta} > 0,\ f \in \mathcal{A}$,*

$$f(z) = z + a_{21}z^2 + a_{31}z^3 + \dots$$

If

$$Re\left[e^{i\theta}\left(\frac{zf'(z)}{f(z)} - 1\right)\right] \leq \frac{|\beta| Re\,\frac{1}{\beta}}{4}, \quad (p \in (0,1)) \tag{29}$$

or

$$Re\left[e^{i\theta}\left(\frac{zf'(z)}{f(z)} - 1\right)\right] \leq \frac{|\beta|}{4}, \quad (p \in [1,\infty)) \tag{30}$$

for all $z \in \mathcal{U}$ and $\theta \in [0, 2\pi)$, then the integral operator H_β is in the class $\mathcal{S}$.

Proof. In Theorem 2 we take $n = 1,\ f_1 = f,\ \gamma_1 = \beta,\ \alpha = 1$. □

Corollary 4. *Let the function $f \in \mathcal{A},\ f(z) = z + a_{21}z^2 + a_{31}z^3 + \dots$*
If

$$Re\left[e^{i\theta}\left(\frac{zf'(z)}{f(z)} - 1\right)\right] \leq \frac{1}{4}, \tag{31}$$

for all $z \in \mathcal{U}$ and $\theta \in [0, 2\pi)$, then the integral operator Alexander given by

$$G(z) = \int_0^z \frac{f(u)}{u} du \tag{32}$$

is in the class $\mathcal{S}$.

Proof. In Theorem 2 we take $n = 1,\ f_1 = f,\ \gamma_1 = 1,\ \alpha = 1$ and we obtain $G \in \mathcal{S}$. □

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