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# COVERING $C_4 \bigoplus e$ BY THE SAME LABEL

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**Abstract**

In this paper we have proven a result for a covered graph with at least one subgraph $C_4 \bigoplus e$. We have also mentioned some observations and conditions for a graph containing $C_4 \bigoplus e$. An algorithm along with the flowchart, that describes the impact of covering a specified $C_4 \bigoplus e$ with a common label is described by us.

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*Keywords:* feasible labeling, covered graph.

## 1 Introduction

A graph G is a pair $(V(G), E(G))$ where $V(G)$ is a nonempty finite set of elements known as vertices and $E(G)$ is family of unordered pairs of elements of $V(G)$ known as edges.
A path is a sequence of vertices and edges of a graph in which neither vertices nor edges are repeated. A cycle $C_n$ is a path that starts from a given vertex and ends at the same vertex containing n vertices[5].
By labeling of graph we mean allotting integers or symbols to the vertices or edges or both, satisfying some conditions. There are different labeling techniques such as sum labeling, graceful labeling, feasible labeling etc.
In [2] feasible labeling with a specific condition is discussed which leads to "Covering a graph".
In [8] we have proven that three graphs, namely Prism, Wheel and Corona graph are covered for specific parameters $k = 2, t = 4$. In this paper we are working on covered graph containing a specific subgraph for the parameters $k = 2, t = 4$.

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This can be considered as an extension of a result proved in [2] for the parameters $k = 2, t = 3$.
Feasible labeling and covered graph are defined as follows: [2] [8].
$G = (V, E)$ be an undirected graph, $k$ and $t$ be positive integers and $\sum$ be the set of symbols. Then a feasible labeling is defined as an assignment of a set $L_v \in \sum$ to each vertex $v \in V$, such that

(i) $\mid L_v \mid \leq k$ for all $v \in V$

(ii) each label $\alpha \in \sum$ is used no more than t times.

An edge $e = \{i, j\}$ is said to be covered by a feasible labeling if $L_i \cap L_j \neq \phi$. A graph $G$ is said to be a covered graph if there exists a feasible labeling that covers each edge $e \in E$. Hence, a graph G is said to be covered if we have an assignment of at most $k$ labels to each vertex of $G$ such that each label is allotted to at most $t$ vertices and there is at least one common label among the labels allotted at the two end points of each edge.

We define some terminologies [5]:

**Definition 1.1.** *Diagonal vertices of $C_4 \bigoplus e$ are the vertices which are the end points of a diagonal 'e'.*

"$C_4 \bigoplus$ e" is the subgraph $C_4$ with a diagonal edge 'e'.
It can be seen in Figure 1.

**Definition 1.2.** *Non-diagonal vertices of $C_4 \bigoplus e$ are the vertices which are not the end points of a diagonal 'e'.*

**Definition 1.3.** *Two distinct $C_4 \bigoplus e$ are said to be adjacent if they share either one edge in common or one vertex in common. Otherwise, they are called non-adjacent.*

If it is not a disjoint component then vertices of $C_4 \bigoplus e$ have additional edges incident on it, such that maximum degree is not greater than 6.

In [2] it is proved that a graph G can be covered if deg $_G(i) \leq k(t-1)$ for all $i \in V$.

In [2] it is also proved that for $k = 2, t = 3$ if $G$ has one or more triangles and if there exists a feasible labeling of G then there exists a feasible labeling in which a triangle is covered by the same label.

Motivated with this result we worked on covered graph containing one or more subgraph "$C_4 \bigoplus$ e" for $k = 2, t = 4$.

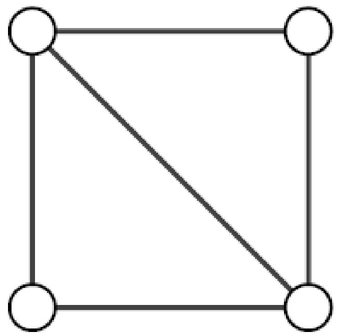


Figure 1: $C_4 \bigoplus e$

An algorithm along with the flowchart, that propagates the impact of covering a specified $C_4 \bigoplus e$ by common label is described by us.

## 2 G has one or more $C_4 \bigoplus e$: Covering a $C_4 \bigoplus e$ by the common label

We prove that for a covered graph G containing $C_4 \bigoplus e$ if none of the $C_4 \bigoplus e$ gets common label for its vertices then we can identify a $C_4 \bigoplus e$ such that common label can be allotted to the vertices of $C_4$ without affecting the outcome of the covering problem.

Since we work for a feasible labeling of graph with $k = 2, t = 4$ ; there exists a feasible labeling for a graph with $\Delta \leq k(t-1) = 6$ [2].

We prove the following propositions:

**Proposition 2.1.** *If degree of each diagonal vertex is 3 and degree of each non-diagonal vertex is 2 then a subgraph $C_4 \bigoplus e$ forms a disjoint component, and we can get a feasible labeling by allotting a common label to all 4 vertices.*

*Proof.* Let $v_1, v_2, v_3, v_4$ be the vertices of $C_4 \bigoplus e$ with $e = \{v_1, v_2\}$

Let deg $v_1 =$ deg $v_2 = 3$ and deg $v_3 =$ deg $v_4 = 2$

The proof is followed as shown in Figure 2. □

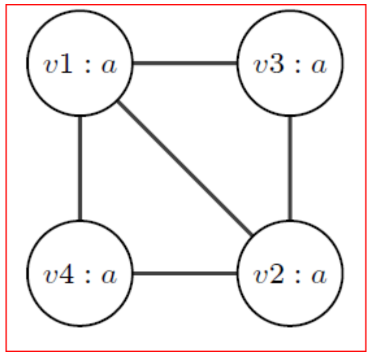



Figure 2: Proof for Proposition 2.1

**Proposition 2.2.** *If degree of all 4 vertices of* $C_4 \bigoplus e$ *is 6 then feasible labeling does not exist.*

*Proof.* Let $v_1, v_2, v_3, v_4$ be the vertices of $C_4 \bigoplus e$ with $e = \{v_1, v_2\}$

Let deg $v_i = 6$ for all $i = 1$ to 4.

For vertices $v_1$ and $v_2$, out of the two labels allotted one can be allotted to the 3 vertices adjacent to $v_1$ and lying on $C_4 \bigoplus e$. The other label can be allotted to remaining three vertices adjacent to $v_1$. Hence, all the edges lying on $v_1$ are covered. Similarly, all the edges lying on $v_2$ can also be covered.

Now for $v_3$ and $v_4$: -

The two edges adjacent to $v_3$ and lying on $C_4 \bigoplus e$ are already covered. Out of the remaining 4 edges, 3 can be covered by allotting a label 'c'. Vertex $v_3$ gets two labels a and c, both are used 4 times. So one edge adjacent to $v_3$ can not be covered. Similarly, one edge adjacent to $v_4$ can not be covered. Hence, the feasible labeling does not exist. This is shown in Figure 3. $\square$

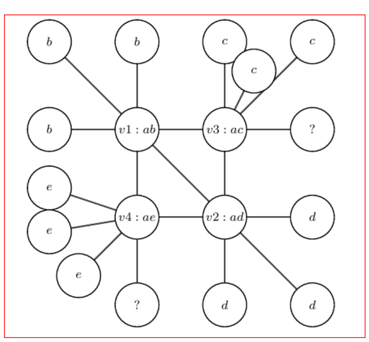


Figure 3: Proof for Proposition 2.2

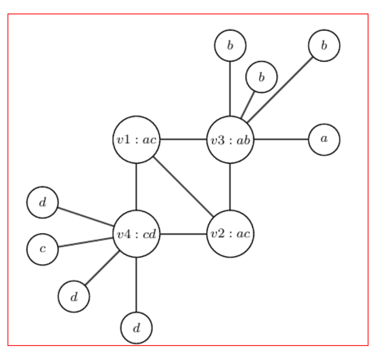


Figure 4: Proof for Proposition 2.3

**Proposition 2.3.** *If degree of non-diagonal vertex is greater than 5 then graph G can be covered but all the vertices of $C_4$ do not get a common label.*

*Proof.* Let $v_1, v_2, v_3, v_4$ be the vertices of $C_4 \bigoplus e$ with $e = \{v_1, v_2\}$.

Let $\deg v_3 = \deg v_4 = 6$. Both the labels allotted to $v_3$ (a and b) gets exhausted in covering the 6 edges adjacent to it. This includes the two vertices $v_1$ and $v_2$ of $C_4$. So vertex $v_4$ can not get any of the label out of a and b. Hence, though the graph G is covered all the vertices of $C_4$ do not get the common label. This can be seen in Figure 4. □

*Remark 2.1*: We have proven the result by considering maximum possible degree sequence (modulo permutations of diagonal vertices and modulo of non-diagonal vertices) of vertices in $C_4 \bigoplus e$, so that the result is obvious for the lower degree sequence.

*Remark 2.2*: We consider the labeling for covered graph G in which none of the $C_4 \bigoplus$ e gets common label and $\Delta \leq 6$.

*Remark 2.3*: If G has more than one $C_4 \bigoplus e$ then either they are adjacent or non-adjacent. These cases are discussed separately in Lemmas 2.1, 2.2 and 2.3.

## 2.1 G has non-adjacent $C_4 \bigoplus e$

In this section, we consider the graph G where no two $C_4 \bigoplus e$ are adjacent.

**Lemma 2.4.** *Let G be a covered graph having a $C_4 \bigoplus e$, with degree of each non-diagonal vertex $< 6$, then G can be covered with a labeling in which a $C_4 \bigoplus e$ subgraph has at least one common label for all the vertices in it.*

*Proof.* Let G has one $C_4 \bigoplus e$, on vertices $v_1, v_2, v_3, v_4$ and $e = \{v_1, v_2\}$. We consider two categories as follows:
Category I- Either $v_1$ or $v_2$ has maximum degree.
Category II- Either $v_3$ or $v_4$ has maximum degree.
As mentioned earlier we consider maximum possible degree sequence so that results are obvious for the lower degree sequence.

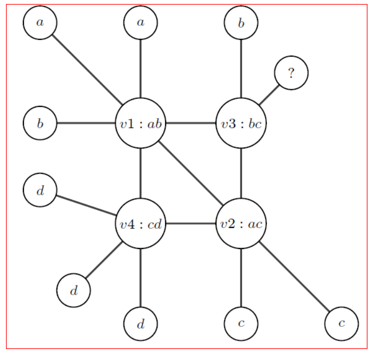



Figure 5: Degree of $v_3$ can not be more than 3.

We can see in Figure 5 that degree of $v_3$ can not be more than 3 as all the labels a, b, c and d are exhausted and $v_3$ has two labels. Hence, we get following cases in Category 1–

Case 1-Let deg $v_1 = 6$, deg $v_2 = 5$, deg $v_4 = 5$, deg $v_3 = 2$

In Figure 6 label 'a' is allotted to vertex $v_5$, label 'b' is allotted to vertex $v_3$. These two labels are exchanged. In Figure 7 we can see that graph G is covered with a different labeling such that

(i) the vertices of $C_4 \bigoplus e, \{v_1, v_2, v_3, v_4\}$, have a common label

(ii) the utilization of the labels 'a', 'b', 'c' and 'd' in Figure 6 and Figure 7 is same.

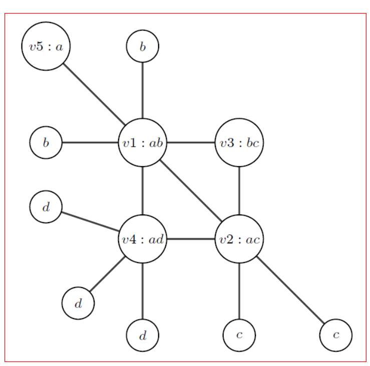



Figure 6: Lemma 2.4 Category 1 case 1:$C_4 \bigoplus e$ without common label.

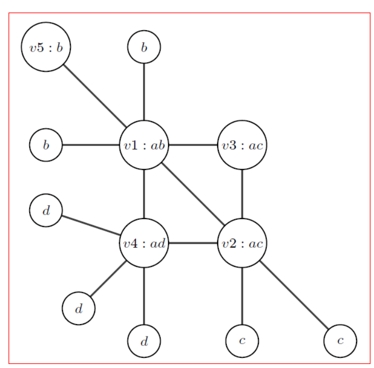



Figure 7: Lemma 2.4 Category 1 case 1: $C_4 \bigoplus e$ with common label.

Now by reducing the degree of $v_2$ we can get maximum possible degree for $v_3$ and $v_4$ as 4 and 5 respectively. This is shown in case 2.

Case 2: Let deg $v_1 = 6$, deg $v_2 = 3$, deg $v_3 = 4$, deg $v_4 = 5$

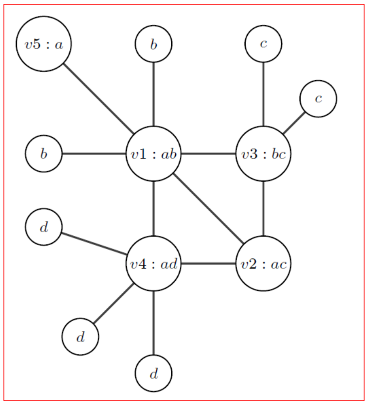



Figure 8: Lemma 2.4 Category 1 case 2: $C_4 \bigoplus e$ without common label.

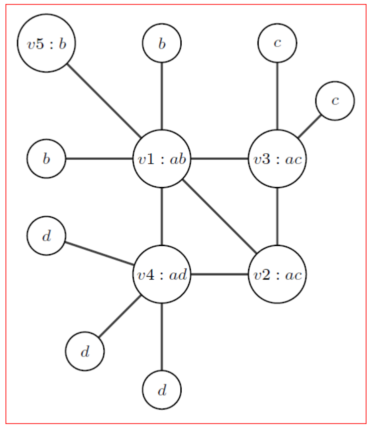



Figure 9: Lemma 2.4 Category 1 case 2: $C_4 \bigoplus e$ with common label.

In Figure 8 label 'a' is allotted to vertex $v_5$, label 'b' is allotted to vertex $v_3$. These two labels are exchanged. In Figure 9 we can see that graph G is covered with a different labeling such that

(i) the vertices of $C_4 \bigoplus e, \{v_1, v_2, v_3, v_4\}$, have a common label and

(ii) the utilization of the labels ‘a’, ‘b’, ‘c’ and ‘d’ in Figure 8 and Figure 9 is the same.

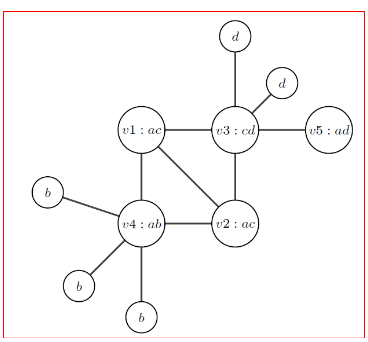



Figure 10: Lemma 2.4 Category II: $C_4 \bigoplus e$ without common label.

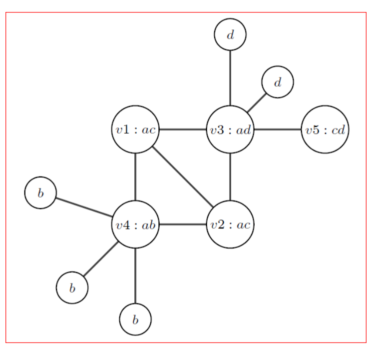



Figure 11: Lemma 2.4 Category II: $C_4 \bigoplus e$ with common label.

This completes the discussion on the maximum degree that vertices $v_1$ and $v_2$ can have.

Category II- Either $v_3$ or $v_4$ has maximum degree.

Here we have only one possibility- Let deg $v_1$ = deg $v_2 = 3$, deg $v_4$ = deg $v_3 = 5$

In Figure 10 label 'a' is allotted to vertex $v_5$, label 'c' is allotted to vertex $v_3$. These two labels are exchanged. In Figure 11 we can see that graph $G$ is covered with a different labeling such that

(i) the vertices of $C_4 \bigoplus e, \{v_1, v_2, v_3, v_4\}$, have a common label and

(ii) the utilization of the labels 'a', 'b', 'c' and 'd' 10 and Figure 11 is same.

□

This completes the discussion on the maximum degree that vertices $v_3$ and $v_4$ can have.

### 2.2 G has two adjacent $C_4 \bigoplus e$ with a common edge

In this section, we consider the graph G with two adjacent $C_4 \bigoplus e$ with a common edge.

**Lemma 2.5.** *Let $G$ be a covered graph having two $C_4 \bigoplus e$ and one edge common in them with degree of each non-diagonal vertex $< 6$, then $G$ can be covered with a labeling in which a $C_4 \bigoplus e$ subgraph has at least one common label for all the vertices in it.*

*Proof.* Let G has two $C_4 \bigoplus e$ as follows:

(i) $C_4$ on $v_1, v_2, v_3, v_4$ with $e_1$ and

(ii) $C_4$ on $v_1, v_2, v_6, v_5$ with $e_2$

with common edge $v_1v_2$

We get two cases:

Case 1: $e_1$ and $e_2$ are adjacent. $e_1 = v_1v_3, e_2 = v_1v_6$

Note that labels a and b are allotted to $v_1$. If $v_3$ gets 'a', $v_6$ must get 'b' and vice versa, else a feasible labeling does not exist. Also, both $v_4$ and $v_5$ should get a common label else a feasible labeling does not exist. Degree of both $v_3$ and $v_6$ can not be more 3 else a feasible labeling does not exist.

Hence, we get following subcases:

Subcase 1: Let deg $v_1 = 6$, deg $v_2 = 3$, deg $v_3 = 3$, deg $v_4 = 2$, deg $v_5 = 5$, deg $v_6 = 3$

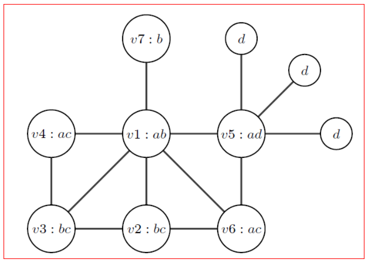



Figure 12: Lemma 2.5 Case 1 Subcase 1 : $C_4 \bigoplus e$ without common label.

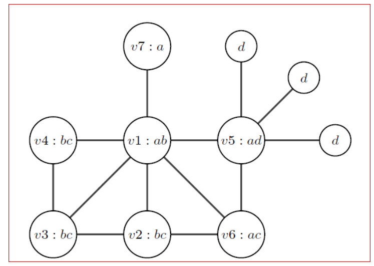



Figure 13: Lemma 2.5 Case 1 Subcase 1 : $C_4 \bigoplus e$ with common label.

In Figure 12 label 'a' is allotted to vertex $v_4$, label 'b' is allotted to vertex $v_7$. These two labels are exchanged. In Figure 13 we can see that graph G is covered with a different labeling such that

(i) the vertices of $C_4 \bigoplus e, \{v_1, v_2, v_6, v_5\}$, have a common label and

(ii) the utilization of the labels 'a', 'b', 'c' and 'd' in Figure 12 and Figure 13 is same.

Subcase 2: Let deg $v_1 = 5$, deg $v_2 = 5$, deg $v_3 = 3$, deg $v_4 = 5$, deg $v_5 = 2$, deg $v_6 = 6$

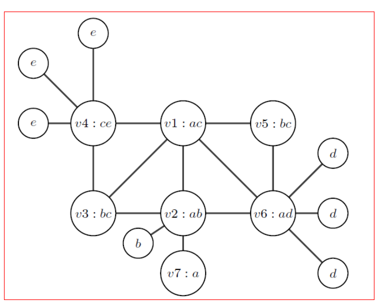



Figure 14: Lemma 2.5 Case 1 Subcase 2: $C_4 \bigoplus e$ without common label.

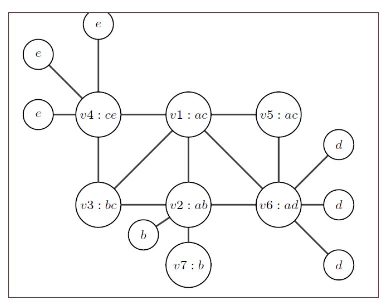



Figure 15: Lemma 2.5 Case 1 Subcase 2: $C_4 \bigoplus e$ with common label.

In Figure 14 label a is allotted to vertex $v_7$, label b is allotted to vertex $v_5$. These two labels are exchanged. In Figure 15 we can see that graph G is covered with a different labeling such that

(i) the vertices of $C_4 \bigoplus e, \{v_1, v_2, v_3, v_4\}$, have a common label and

(ii) the utilization of the labels 'a', 'b', 'c', 'd' and 'e' in Figure 14 and Figure 15 is same.

Case 2: $e_1$ and $e_2$ are non-adjacent. $e_1 = v_1v_3, e_2 = v_2v_5$.

Subcase 1: Let deg $v_1 = 4$, deg $v_2 = 5$, deg $v_3 = 4$, deg $v_4 = 2$, deg $v_5 = 6$, deg $v_6 = 2$.

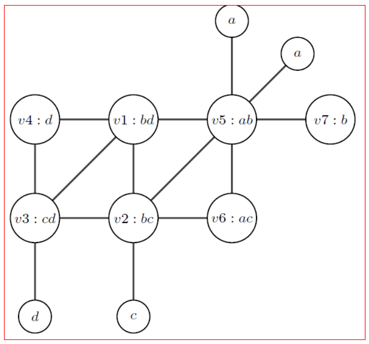



Figure 16: Lemma 2.5 Case 2 Subcase 1: $C_4 \bigoplus e$ without common label.

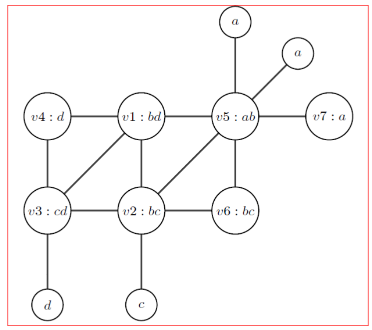



Figure 17: Lemma 2.5 Case 2 Subcase 1: $C_4 \bigoplus e$ with common label.

In Figure 16 label 'a' is allotted to vertex $v_6$, label 'b' is allotted to vertex $v_7$. These two labels are exchanged. In Figure 17 we can see that graph G is covered with a different labeling such that

(i) the vertices of $C_4 \bigoplus e, \{v_1, v_2, v_5, v_6\}$, have a common label and

(ii) the utilization of the labels 'a', 'b', 'c' and 'd' in Figure 16 and Figure 17 is same.

Subcase 2: Let deg $v_1 = 6$, deg $v_2 = 4$, deg $v_3 = 3$, deg $v_4 = 5$, deg $v_5 = 3$, deg $v_6 = 5$.

In Figure 18 label 'a' is allotted to vertex $v_2$, label 'b' is allotted to vertex $v_1$. These two labels are exchanged. In Figure 19 we can see that graph G is covered with a different labeling such that

(i) the vertices of $C_4 \bigoplus e, \{v_1, v_2, v_3, v_4\}$, have a common label and

(ii) the utilization of the labels 'a', 'b', 'c' and 'd' in Figure 18 and Figure 19 is same.

□

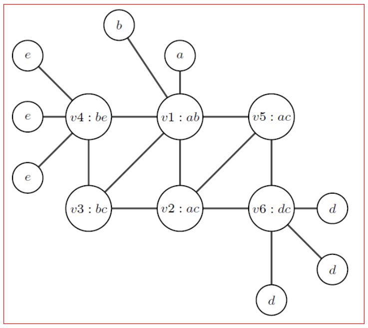



Figure 18: Lemma 2.5 Case 2 Subcase 2 : $C_4 \bigoplus e$ without common label.

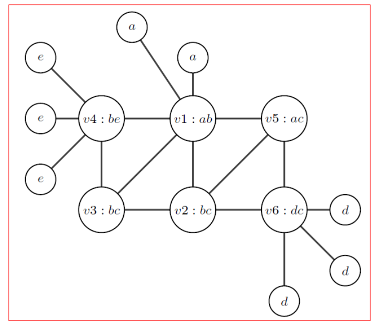



Figure 19: Lemma 2.5 Case 2 Subcase 2 : $C_4 \bigoplus e$ with common label.

### 2.3 G has two adjacent $C_4 \bigoplus e$ with one common vertex.

**Lemma 2.6.** *Let $G$ be a covered graph having two adjacent $C_4 \bigoplus e$ with one vertex common in them with degree of each non-diagonal vertex $< 6$, then $G$ can be covered with a labeling in which a $C_4 \bigoplus e$ subgraph has at least one common label for all the vertices in it.*

*Proof.* Let $\{v_1, v_2, v_3, v_4\}$ with $e = v_1v_2$ and $\{v_1, v_5, v_6, v_7\}$ with $e = v_1v_6$ be the two $C_4 \bigoplus e$ with a common vertex $v_1$. We note that only $v_1$ can have degree 6 and only one vertex from each of $C_4 \bigoplus e$ can have degree 5. Also, deg $v_3$ = deg $v_6 = 3$, else G can not have feasible labeling.

Hence, we get the following cases:

Case 1: Let deg $v_1 = 6$, deg $v_2 = 3$, deg $v_3 = 5$, deg $v_4 = 2$, deg $v_5 = 2$, deg $v_6 = 3$, deg $v_7 = 5$.

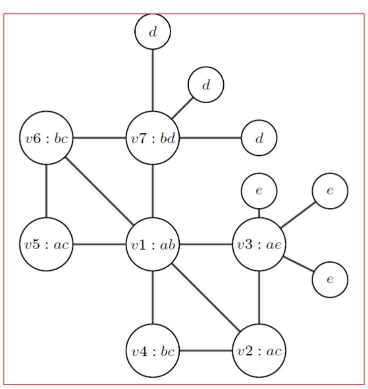



Figure 20: Lemma 2.6 Case 1: $C_4 \bigoplus e$ without common label.

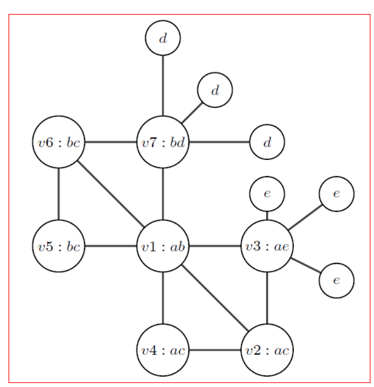



Figure 21: Lemma 2.6 Case 1: $C_4 \bigoplus e$ with common label.

In Figure 20 label 'a' is allotted to vertex $v_6$, label 'b' is allotted to vertex $v_2$. These two labels are exchanged. In Figure 21 we can see that graph G is covered with a different labeling such that

(i) the vertices of $C_4 \bigoplus e, \{v_1, v_2, v_3, v_4\}$, have a common label and

(ii) the utilization of the labels 'a', 'b', 'c', 'd' and 'e' in Figure 20 and Figure 21 is same.

Case 2: Let deg $v_1 = 6$, deg $v_2 = 4$, deg $v_3 = 2$, deg $v_4 = 2$, deg $v_5 = 2$, deg $v_6 = 4$, deg $v_7 = 2$.

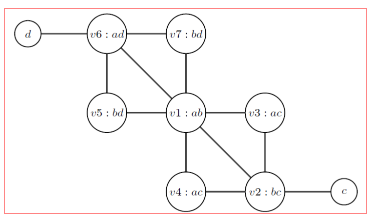



Figure 22: Lemma 2.6 Case 2: $C_4 \bigoplus e$ without common label.

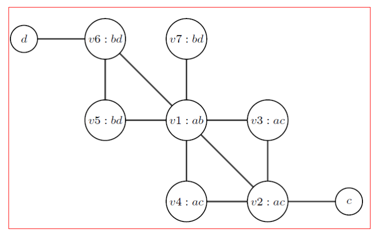



Figure 23: Lemma 2.6 Case 2: $C_4 \bigoplus e$ with common label.

In Figure 22 label 'b' is allotted to vertex $v_2$, label 'a' is allotted to vertex $v_6$. These two labels are exchanged. In Figure 23 we can see that graph G is covered with a different labeling such that

(i) the vertices of $C_4 \bigoplus e, \{v_1, v_3, v_2, v_4\}$, have a common label and

(ii) the utilization of the labels 'a', 'b', 'c' and 'd' in Figure 22 and Figure 23 is same.

□

From lemmas 2.1, 2.2, 2.3 and above observation we get the following theorem:

**Theorem 2.7.** *Let G be a covered graph having one or more $C_4 \bigoplus e$ with three or more of them not having a common vertex and with degree of each non-diagonal vertex $< 6$, then G can be covered with a labeling in which $C_4 \bigoplus e$ subgraph has at least one common label for all the vertices in it.*

# 3 Algorithm

*Remark 3.1*: The proof of Theorem, in fact implies that if two $C_4 \bigoplus e$ are adjacent and if the graph is covered then G can be covered with a labeling in which one of the two $C_4 \bigoplus e$ has at least one common label for all its vertices in it.

Although Theorem 2.4 guarantees the existence of a $C_4 \bigoplus e$ that is covered by a common label in a feasible labeling (if one exists), finding a $C_4 \bigoplus e$ with this property is an issue that remains to be addressed. We now describe an algorithm that propagates the impact of covering a specified $C_4 \bigoplus e$ by a common label. The output of the algorithm is a binary signal: An output 1 implies that the vertices of the specified $C_4 \bigoplus e$ can be assigned a common label without impacting the outcome of the covering problem, An output of 0 indicates otherwise. In the algorithm vertices are classified into three types: single labeled vertices, double labeled vertices and no labeled vertices. We now define the following terms:

**Definition 3.1.** *A single labeled vertex is a vertex for which only one label remains to be allotted. The other label has been previously allotted during the propagation and has been completely used (i.e. has been assigned at t vertices).*

**Definition 3.2.** *A double labeled vertex is a vertex for which both its labels have been previously allotted during the propagation. Furthermore, both the labels have been completely used (i.e. each label has been allotted at t vertices).*

**Definition 3.3.** *A no marked vertex is a vertex for which both its labels remain to be allotted.*

During the propagation several vertices and edges may be removed. Therefore, at any point of time in the procedure, we refer to the graph as the residual graph.

## 3.1 Algorithm

Input: A $C_4 \bigoplus e$ on vertices $\{i, j, k, l\}$ in G.

Output: 1 (Success) or 0 (Failure). An output of 1 denotes that vertices i, j, k and l can be allotted a common label without affecting the outcome of the covering problem. An output of 0 indicates otherwise.

Step 1: Allot label a to the vertices i, j, k and l.

Step 2: Denote vertices i, j k and l as single labeled vertices and delete edges $\{i, j\}, \{j, k\}, \{k, l\}, \{l, i\} and \{i, k\}$. This corresponds to the situation where vertices i, j, k and l share a common label. Thus, only one label remains to be allotted to these four vertices. Since the common label is used at four vertices, it can not be used further.

Step 3: Consider a single labeled vertex I:

(a) If the degree of vertex I in the residual graph is less than or equal to 3 then allot a label to I and its neighbor vertices and

i) denote vertex I as a double labeled vertex. ii) denote each marked neighbor of vertex I as double labeled vertex. iii) denote each no labeled vertex of I as single labeled vertex, and iv) delete the edges incident on I.

(b) If the degree of vertex I in the residual graph is 4 or more, then return 0 (Failure)

Step 4: Consider a double labeled vertex I: (a) If the degree of vertex I is 0 or more, then return 0 (Failure) (b) If the degree of vertex I is 0, delete vertex I.

Step 5: If there exists a single labeled vertex with degree at least 3, go to Step 3.

Otherwise, return 1 (Success) ; this corresponds to the situation when each vertex in the residual graph with degree 1 is either a no labeled vertex or a single labeled vertex with degree.

*Remark 3.2*: If two $C_4 \bigoplus e$ are adjacent, then we can apply the Algorithm on one $C_4 \bigoplus e$ and then immediately on the other. It follows from Remark 3.1 that the algorithm must return a 1 (i.e. Success) on at least one of these two processes for otherwise no feasible labeling exists.

### 3.2 Illustrative example

Consider the graph G shown in Figure 24.

Input - $C_4 \bigoplus e$ on vertices $\{v_1, v_2, v_3, v_4\}$ with $e = \{v_1, v_2\}$

Step 1 - Label 'a' is allotted to $v_1, v_2, v_3, v_4$ as shown in Figure 25.

Step 2 - $v_1, v_2, v_3, v_4$ are single labeled vertices. Delete edges $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_1, v_4\}$, $\{v_3, v_2\}$, $\{v_4, v_2\}$ as shown in Figure 26.

Step 3 - Consider single labeled vertices and their degrees in residual graph.

Refer Figure 27.

I $= v_1$, deg $v_1 = 3$.

Label 'b' is allotted to $v_1, v_5, v_6, v_7$.

Delete edges $\{v_1, v_5\}, \{v_1, v_6\}, \{v_1, v_7\}$.

I $= v_4$, deg $v_4 = 3$.

Label 'c' is allotted to $v_4, v_{11}, v_{12}, v_{13}$

Delete edges $\{v_4, v_{11}\}, \{v_4, v_{12}\}, \{v_4, v_{13}\}$.

I $= v_2$, deg $v_2 = 2$.

Label 'd' is allotted to $v_{14}, v_{15}$

Delete edges $\{v_2, v_{14}\}, \{v_2, v_{15}\}$.

I $= v_7$, deg $v_7 = 3$

Label 'e' is allotted to $v_7, v_8, v_9, v_{10}$.

Delete edges $\{v_7, v_8\}, \{v_7, v_9\}, \{v_7, v_{10}\}$.

Step 4 - Consider a double labeled vertex I. Refer Figure 27.

I $= v_1$ deg $v_1 = 0$. Delete $v_1$.

I $= v_2$ deg $v_2 = 0$. Delete $v_2$.

I $= v_4$ deg $v_4 = 0$. Delete $v_4$.

I $= v_7$ deg $v_7 = 0$. Delete $v_7$.

Step 5 - No single labeled vertex with degree $\geq 3$.

Therefore, return 1. Hence, success.
Figure 28 shows feasible labeling of Graph G.

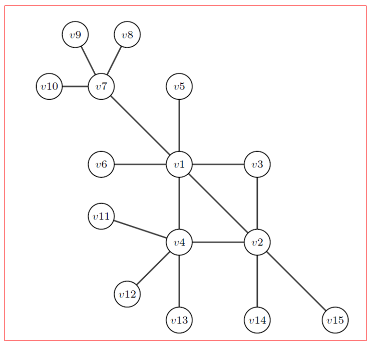



Figure 24: Graph G

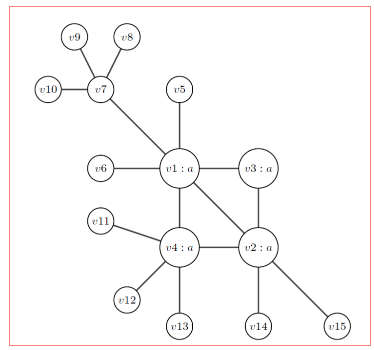



Figure 25: Step 1 of Algorithm

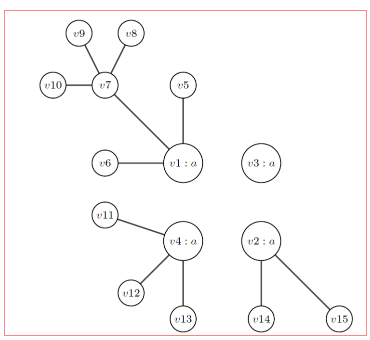



Figure 26: Step 2 of Algorithm

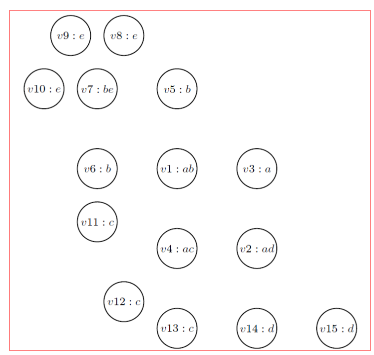



Figure 27: Step 3 and 4 of Algorithm

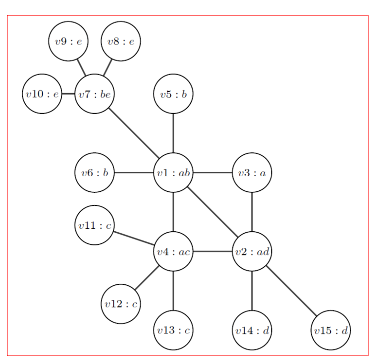
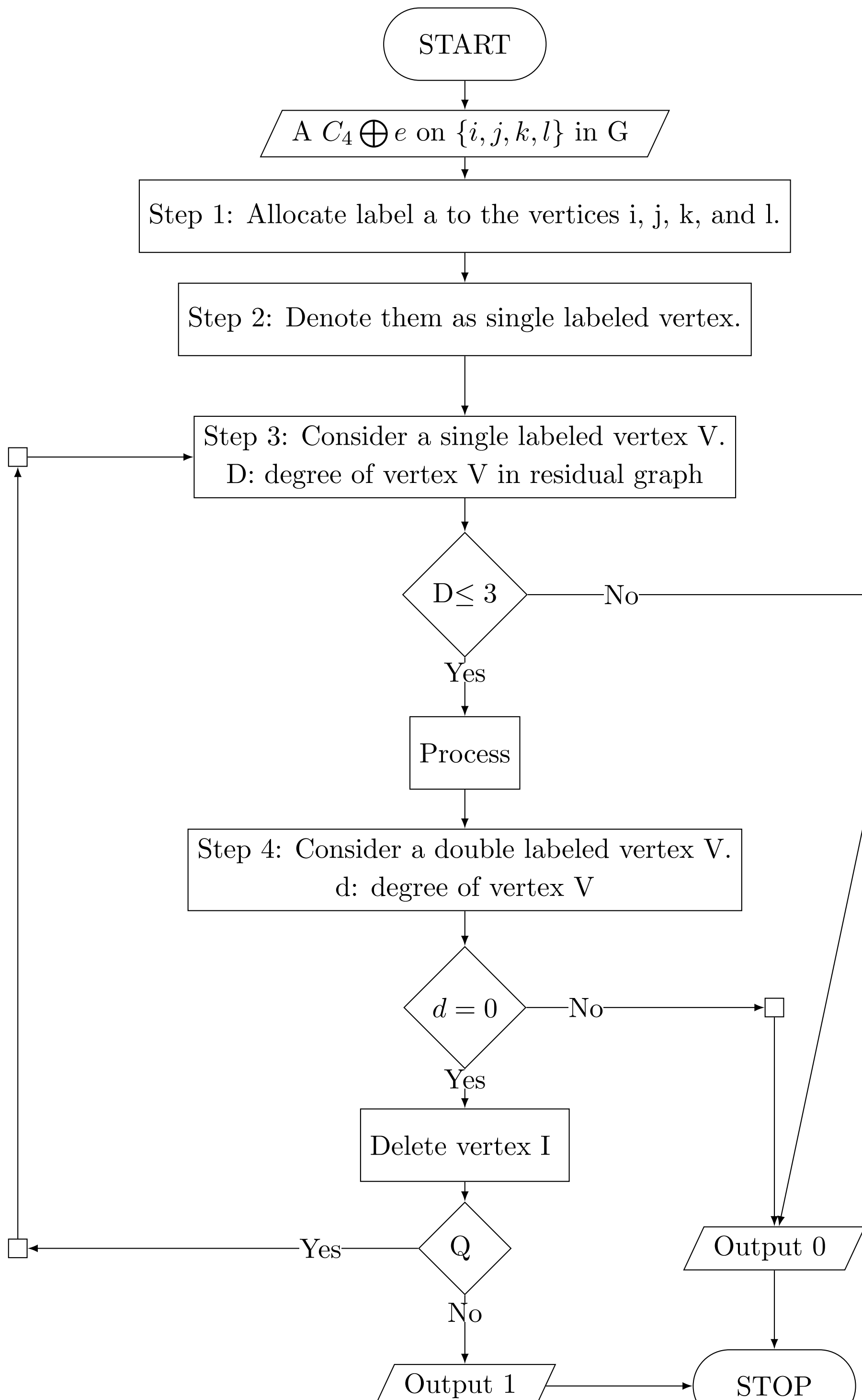



Figure 28: Feasible labeling of Graph G

### 3.3 Flowchart




KEY:

Process: (i) Denote vertex I as a double labeled vertex.

ii) Denote each single labeled neighbor of vertex I as double labeled vertex.
iii) Denote each no labeled vertex of I as single labeled, and
iv) Delete the edges incident on I
Q: Is there a single labeled vertex with degree $\geq 3$?

## 4 Conclusions and future extensions

In this paper we proved that if G is a covered graph having one or more $C_4 \bigoplus e$, then G can be covered with a labeling in which $C_4 \bigoplus e$ subgraph has at least one common label for all the vertices in it. Future extension of our work is to consider the problem for $k = 2$ and $t \geq 5$ in any general graph G.

In pair programming two programmers are assigned at each workstation, where they work on the same piece of code. One member types the code while the other continually reviews the work. It gives higher software quality and improved productivity.

The graph G represents software system consisting of several modules, which are represented by vertices. A pair of vertices is connected by an edge if two modules are related with same development activity such as sharing a code or one module using the other as subroutine. Programmers are represented by labels. Each programmer will work on at most a specified number of modules. The feasible labeling mentioned in this paper indicates that there is an assignment of a pair of programmers to the vertices with the following property: for each edge $(i, j)$ of $G$, there should be at least one common programmer among those allotted at vertices i and j. If the theorem proved in this paper is applied to a software system it ensures that a part of it represented by $C_4 \bigoplus e$ can be lead by a common programmer. This can develop the leadership quality and better team work. And hence better utilization of manpower.

## References

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