Bulletin of the *Transilvania* University of Braşov • Vol 12(61), No. 2 - 2019
Series III: Mathematics, Informatics, Physics, 199-212
https://doi.org/10.31926/but.mif.2019.12.61.2.1

CR-HYPERSURFACES OF CONFORMAL KENMOTSU MANIFOLDS WITH ξ-PARALLEL NORMAL JACOBI OPERATOR

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Abstract

In this paper, we study CR-hypersurfaces of a conformal Kenmotsu manifold with a ξ-parallel or Lie ξ-parallel normal Jacobi operator.

2000 *Mathematics Subject Classification:* 53C25, 53C40.
Key words: conformal Kenmotsu manifold, Jacobi operator.

1 Introduction

In [9], Kenmotsu defined and studied a new class of almost contact manifolds called Kenmotsu manifolds.
Let (M, J, g) be an almost Hermitian manifold of dimension $2n$, where J denotes the almost complex structure and g the Hermitian metric. Then (M, J, g) is called a locally conformal Kaehler manifold if for each point p of M there exists an open neighborhood U of p and a positive function f_U on U so that the local metric $g_U = \exp(-f)g_{|U}$ is Kaehlerian. If $U = M$, then manifold (M, J, g) is said to be a globally conformal Kaehler manifold. The 1-form $\omega = df$ is called the Lee form and its metrically equivalent vector field $\omega^\sharp = grad\ f$, where $\sharp$ means the rising of the indices with respect to g, namely $g(X, \omega^\sharp) = \omega(X)$ for all X tangent to M, is called Lee vector field [8]. Submanifolds of locally conformal Kaehler manifolds with parallel Lee form have been studied by several authors (see, for instance, [11]).
We have introduced conformal Kenmotsu manifolds by using an idea of globally conformal Kaehler manifolds. Also, we have given an example of a conformal Kenmotsu manifold that is not Kenmotsu. Hence the category of conformal Kenmotsu manifolds and Kenmotsu manifolds is not the same.
The definition of a conformal Kenmotsu manifold is as follows.

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A $(2n+1)$-dimensional smooth manifold M with almost contact metric structure (φ, η, ξ, g) is called a conformal Kenmotsu manifold if there exists a positive smooth function $f : M \to \mathbb{R}$ so that

$$\tilde{g} = \exp(f)g, \qquad \tilde{\xi} = (\exp(-f))^{\frac{1}{2}}\xi, \qquad \tilde{\eta} = (\exp(f))^{\frac{1}{2}}\eta, \qquad \tilde{\varphi} = \varphi$$

is a Kenmotsu structure on M (see [1]-[3]).

In fact, manifold M with almost contact metric structure (φ, η, ξ, g) is not Kenmotsu, but with a conformal change of the metric g, that is, $\tilde{g} = \exp(f)g$ is Kenmotsu. Thus, there exist two structures (φ, η, ξ, g) and $(\varphi, \tilde{\eta}, \tilde{\xi}, \tilde{g})$ on M. Let $\tilde{\nabla}$ and ∇ be the Riemannian connections of M with respect to metrics $\tilde{g}$ and g, respectively and $\tilde{R}$ and R denote the curvature tensors of $\tilde{\nabla}$ and ∇, respectively. We have calculated the relation between $\tilde{R}$ and R as

$$\begin{aligned}
\exp(-f)\tilde{g}(\tilde{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) \\
&+ \frac{1}{2}\{B(X,Z)g(Y,W) - B(Y,Z)g(X,W) \\
&+ B(Y,W)g(X,Z) - B(X,W)g(Y,Z)\} \\
&+ \frac{1}{4}\|\omega^\sharp\|^2\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\}
\end{aligned}$$

for all vector fields X, Y, Z, W on M, where

$$B := \nabla\omega - \frac{1}{2}\omega \otimes \omega.$$

In [10], Kobayashi has proven: let M be a submanifold of a Kenmotsu manifold $\tilde{M}$ such that the structural vector field $\xi\mid_M$ is tangent to M, then

$$\nabla_X\xi = X - \eta(X)\xi, \qquad h(X,\xi) = 0$$

for each vector field X tangent to M, where ∇ and h are the Riemannian connection and the second fundamental form of M, respectively.

In this paper as a generalization of these results, we state Lemmas 3.1 and 3.2 for a submanifold of a conformal Kenmotsu manifold.

In quaternionic space forms Berndt [5] has introduced the notion of normal Jacobi operator $\bar{R}_N(X) = \bar{R}(X,N)N \in End\ T_xM$, $x \in M$ for every real hypersurface M in a quaternionic projective space $\mathbb{Q}P^m$ or in a quaternionic hyperbolic space $\mathbb{Q}H^m$, where $\bar{R}$ denotes the curvature tensor of the ambient space. He has also shown in [5] that the curvature adaptedness, that is, the normal Jacobi operator $\bar{R}_N$ commuting with the shape operator of M, is equivalent to the fact that distributions $\mathcal{D}$ and $\mathcal{D}^\perp = Span\{\xi_1, \xi_2, \xi_3\}$ are invariant by the shape operator, where $T_xM = \mathcal{D} \oplus \mathcal{D}^\perp$, $x \in M$. Motivated by this study, we present the following problem:

Can we characterize CR-hypersurfaces in conformal Kenmotsu manifolds with a ξ-parallel or Lie ξ-parallel normal Jacobi operator such that the structural vector field ξ is tangent and the Lee vector field $\omega^\sharp$ is either tangent or normal to the

CR-hypersurface?
Before considering the answer of the above question, an example for the existence of CR-hypersurfaces in conformal Kenmotsu manifolds tangent to ξ and either tangent or normal to $\omega^\sharp$ is constructed.
Corresponding to the above problem, we give the following theorems.

- Let $\acute{M}$ be a CR-hypersurface of a conformal Kenmotsu manifold M with a ξ-parallel normal Jacobi operator R_N such that $\omega^\sharp \mid_{\acute{M}}$ is normal to $\acute{M}$. Then $\acute{M}$ is totally umbilic with scalar shape operator $\frac{1}{2}id$ iff $\tilde{R}_N$ is ξ-parallel.
- Let $\acute{M}$ be a CR-hypersurface of a conformal Kenmotsu manifold M with a Lie ξ-parallel normal Jacobi operator R_N such that $\omega^\sharp \mid_{\acute{M}}$ is normal to $\acute{M}$. Then $\acute{M}$ is totally umbilic with scalar shape operator $\frac{1}{2}id$ iff $\tilde{R}_N$ is ξ-parallel.
- Let $\acute{M}$ be a CR-hypersurface of a conformal Kenmotsu manifold M with a ξ-parallel normal Jacobi operator R_N such that $\omega^\sharp \mid_{\acute{M}}$ is tangent to $\acute{M}$ and parallel on $\acute{M}$. Then $\omega^\sharp$ is an eigen vector field with eigen value $-\exp(f)$ for $\tilde{R}_N$ and $-\exp(f) - \frac{1}{2}(\omega(\nabla_N N) - \frac{1}{2} \parallel \omega^\sharp \parallel^2)$ for R_N and $\tilde{R}_N$ cannot be ξ-parallel.

One can find the above results in Theorems 1, 2 and 3 and Corollary 1.
The present paper is organized as follows. In Section 2, we recall some definitions and notions about conformal Kenmotsu manifolds. Section 3 gives some preliminary lemmas on submanifolds of a conformal Kenmotsu manifold. Also, we present an example for the existence of submanifolds in conformal Kenmotsu manifolds tangent to ξ and either tangent or normal to $\omega^\sharp$. Section 4 deals with the study of submanifolds in conformal Kenmotsu manifolds with a ξ-parallel or Lie ξ-parallel normal Jacobi operator.

2 Conformal Kenmotsu manifolds

A $(2n+1)$-dimensional differentiable manifold M is an almost contact metric manifold, if it admits an almost contact metric structure (φ, ξ, η, g) consisting of a tensor field φ of type $(1,1)$, a vector field ξ, a 1-form η and a Riemannian metric g satisfying the following properties:

$$\varphi^2 = -Id + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y),$$
$$\varphi\xi = 0, \qquad \eta o \varphi = 0, \qquad \eta(X) = g(X, \xi)$$

for all vector fields X, Y on M [6].
An almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is said to be a Kenmotsu manifold and an α-Kenmotsu manifold if the following relations

$$(\nabla_X \varphi)Y = -g(X, \varphi Y)\xi - \eta(Y)\varphi X \tag{1}$$

and

$$(\nabla_X \varphi)Y = \alpha\{-g(X, \varphi Y)\xi - \eta(Y)\varphi X\} \tag{2}$$

hold on M, respectively, where ∇ denotes the Riemannian connection of g and α is a constant function on M. From (1) for a Kenmotsu manifold, we have

$$\nabla_X \xi = X - \eta(X)\xi. \tag{3}$$

For a Kenmotsu manifold, we also have

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X \tag{4}$$

for all vector fields X, Y tangent to M, where R is the curvature tensor of M (see [9]).

A $(2n+1)$-dimensional smooth manifold M with almost contact metric structure (φ, η, ξ, g) is called a conformal Kenmotsu manifold if there exists a positive smooth function $f : M \to \mathbb{R}$ so that

$$\tilde{g} = \exp(f)g, \qquad \tilde{\xi} = (\exp(-f))^{\frac{1}{2}}\xi, \qquad \tilde{\eta} = (\exp(f))^{\frac{1}{2}}\eta, \qquad \tilde{\varphi} = \varphi$$

is a Kenmotsu structure on M (see [1]-[3]).

Let M be a conformal Kenmotsu manifold, with $\tilde{\nabla}$ and ∇ denoting the Riemannian connections of M with respect to metrics $\tilde{g}$ and g, respectively. Using the Koszul formula, one can simply obtain the following relation between $\tilde{\nabla}$ and ∇:

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\{\omega(X)Y + \omega(Y)X - g(X,Y)\omega^\sharp\} \tag{5}$$

for all vector fields X, Y on M, where $\omega(X) = g(grad\, f, X) = X(f)$. Note that the vector field $\omega^\sharp = grad\ f$ is called the Lee vector field of the conformal Kenmotsu manifold M. Then from $\eta(X) = g(X, \xi)$, we have the equality $\eta(\omega^\sharp) = \omega(\xi)$. Although $\omega(\omega^\sharp) = \|\omega^\sharp\|^2$, it is not necessarily $\|\omega^\sharp\|^2 = 1$, that is, $\omega^\sharp$ is not necessarily a unit vector field.

Assuming that $\tilde{R}$ and R are the curvature tensors of $(M, \varphi, \tilde{\eta}, \tilde{\xi}, \tilde{g})$ and $(M, \varphi, \eta, \xi, g)$, respectively. We have the following relation between $\tilde{R}$ and R:

$$\begin{aligned} \exp(-f)\tilde{g}(\tilde{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) \\ &+ \frac{1}{2}\{B(X,Z)g(Y,W) - B(Y,Z)g(X,W) \\ &+ B(Y,W)g(X,Z) - B(X,W)g(Y,Z)\} \\ &+ \frac{1}{4}\|\omega^\sharp\|^2\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\} \end{aligned} \tag{6}$$

for all vector fields X, Y, Z, W on M, where B satisfies

$$B := \nabla\omega - \frac{1}{2}\omega \otimes \omega. \tag{7}$$

Obviously, B is a symmetric tensor field of type (0,2). On the other hand, from equations (1), (3) and (5), we get

$$\begin{aligned} (\nabla_X\varphi)Y &= (\exp(f))^{\frac{1}{2}}\{-g(X,\varphi Y)\xi - \eta(Y)\varphi X\} \\ &- \frac{1}{2}\{\omega(\varphi Y)X - \omega(Y)\varphi X + g(X,Y)\varphi\omega^\sharp - g(X,\varphi Y)\omega^\sharp\}, \end{aligned} \tag{8}$$

$$\nabla_X\xi = (\exp(f))^{\frac{1}{2}}\{X - \eta(X)\xi\} - \frac{1}{2}\{\omega(\xi)X - \eta(X)\omega^\sharp\} \tag{9}$$

for all vector fields X, Y on M.
Note that if function f is constant on the conformal Kenmotsu manifold M, i.e. $\omega^\sharp = 0$, then M is an α-Kenmotsu manifold in view of (2) and (8). In this paper, we suppose that the conformal Kenmotsu manifold M is non-α-Kenmotsu (and hence non-Kenmotsu), that is, f is non-constant, so $\omega^\sharp$ is a non-zero vector field on M.

3 Submanifolds of conformal Kenmotsu manifolds

Let $(\acute{M}, \acute{g})$ be an m-dimensional submanifold into a $(2n+1)$-dimensional conformal Kenmotsu manifold (M, g). The Gauss and Weingarten formulas are given by

$$\nabla_X Y = \acute{\nabla}_X Y + h(X, Y), \qquad \nabla_X N = -A_N X + \nabla^\perp_X N$$

for all vector fields X, Y tangent to $\acute{M}$ and each vector field N normal to $\acute{M}$, where $\acute{\nabla}$ is the Riemannian connection of $\acute{M}$ determined by the induced metric $\acute{g}$ and $\nabla^\perp$ is the normal connection on $T^\perp\acute{M}$ of $\acute{M}$. It is known that $g(h(X, Y), N) = \acute{g}(A_N X, Y)$, where A_N is the shape operator of $\acute{M}$ with respect to unit normal vector field N.
In this paper, we assume that $\xi\mid_{\acute{M}}$ is tangent to $\acute{M}$.

3.1 Example

In this subsection, we construct an example of a five-dimensional conformal Kenmotsu manifold which is not Kenmotsu. Also, we present two submanifolds M_1 and M_2 in M such that the structural vector field ξ is tangent to both M_1 and M_2 and the Lee vector field $\omega^\sharp$ is tangent to M_1 and normal to M_2.
We consider the five-dimensional manifold

$$M = \{(x_1, x_2, y_1, y_2, z) \in \mathbb{R}^5 \mid x_1 > 0, z \neq 0\},$$

where (x_1, x_2, y_1, y_2, z) are the standard coordinates in $\mathbb{R}^5$. We choose the vector fields

$$e_1 = \exp(-z)\frac{\partial}{\partial x_1}, \qquad e_2 = \exp(-z)\frac{\partial}{\partial x_2}, \qquad e_3 = \exp(-z)\frac{\partial}{\partial y_1},$$
$$e_4 = \exp(-z)\frac{\partial}{\partial y_2}, \qquad e_5 = (\exp(x_1))^{\frac{1}{2}}\frac{\partial}{\partial z},$$

which are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = \exp(-x_1), \quad g(e_5, e_5) = 1$$

and the remaining $g(e_i, e_j) = 0$, $i, j : 1, \cdots, 5$. Let η be the 1-form defined by $\eta(X) = g(X, e_5)$ for each vector field X on M. Thus, we have

$$\eta(e_1) = 0, \quad \eta(e_2) = 0, \quad \eta(e_3) = 0, \quad \eta(e_4) = 0, \quad \eta(e_5) = 1.$$

We define the $(1,1)$-tensor field φ as

$$\varphi e_1 = e_3, \quad \varphi e_2 = e_4, \quad \varphi e_3 = -e_1, \quad \varphi e_4 = -e_2, \quad \varphi e_5 = 0.$$

Then using the linearity of φ and g, we have

$$\varphi^2 X = -X + \eta(X)e_5, \qquad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for all vector fields X, Y on M. Thus, for $e_5 = \xi$, (φ, ξ, η, g) defines an almost contact metric structure on M. Moreover, by the definition of bracket on manifolds we get

$$[e_1, e_5] = (\exp(x_1))^{\frac{1}{2}} e_1 + \frac{1}{2}\exp(-z)e_5, \qquad [e_2, e_5] = (\exp(x_1))^{\frac{1}{2}} e_2,$$
$$[e_3, e_5] = (\exp(x_1))^{\frac{1}{2}} e_3, \qquad [e_4, e_5] = (\exp(x_1))^{\frac{1}{2}} e_4$$

and the remaining $[e_i, e_j] = 0$, $i, j : 1, \cdots, 5$. The Riemannian connection ∇ of metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &- g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]), \end{aligned}$$

which is known as Koszul formula. By using this formula, we obtain

$$\begin{aligned}
&\nabla_{e_1} e_1 = -\frac{1}{2}\exp(-z)e_1 - (\exp(-x_1))^{\frac{1}{2}} e_5, \quad \nabla_{e_1} e_2 = -\frac{1}{2}\exp(-z)e_2, \\
&\nabla_{e_1} e_3 = -\frac{1}{2}\exp(-z)e_3, \qquad \nabla_{e_1} e_4 = -\frac{1}{2}\exp(-z)e_4, \\
&\nabla_{e_1} e_5 = (\exp(x_1))^{\frac{1}{2}} e_1, \qquad \nabla_{e_2} e_1 = -\frac{1}{2}\exp(-z)e_2, \\
&\nabla_{e_2} e_2 = \frac{1}{2}\exp(-z)e_1 - (\exp(-x_1))^{\frac{1}{2}} e_5, \quad \nabla_{e_2} e_5 = (\exp(x_1))^{\frac{1}{2}} e_2, \\
&\nabla_{e_3} e_1 = -\frac{1}{2}\exp(-z)e_3, \quad \nabla_{e_3} e_3 = \frac{1}{2}\exp(-z)e_1 - (\exp(-x_1))^{\frac{1}{2}} e_5, \\
&\nabla_{e_3} e_5 = (\exp(x_1))^{\frac{1}{2}} e_3, \qquad \nabla_{e_4} e_1 = -\frac{1}{2}\exp(-z)e_4, \\
&\nabla_{e_4} e_4 = \frac{1}{2}\exp(-z)e_1 - (\exp(-x_1))^{\frac{1}{2}} e_5, \quad \nabla_{e_5} e_1 = -\frac{1}{2}\exp(-z)e_5, \\
&\nabla_{e_5} e_5 = \frac{1}{2}\exp(x_1 - z)e_1, \qquad \nabla_{e_4} e_5 = (\exp(x_1))^{\frac{1}{2}} e_4
\end{aligned}$$

and the remaining $\nabla_{e_i} e_j = 0$, $i, j : 1, \cdots, 5$. By the following conformal change

$$\tilde{g} = \exp(x_1)g, \qquad \tilde{\xi} = (\exp(-x_1))^{\frac{1}{2}}\xi, \qquad \tilde{\eta} = (\exp(x_1))^{\frac{1}{2}}\eta, \qquad \tilde{\varphi} = \varphi,$$

it can be easily considerd that $(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a Kenmotsu manifold (see [7]). Thus, $(M, \varphi, \xi, \eta, g)$ is a conformal Kenmotsu manifold but is not Kenmotsu, Since we have

$$(\nabla_X \varphi)Y \neq -g(X, \varphi Y)\xi - \eta(Y)\varphi X$$

for some vector fields X, Y on M (for instance, $(\nabla_{e_4}\varphi)e_2 \neq -g(e_4, \varphi e_2)\xi - \eta(e_2)\varphi e_4$).

Suppose $M_1 = \{(x_1, y_1, y_2, z) \in \mathbb{R}^4 \mid (x_1, y_1, y_2, z) \neq 0\}$ is a four-dimensional submanifold of M with the isometric immersion defined by

$$\begin{aligned} &\iota_1 : M_1 \to M \\ &\iota(x_1, y_1, y_2, z) = (x_1, 0, y_1, y_2, z), \end{aligned}$$

where (x_1, y_1, y_2, z) are the standard coordinates in $\mathbb{R}^4$. We choose the vector fields

$$\begin{aligned} e_1 &= \exp(-z)\frac{\partial}{\partial x_1}, & e_3 &= \exp(-z)\frac{\partial}{\partial y_1}, \\ e_4 &= \exp(-z)\frac{\partial}{\partial y_2}, & e_5 &= (\exp(x_1))^{\frac{1}{2}}\frac{\partial}{\partial z}, \end{aligned}$$

which are linearly independent at each point of M_1. Then, e_1, e_3, e_4 and e_5 form a basis for the tangent space of M_1 and e_2 spans the normal space of M_1 in M. Let g_1 be the induced metric on M_1. Thus, we have

$$g_1(e_1, e_1) = g_1(e_3, e_3) = g_1(e_4, e_4) = \exp(-x_1), \qquad g_1(e_5, e_5) = 1.$$

Using $\omega(Y) = Y(x_1)$, for each vector field Y on M, it can be easily calculated that

$$\omega(e_1) = e_1(x_1) = \exp(-z),\ \ \omega(e_2) = 0,\ \ \omega(e_3) = 0,\ \ \omega(e_4) = 0,\ \ \omega(e_5) = 0.$$

We see that M_1 is a hypersurface of the conformal Kenmotsu manifold M such that $\omega^\sharp \mid_{M_1}$ and $\xi \mid_{M_1}$ are tangent to $\acute{M}$.

Now, let $M_2 = \{(x_2, y_1, y_2, z) \in \mathbb{R}^4 \mid (x_2, y_1, y_2, z) \neq 0\}$ be a four-dimensional submanifold of M with the isometric immersion defined by

$$\begin{aligned} &\iota_2 : (M_2, g_2) \to (M, g) \\ &\iota_2(x_2, y_1, y_2, z) = (2, x_2, y_1, y_2, z), \end{aligned}$$

where (x_2, y_1, y_2, z) are the standard coordinates in $\mathbb{R}^4$. We choose the vector fields

$$\begin{aligned} e_2 &= \exp(-z)\frac{\partial}{\partial x_2}, & e_3 &= \exp(-z)\frac{\partial}{\partial y_1}, \\ e_4 &= \exp(-z)\frac{\partial}{\partial y_2}, & e_5 &= \exp(1)\frac{\partial}{\partial z}, \end{aligned}$$

which are linearly independent at each point of M_2. Then, e_2, e_3, e_4 and e_5 form a basis for the tangent space of M_2 and e_1 spans the normal space of M_2 in M. Suppose g_2 is the induced metric on M_2. Then, we have

$$g_2(e_2, e_2) = g_2(e_3, e_3) = g_2(e_4, e_4) = \exp(-2), \qquad g_2(e_5, e_5) = 1.$$

Thus, M_2 is a hypersurface of the conformal Kenmotsu manifold M such that $\xi\mid_{M_2}$ and $\omega^\sharp\mid_{M_2}$ are tangent and normal to M_2, respectively, in view of the values $\omega(e_i)$ for all $i:1,\cdots,5$.
Now, we give some preliminary lemmas on the submanifold $\acute{M}$ of the conformal Kenmotsu manifold M tangent to ξ and either tangent or normal to $\omega^\sharp$.

Lemma 1. *[3] Let $\acute{M}$ be a submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp\mid_{\acute{M}}$ is normal to $\acute{M}$. Then*

$$B(X,Y)=-\omega(h(X,Y)), \tag{10}$$
$$h(X,\xi)=\frac{1}{2}\eta(X)\omega^\sharp, \tag{11}$$
$$\acute{\nabla}_X\xi=(\exp(f))^{\frac{1}{2}}\{X-\eta(X)\xi\} \tag{12}$$

for all vector fields X,Y tangent to $\acute{M}$.

Proof. From (7) we have

$$B(X,Y)=(\nabla_X\omega)Y-\frac{1}{2}\omega(X)\omega(Y)=\nabla_X(\omega(Y))-\omega(\nabla_XY)-\frac{1}{2}\omega(X)\omega(Y)$$

for all X,Y tangent to $\acute{M}$. Since $\omega^\sharp\mid_{\acute{M}}$ is normal to $\acute{M}$, the above equation can be written as

$$B(X,Y)=-\omega(\nabla_XY)$$

for all X,Y on $\acute{M}$. Then by the use of the Gauss formula we obtain (10).
Taking $Y=\xi$ in the Gauss formula and using (9), we have

$$\acute{\nabla}_X\xi+h(X,\xi)=\nabla_X\xi=(\exp(f))^{\frac{1}{2}}\{X-\eta(X)\xi\}-\frac{1}{2}\{\omega(\xi)X-\eta(X)\omega^\sharp\}$$

for each X tangent to $\acute{M}$. Since $\omega^\sharp\mid_{\acute{M}}$ is normal to $\acute{M}$, comparing the tangential part and the normal part in the above equation, we obtain (11) and (12). □

Lemma 2. *[3] Let $\acute{M}$ be a submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp\mid_{\acute{M}}$ is tangent to $\acute{M}$. Then*

$$B(X,Y)=\acute{g}(\acute{\nabla}_X\omega^\sharp,Y)-\frac{1}{2}\omega(X)\omega(Y), \tag{13}$$
$$h(X,\xi)=0, \tag{14}$$
$$\acute{\nabla}_X\xi=(\exp(f))^{\frac{1}{2}}\{X-\eta(X)\xi\}-\frac{1}{2}\{\omega(\xi)X-\eta(X)\omega^\sharp\} \tag{15}$$

for all vector fields X,Y tangent to $\acute{M}$.

Proof. Similarly to Lemma 1, equations (13), (14) and (15) are immediate results of (7), (9) and the Gauss formula. □

Lemma 3. *[3] Let $\acute{M}$ be a submanifold of a conformal Kenmotsu manifold M such that $\omega^\sharp\mid_{\acute{M}}$ is tangent to $\acute{M}$ and parallel on $\acute{M}$. Then*

$$\omega(\xi) \neq 0. \tag{16}$$

Proof. The proof of relation (16) is given by contradiction. Suppose $\omega(\xi) = 0$. Taking the covariant differentation of $\omega(\xi) = 0$ with respect to ξ and using $\acute{\nabla}\omega^\sharp = 0$, we obtain

$$\acute{g}(\acute{\nabla}_\xi \xi, \omega^\sharp) = 0.$$

Using (15) in the above equation, we get

$$\|\omega^\sharp\|^2 = \omega(\xi)^2.$$

Since we have assumed that $\omega(\xi) = 0$, from the above equation it follows that $\|\omega^\sharp\|^2 = 0$ which contradicts the hypothesis $\omega^\sharp \neq 0$. Hence (16) holds on $\acute{M}$. □

4 CR-hypersurfaces with a ξ-parallel normal Jacobi operator

An m-dimensional Riemannian submanifold $\acute{M}$ of a conformal Kenmotsu manifold M is called a CR-submanifold [4] if ξ is tangent to $\acute{M}$ and there exists a differentiable distribution $D : x \in \acute{M} \longrightarrow D_x \subset T_x\acute{M}$ such that
(1) the distribution D_x is invariant under φ, that is, $\varphi(D_x) \subset D_x$ for each $x \in \acute{M}$;
(2) the complementary orthogonal distribution $D^\perp : x \in \acute{M} \longrightarrow D_x^\perp \subset T_x\acute{M}$ of D is anti-invariant under φ, that is, $\varphi D_x^\perp \subset T_x^\perp \acute{M}$ for all $x \in \acute{M}$, where $T_x\acute{M}$ and $T_x^\perp\acute{M}$ are the tangent space and the normal space of $\acute{M}$ at x, respectively.
Now, assume $\acute{M}$ is a hypersurface of a conformal Kenmotsu manifold M such that the vector field ξ always belongs to the tangent space of $\acute{M}$. Let $\acute{g}$ be the induced metric on $\acute{M}$. Also, let N be a unit normal vector field belonging to the normal space of $\acute{M}$. We put $\varphi N = -U$. Clearly U is a unit tangent vector field on $\acute{M}$. We denote by $D^\perp = span\{U, \xi\}$ the 2-dimensional distribution generated by U, ξ and by D the orthogonal complement of $D^\perp$ in $T\acute{M}$. Thus, we have the following decompositions

$$TM = D \oplus D^\perp \oplus span\{N\}, \tag{17}$$
$$T\acute{M} = D \oplus D^\perp, \tag{18}$$

hence $\acute{M}$ is a CR-hypersurface of M.
Let $\acute{M}$ be a CR-hypersurface of a conformal Kenmotsu manifold M. Denote by ∇ and $\acute{\nabla}$ the Riemannian connection of M and the induced Riemannian connection of $\acute{M}$, respectively. By using (17) and (18), the Gauss and Wiengarten formulas are

$$\nabla_X Y = \acute{\nabla}_X Y + h(X, Y),$$
$$\nabla_X N = -AX$$

for all X, Y tangent to $\acute{M}$, where A is the shape operator of $\acute{M}$ with respect to the unit normal vector field N. It is known that $h(X,Y) = \acute{g}(AX,Y)N$, for all vector fields X, Y on $\acute{M}$.
In the usual way, by using (6) we derive the Codazzi equation as

$$\begin{aligned}\acute{g}((\acute{\nabla}_X A)Y - (\acute{\nabla}_Y A)X, Z) &= \exp(-f)\tilde{g}(\tilde{R}(X,Y)Z, N) \\ &+ \frac{1}{2}\{B(X,N)\acute{g}(Y,Z) - B(Y,N)\acute{g}(X,Z)\}\end{aligned} \tag{19}$$

for all vector fields X, Y, Z tangent to $\acute{M}$.
Let (M, g) be a Riemannian manifold. The Jacobi operator R_X, for each tangent vector field X at $x \in M$, is defined by

$$(R_X Y)(x) = (R(Y,X)X)(x),$$

for each Y orthogonal to X at $x \in M$. It becomes a self adjoint endomorphism of the tangent bundle TM of M, where R denotes the curvature tensor of (M, g). Then the normal Jacobi operator $R_N : T\acute{M} \longrightarrow T\acute{M}$ for the unit normal vector field N of a CR-hypersurface $\acute{M}$ in a conformal Kenmotsu manifold M can be obtained from (6) by putting $Y = Z = N$. Hence, we have

$$\begin{aligned}\acute{g}(R_N(X), Y) &= \exp(-f)\tilde{g}(\tilde{R}_N(X), Y) \\ &+ \frac{1}{2}\{B(N,N)\acute{g}(X,Y) + B(X,Y)\} + \frac{1}{4}\|\omega^\sharp\|^2\, \acute{g}(X,Y)\end{aligned} \tag{20}$$

for all vector fields X, Y on $\acute{M}$. Making use of (4) and the definition of a conformal Kenmotsu manifold, we can write

$$\begin{aligned}\tilde{R}_N\xi = (\exp(f))^{\frac{1}{2}}\tilde{R}_N\tilde{\xi} &= (\exp(f))^{\frac{1}{2}}(-\tilde{g}(N,N)\tilde{\xi} + \tilde{\eta}(N)N) \\ &= -\tilde{g}(N,N)\xi = -\exp(f)g(N,N)\xi \\ &= -\exp(f)\xi.\end{aligned} \tag{21}$$

Now, we have the following results:

Theorem 1. *Let $\acute{M}$ be a CR-hypersurface of a conformal Kenmotsu manifold M with a ξ-parallel normal Jacobi operator R_N such that $\omega^\sharp\mid_{\acute{M}}$ is normal to $\acute{M}$. Then $\acute{M}$ is totally umbilic with scalar shape operator $\frac{1}{2}id$ iff $\tilde{R}_N$ is ξ-parallel.*

Proof. Since $\omega^\sharp\mid_{\acute{M}}$ is orthogonal to $\acute{M}$, we put $\omega^\sharp = N$. Then from (7) and the Weingarten formula, we have

$$B(N,N) = -\frac{1}{2}, \tag{22}$$

$$B(X,N) = 0, \tag{23}$$

$$B(X,Y) = -\acute{g}(AX,Y) \tag{24}$$

for all vector fields X, Y tangent to $\acute{M}$. By the use of (10) and (22) in (20), we obtain

$$R_N(X) = \tilde{R}_N(X) - \frac{1}{2}AX \tag{25}$$

for each vector field X on $\acute{M}$. Taking the covariant differentiation of (25) and removing the similar sentences, we get

$$(\acute{\nabla}_\xi R_N)X = (\acute{\nabla}_\xi \tilde{R}_N)X - \frac{1}{2}(\acute{\nabla}_\xi A)X \tag{26}$$

for each vector field X on $\acute{M}$. From (19) we obtain

$$\begin{aligned}(\acute{\nabla}_\xi A)X &= (\acute{\nabla}_X A)\xi + \frac{1}{2}\{B(X,N)\xi - B(\xi,N)X\} \\ &= \acute{\nabla}_X A\xi - A\acute{\nabla}_X\xi + \frac{1}{2}\{B(X,N)\xi - B(\xi,N)X\}. \end{aligned} \tag{27}$$

From (11) it follows that $A\xi = \frac{1}{2}\xi$. Thus, putting (12) and (23) in (27), we find

$$(\acute{\nabla}_\xi A)X = (\exp(f))^{\frac{1}{2}}(\frac{1}{2}X - AX) \tag{28}$$

for each vector field X on $\acute{M}$. Since $\acute{\nabla}_\xi R_N = 0$, by the use of (28) in (26), we obtain

$$(\acute{\nabla}_\xi \tilde{R}_N)X = (\exp(f))^{\frac{1}{2}}(\frac{1}{2}X - AX)$$

for each vector field X on $\acute{M}$. The above equation completes the proof of the theorem. □

Theorem 2. *Let $\acute{M}$ be a CR-hypersurface of a conformal Kenmotsu manifold M with a Lie ξ-parallel normal Jacobi operator R_N such that $\omega^\sharp\mid_{\acute{M}}$ is normal to $\acute{M}$. Then $\acute{M}$ is totally umbilic with scalar shape operator $\frac{1}{2}id$ iff $\tilde{R}_N$ is ξ-parallel.*

Proof. Since the normal Jacobi operator of $\acute{M}$ is Lie ξ-parallel, we have

$$\begin{aligned} 0 = (L_\xi R_N)X &= L_\xi R_N X - R_N(L_\xi X) \\ &= (\acute{\nabla}_\xi R_N)X - \acute{\nabla}_{R_N(X)}\xi + R_N(\acute{\nabla}_X\xi) \end{aligned} \tag{29}$$

for each vector field X on $\acute{M}$, where L_ξ shows the Lie derivative relative to ξ. From (12), (21) and (25), we get

$$-\acute{\nabla}_{R_N(X)}\xi + R_N(\acute{\nabla}_X\xi) = 0 \tag{30}$$

for each vector field X on $\acute{M}$. Substituting (26) and (30) in (29), it follows that

$$(\acute{\nabla}_\xi \tilde{R}_N)X - \frac{1}{2}(\acute{\nabla}_\xi A)X = 0. \tag{31}$$

Putting (28) in (31), we get

$$(\acute{\nabla}_\xi \tilde{R}_N)X - \frac{1}{2}(\exp(f))^{\frac{1}{2}}(\frac{1}{2}X - AX) = 0$$

for each vector field X on $\acute{M}$. Hence, the above equation completes the proof of the theorem. □

Theorem 3. *Let $\acute{M}$ be a CR-hypersurface of a conformal Kenmotsu manifold M with a ξ-parallel normal Jacobi operator R_N such that $\omega^\sharp\mid_{\acute{M}}$ is tangent to $\acute{M}$ and parallel on $\acute{M}$. Then $\omega^\sharp\mid_{\acute{M}}$ is an eigen vector field corresponding to eigen value $-\exp(f)$ of $\tilde{R}_N$ and $-\exp(f) - \frac{1}{2}(\omega(\nabla_N N) - \frac{1}{2}\parallel \omega^\sharp \parallel^2)$ of R_N.*

Proof. From (7), we can write

$$B(N,N) = -\omega(\nabla_N N), \tag{32}$$

$$B(X,Y) = -\frac{1}{2}\omega(X)\omega(Y) \tag{33}$$

for all vector fields X, Y tangent to $\acute{M}$. Making use of (32) and (33) in (20), we have

$$R_N(X) = \tilde{R}_N(X) - \frac{1}{2}\{\omega(\nabla_N N)X + \frac{1}{2}\omega(X)\omega^\sharp - \parallel \omega^\sharp \parallel^2 X\}. \tag{34}$$

Taking the covariant differentiation of (34), we get

$$(\acute{\nabla}_\xi R_N)X = \acute{\nabla}_\xi R_N(X) - R_N \acute{\nabla}_\xi X = (\acute{\nabla}_\xi \tilde{R}_N)X - \frac{1}{2}\omega(\acute{\nabla}_\xi \nabla_N N)X$$

for each vector field X tangent to $\acute{M}$. Since R_N is ξ-parallel, the above equation implies

$$(\acute{\nabla}_\xi \tilde{R}_N)X = \frac{1}{2}\omega(\acute{\nabla}_\xi \nabla_N N)X \tag{35}$$

for each vector field X tangent to $\acute{M}$. By (21), we find

$$\begin{aligned}\acute{g}((\acute{\nabla}_\xi \tilde{R}_N)\xi, \xi) &= \acute{g}(\acute{\nabla}_\xi(\tilde{R}_N\xi), \xi) - \acute{g}(\tilde{R}_N(\acute{\nabla}_\xi \xi), \xi) \\ &= \acute{g}(\acute{\nabla}_\xi(-\exp(f)\xi), \xi) - \acute{g}(\tilde{R}_N(\acute{\nabla}_\xi \xi), \xi) \\ &= -\exp(f)\omega(\xi) - \acute{g}(\tilde{R}_N(\acute{\nabla}_\xi \xi), \xi).\end{aligned}$$

Using (15) in the above equation, it follows that

$$\acute{g}((\acute{\nabla}_\xi \tilde{R}_N)\xi, \xi) = -\exp(f)\omega(\xi) + \frac{1}{2}\omega(\xi)\acute{g}(\tilde{R}_N\xi, \xi) - \frac{1}{2}\acute{g}(\tilde{R}_N\omega^\sharp, \xi).$$

As $\tilde{R}_N$ is symmetric, the above equation and (21) yield

$$\begin{aligned}\acute{g}((\acute{\nabla}_\xi \tilde{R}_N)\xi, \xi) &= -\exp(f)\omega(\xi) + \frac{1}{2}\omega(\xi)\acute{g}(\tilde{R}_N\xi, \xi) - \frac{1}{2}\acute{g}(\tilde{R}_N\xi, \omega^\sharp). \\ &= -\exp(f)\omega(\xi).\end{aligned} \tag{36}$$

Puting $X = \xi$ in (35) and taking the inner product of the obtained relation with ξ and using (36), we have

$$\frac{1}{2}\omega(\acute{\nabla}_\xi \nabla_N N) = -\exp(f)\omega(\xi). \tag{37}$$

substituting (37) in (35), we get

$$(\acute{\nabla}_\xi \tilde{R}_N)X = -\exp(f)\omega(\xi)X \tag{38}$$

for each vector field X tangent to $\acute{M}$. Taking $X = \xi$ in the above equation and using (21) and (15), we can write

$$\begin{aligned}
-\exp(f)\omega(\xi)\xi = (\acute{\nabla}_\xi \tilde{R}_N)\xi &= \acute{\nabla}_\xi(\tilde{R}_N\xi) - \tilde{R}_N(\acute{\nabla}_\xi\xi) \\
&= -\acute{\nabla}_\xi(\exp(f)\xi) - \tilde{R}_N(-\frac{1}{2}(\omega(\xi)\xi - \omega^\sharp)) \\
&= -\exp(f)\omega(\xi)\xi - \exp(f)\acute{\nabla}_\xi\xi + \frac{1}{2}\omega(\xi)\tilde{R}_N\xi \\
&- \frac{1}{2}\tilde{R}_N\omega^\sharp \\
&= -\exp(f)\omega(\xi)\xi + \frac{1}{2}\exp(f)(\omega(\xi)\xi - \omega^\sharp) \\
&- \frac{1}{2}\exp(f)\omega(\xi)\xi - \frac{1}{2}\tilde{R}_N\omega^\sharp.
\end{aligned}$$

The above equation implies

$$\tilde{R}_N\omega^\sharp = -\exp(f)\omega^\sharp.$$

The above relation shows that $\omega^\sharp \mid_{\acute{M}}$ is an eigen vector field of $\tilde{R}_N$ corresponding to eigen value $-\exp(f)$. Moreover, making use of the above relation in (34) we see that $\omega^\sharp$ is an eigen vector field of R_N corresponding to eigen value $-\exp(f) - \frac{1}{2}(\omega(\nabla_N N) - \frac{1}{2} \parallel \omega^\sharp \parallel^2)$. □

Corollary 1. *Let $\acute{M}$ be a CR-hypersurface of a conformal Kenmotsu manifold M with a ξ-parallel normal Jacobi operator R_N such that $\omega^\sharp \mid_{\acute{M}}$ is tangent to $\acute{M}$ and parallel on $\acute{M}$. Then $\tilde{R}_N$ cannot be ξ-parallel.*

Proof. It is an immediate result of (16) and (38). □

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