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EXTREMAL FUNCTIONS AND APPROXIMATE INVERSION FORMULAS FOR WEIGHTED HARDY SPACES

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Abstract

In this paper we consider the weighted Hardy space $\mathscr{H}_\beta$. This space which gives a generalization of some Hilbert spaces of analytic functions on the unit disk like, the Hardy space $\mathscr{H}$, the weighted Bergman space $\mathscr{B}_\nu$ and the weighted Dirichlet space $\mathscr{D}_\nu$, it plays a background to our contribution. Especially, we examine the extremal functions for the primitive operator $Pf(z) := \int_{[0,z]} f(w)\mathrm{d}w$, where $[0, z] = \{tz,\ t \in [0, 1]\}$; and we deduce approximate inversion formulas for the operator P on the weighted Hardy space $\mathscr{H}_\beta$.

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1 Introduction

The spaces of analytic functions on the unit disk are one of the complex analysis tools used in harmonic analysis [1, 3, 4, 8, 18]. Recently, Soltani proved some versions of uncertainty principle in the context of weighted Hardy space, weighted Bergman space and weighted Dirichlet space (see [13, 14, 15, 16, 17]). In this paper we are going to examine the theory of extremal functions in the context of weighted Hardy spaces.

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. The weighted Hardy space $\mathscr{H}_\beta$ (see [2, 8, 16, 17]) is the set of all analytic functions f in $\mathbb{D}$, with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, such that

$$\|f\|^2_{\mathscr{H}_\beta} := \sum_{n=0}^{\infty} \beta_n |a_n|^2 < \infty,$$

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where $\beta = \{\beta_n\}$ is a positive sequence so that $\limsup_{n\to\infty}(\beta_n)^{-1/n} = 1$.

The space $\mathscr{H}_\beta$ is a reproducing kernel Hilbert space (RKHS) that gives a generalization of some Hilbert spaces of analytic functions in the unit disk $\mathbb{D}$ like, the Hardy space $\mathscr{H}$ (see [8, 18, 19]), the weighted Bergman space $\mathscr{B}_\nu$ (see [4, 5, 14]), and the weighted Direchlet space $\mathscr{D}_\nu$ (see [1, 3, 13, 15]).

Let $P : \mathscr{H}_\beta \to \mathscr{H}_\beta$ be the primitive operator given by

$$Pf(z) := \int_{[0,z]} f(w)\mathrm{d}w, \quad f \in \mathscr{H}_\beta,$$

where $[0, z] = \{tz,\ t \in [0, 1]\}$. This operator has many applications in logic and theoretical computer science.

The main goal of the paper is to find the minimizer (denoted by $F^*_{\lambda,P}(h)$) for the extremal problem:

$$\inf_{f\in\mathscr{H}_\beta}\Big\{\lambda\|f\|^2_{\mathscr{H}_\beta} + \|Pf - h\|^2_{\mathscr{H}_\beta}\Big\},$$

where $h \in \mathscr{H}_\beta$ and $\lambda > 0$. We prove that the extremal function $F^*_{\lambda,P}(h)$ is given by

$$F^*_{\lambda,P}(h)(z) = \langle h, \Psi_z\rangle_{\mathscr{H}_\beta},$$

where

$$\Psi_z(w) = \sum_{n=1}^{\infty} \frac{n(\overline{z})^{n-1}w^n}{\lambda n^2\beta_{n-1} + \beta_n}, \quad w \in \mathbb{D}.$$

Moreover, we establish approximate inversion formulas for the primitive operator P on the weighted Hardy space $\mathscr{H}_\beta$. A pointwise approximate inversion formulas for the operator P are also discussed.

The paper is organized as follows. In Section 2 we recall some results about the weighted Hardy space $\mathscr{H}_\beta$. In Section 3 we examine the extremal functions for the primitive operator P. In Section 4, we establish approximate inversion formulas for the operator P on the weighted Hardy space $\mathscr{H}_\beta$. In the last section, we summarize the obtained results and describe the open questions.

2 Weighted Hardy space

We consider a sequence $\beta = \{\beta_n\}$, with $\beta_n > 0$, such that

$$\limsup_{n\to\infty}(\beta_n)^{-1/n} = 1.$$

The weighted Hardy space $\mathscr{H}_\beta$ (see [2, 8, 16, 17]) is the set of all analytic functions f in the disk $\mathbb{D}$, with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, such that

$$\|f\|^2_{\mathscr{H}_\beta} := \sum_{n=0}^{\infty} \beta_n|a_n|^2 < \infty.$$

It is a Hilbert space when equipped with the inner product

$$\langle f, g\rangle_{\mathscr{H}_\beta} = \sum_{n=0}^{\infty} \beta_n a_n \overline{b_n},$$

where $f, g \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. The set $\left\{\frac{z^n}{\sqrt{\beta_n}}\right\}_{n=0}^{\infty}$ forms a Hilbert's basis for the space $\mathscr{H}_\beta$. The function $K_{\mathscr{H}_\beta,z}$, $z \in \mathbb{D}$, given by

$$K_{\mathscr{H}_\beta,z}(w) := \sum_{n=0}^{\infty} \frac{(\overline{z}w)^n}{\beta_n}, \quad w \in \mathbb{D},$$

is a reproducing kernel for the weighted Hardy space $\mathscr{H}_\beta$.

If $\beta_n = 1$ the corresponding weighted Hardy space is the Hardy space $\mathscr{H}$ (see [8, 18, 19]). This Hilbert space has the reproducing kernel

$$K_{\mathscr{H},z}(w) = \sum_{n=0}^{\infty} (\overline{z}w)^n = \frac{1}{1-\overline{z}w}, \quad w, z \in \mathbb{D}.$$

If $\beta_n = \frac{n!}{(\nu+2)_n}$, where $\nu > -1$ and $(a)_n = \frac{\Gamma(n+a)}{\Gamma(a)}$, the corresponding weighted Hardy space is the weighted Bergman space $\mathscr{B}_\nu$ (see [5, 14]). This Hilbert space has the reproducing kernel

$$K_{\mathscr{B}_\nu,z}(w) := \frac{1}{(1-\overline{z}w)^{\nu+2}}, \quad w, z \in \mathbb{D}.$$

If $\beta_0 = 1$ and $\beta_n = (\nu+1)\frac{nn!}{(\nu+1)_n}$, $n \geq 1$, where $\nu \geq 0$, the corresponding weighted Hardy space is the weighted Dirichlet space $\mathscr{D}_\nu$ (see [15]). This Hilbert space has the reproducing kernel

$$K_{\mathscr{D}_\nu,z}(w) := 1 + \frac{1}{\nu+1} \sum_{n=1}^{\infty} \frac{(\nu+1)_n}{nn!} (\overline{z}w)^n, \quad w, z \in \mathbb{D}.$$

In the next of this paper, we suppose that there exists a constant $c > 0$ such that the sequence $\{\beta_n\}$ satisfies the condition

$$n^2 \frac{\beta_{n-1}}{\beta_n} \geq c, \quad \forall\, n \geq 1. \tag{1}$$

The condition (1) is verified in the precedent three cases: in the Hardy space $\mathscr{H}$, weighted Bergman space $\mathscr{B}_\nu$ and in the weighted Dirichlet space $\mathscr{D}_\nu$.

3 Primitive operator

Let H be a Hilbert space, and let $T : \mathscr{H}_\beta \to H$ be a bounded linear operator from $\mathscr{H}_\beta$ into H. Let $\lambda > 0$. We denote by $\langle ., .\rangle_{\lambda,\mathscr{H}_\beta}$ the inner product defined on the space $\mathscr{H}_\beta$ by

$$\langle f, g\rangle_{\lambda,\mathscr{H}_\beta} := \lambda \langle f, g\rangle_{\mathscr{H}_\beta} + \langle Tf, Tg\rangle_H.$$

The two norms $\|.\|_{\mathscr{H}_\beta}$ and $\|.\|_{\lambda,\mathscr{H}_\beta}$ are equivalent. In particular, we have

$$|f(z)| \leq \|f\|_{\lambda,\mathscr{H}_\beta} \left[\frac{K_{\mathscr{H}_\beta,z}(z)}{\lambda}\right]^{1/2}, \quad f \in \mathscr{H}_\beta, z \in \mathbb{D}.$$

Then the space $\mathscr{H}_\beta$, equipped with the norm $\|.\|_{\lambda,\mathscr{H}_\beta}$ has a reproducing kernel $K_{\lambda,\mathscr{H}_\beta,z}$. Therefore, we have the functional equation

$$(\lambda I + T^*T)K_{\lambda,\mathscr{H}_\beta,z} = K_{\mathscr{H}_\beta,z}, \quad z \in \mathbb{D}, \tag{2}$$

where I is the unit operator and $T^* : H \longrightarrow \mathscr{H}_\beta$ is the adjoint of T.

For any $h \in H$ and for any $\lambda > 0$, we define the extremal function $F^*_{\lambda,T}(h)$ by

$$F^*_{\lambda,T}(h)(z) = \langle h, TK_{\lambda,\mathscr{H}_\beta,z}\rangle_H, \quad z \in \mathbb{D}.$$

Then by (2) we deduce that

$$\begin{aligned} F^*_{\lambda,T}(h)(z) &= \langle T^*h, K_{\lambda,\mathscr{H}_\beta,z}\rangle_{\mathscr{H}_\beta} \\ &= \langle T^*h, (\lambda I + T^*T)^{-1}K_{\mathscr{H}_\beta,z}\rangle_{\mathscr{H}_\beta} \\ &= \langle (\lambda I + T^*T)^{-1}T^*h, K_{\mathscr{H}_\beta,z}\rangle_{\mathscr{H}_\beta}. \end{aligned}$$

Hence

$$F^*_{\lambda,T}(h)(z) = (\lambda I + T^*T)^{-1}T^*h(z), \quad z \in \mathbb{D}. \tag{3}$$

The extremal function $F^*_{\lambda,T}(h)$ is the unique solution (see [9], Theorem 2.5, Section 2) of the Tikhonov regularization problem

$$\inf_{f\in\mathscr{H}_\beta}\Big\{\lambda\|f\|^2_{\mathscr{H}_\beta} + \|Tf - h\|^2_H\Big\}.$$

Let P be the primitive operator defined for $f \in \mathscr{H}_\beta$ by

$$Pf(z) := \int_{[0,z]} f(w)\mathrm{d}w,$$

where $[0, z] = \{tz,\ t \in [0, 1]\}$. If $f(z) = \sum_{n=0}^\infty a_n z^n$, then

$$Pf(z) = \sum_{n=1}^\infty \frac{a_{n-1}}{n} z^n. \tag{4}$$

From (1), the operator P maps continuously from $\mathscr{H}_\beta$ into $\mathscr{H}_\beta$, and

$$\|Pf\|_{\mathscr{H}_\beta} \leq \frac{1}{\sqrt{c}}\|f\|_{\mathscr{H}_\beta}.$$

Building on the ideas of Saitoh [9, 10, 11] we examine the extremal function associated with the primitive operator P.

Theorem 1. *(i) For $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we have*

$$P^* f(z) = \sum_{n=0}^{\infty} \frac{\beta_{n+1}}{(n+1)\beta_n} a_{n+1} z^n,$$

$$P^* P f(z) = \sum_{n=0}^{\infty} \frac{\beta_{n+1}}{(n+1)^2\beta_n} a_n z^n.$$

(ii) For any $h \in \mathscr{H}_\beta$ and for any $\lambda > 0$, the problem

$$\inf_{f\in\mathscr{H}_\beta} \left\{ \lambda \|f\|^2_{\mathscr{H}_\beta} + \|Pf - h\|^2_{\mathscr{H}_\beta} \right\}$$

has a unique extremal function given by

$$F^*_{\lambda,P}(h)(z) = \langle h, \Psi_z \rangle_{\mathscr{H}_\beta},$$

where

$$\Psi_z(w) = \sum_{n=1}^{\infty} \frac{n(\overline{z})^{n-1} w^n}{\lambda n^2 \beta_{n-1} + \beta_n}, \quad w \in \mathbb{D}.$$

Proof. (i) If $f, g \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$\langle Pf, g \rangle_{\mathscr{H}_\beta} = \sum_{n=1}^{\infty} \beta_n \frac{a_{n-1}}{n} \overline{b_n} = \sum_{n=0}^{\infty} \beta_{n+1} \frac{a_n}{n+1} \overline{b_{n+1}} = \langle f, P^* g \rangle_{\mathscr{H}_\beta},$$

where

$$P^* g(z) = \sum_{n=0}^{\infty} \frac{\beta_{n+1}}{(n+1)\beta_n} b_{n+1} z^n.$$

And therefore

$$P^* P f(z) = \sum_{n=0}^{\infty} \frac{\beta_{n+1}}{(n+1)^2\beta_n} a_n z^n.$$

(ii) We put $h(z) = \sum_{n=0}^{\infty} h_n z^n$ and $F^*_{\lambda,P}(h)(z) = \sum_{n=0}^{\infty} c_n z^n$. From (3) we have $(\lambda I + P^* P) F^*_{\lambda,P}(h)(z) = P^* h(z)$. By (i) we deduce that

$$c_n = \frac{(n+1)\beta_{n+1} h_{n+1}}{\lambda(n+1)^2 \beta_n + \beta_{n+1}}, \quad n \in \mathbb{N}.$$

Thus

$$F^*_{\lambda,P}(h)(z) = \sum_{n=1}^{\infty} \frac{n\beta_n h_n}{\lambda n^2 \beta_{n-1} + \beta_n} z^{n-1} = \langle h, \Psi_z \rangle_{\mathscr{H}_\beta}, \tag{5}$$

where

$$\Psi_z(w) = \sum_{n=1}^{\infty} \frac{n(\overline{z})^{n-1} w^n}{\lambda n^2 \beta_{n-1} + \beta_n}.$$

The theorem is proved. □

If $\mathscr{H}_\beta$ is the Hardy space $\mathscr{H}$. For $f \in \mathscr{H}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have

$$P^* f(z) = \sum_{n=0}^{\infty} \frac{a_{n+1}}{n+1} z^n, \quad P^* P f(z) = \sum_{n=0}^{\infty} \frac{a_n}{(n+1)^2} z^n.$$

And for any $h \in \mathscr{H}$ and for any $\lambda > 0$, one has

$$F^*_{\lambda,P}(h)(z) = \langle h, \Psi_z \rangle_{\mathscr{H}},$$

where

$$\Psi_z(w) = \sum_{n=1}^{\infty} \frac{n(\overline{z})^{n-1} w^n}{\lambda n^2 + 1}.$$

If $\mathscr{H}_\beta$ is the weighted Bergman space $\mathscr{B}_\nu$. For $f \in \mathscr{B}_\nu$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have

$$P^* f(z) = \sum_{n=0}^{\infty} \frac{a_{n+1}}{n+\nu+2} z^n,$$

$$P^* P f(z) = \sum_{n=0}^{\infty} \frac{a_n}{(n+1)(n+\nu+2)} z^n.$$

And for any $h \in \mathscr{B}_\nu$ and for any $\lambda > 0$, one has

$$F^*_{\lambda,P}(h)(z) = \langle h, \Psi_z \rangle_{\mathscr{A}_\nu},$$

where

$$\Psi_z(w) = \sum_{n=1}^{\infty} \frac{n(\nu+2)_n (\overline{z})^{n-1} w^n}{n![\lambda n(n+\nu+1)+1]}.$$

If $\mathscr{H}_\beta$ is the weighted Dirichlet space $\mathscr{D}_\nu$. For $f \in \mathscr{D}_\nu$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have

$$P^* f(z) = a_1 + \sum_{n=1}^{\infty} \frac{(n+1)a_{n+1}}{n(n+\nu+1)} z^n,$$

$$P^* P f(z) = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{n(n+\nu+1)} z^n.$$

And for any $h \in \mathscr{D}_\nu$ and for any $\lambda > 0$, one has

$$F^*_{\lambda,P}(h)(z) = \langle h, \Psi_z \rangle_{\mathscr{D}_\nu},$$

where

$$\Psi_z(w) = \frac{w}{\lambda+1} + \frac{1}{\nu+1} \sum_{n=2}^{\infty} \frac{(\nu+1)_n (\overline{z})^{n-1} w^n}{n![\lambda(n-1)(n+\nu)+1]}.$$

4 Approximate inversion formulas

In this section we establish the estimate properties of the extremal function $F^*_{\lambda,P}(h)(z)$, and we deduce approximate inversion formulas for the primitive operator P. These formulas are the analogous of Calderón's reproducing formulas for the Fourier type transforms [6, 7, 12]. A pointwise approximate inversion formulas for the operator P are also discussed.

The extremal function $F^*_{\lambda,P}(h)$ given by (5) satisfies the following properties.

Lemma 1. *If $\lambda > 0$ and $h \in \mathscr{H}_\beta$, then*

(i) $|F^*_{\lambda,P}(h)(z)| \le \frac{1}{2\sqrt{\lambda}}(K_{\mathscr{H}_\beta,z}(z))^{1/2}\|h\|_{\mathscr{H}_\beta}$,

(ii) $|PF^*_{\lambda,P}(h)(z)| \le \frac{1}{2\sqrt{\lambda c}}(K_{\mathscr{H}_\beta,z}(z))^{1/2}\|h\|_{\mathscr{H}_\beta}$,

(iii) $\|F^*_{\lambda,P}(h)\|_{\mathscr{H}_\beta} \le \frac{1}{2\sqrt{\lambda}}\|h\|_{\mathscr{H}_\beta}$.

Proof. Let $\lambda > 0$ and $h \in \mathscr{H}_\beta$ with $h(z) = \sum_{n=0}^{\infty} h_n z^n$. From (5) we have

$$|F^*_{\lambda,P}(h)(z)| \le \|\Psi_z\|_{\mathscr{H}_\beta}\|h\|_{\mathscr{H}_\beta}.$$

Using the fact that $(x+y)^2 \ge 4xy$ we obtain

$$\|\Psi_z\|^2_{\mathscr{H}_\beta} = \sum_{n=1}^{\infty} \beta_n \left[\frac{n|z|^{n-1}}{\lambda n^2\beta_{n-1} + \beta_n}\right]^2 \le \frac{1}{4\lambda}\sum_{n=0}^{\infty}\frac{|z|^{2n}}{\beta_n} = \frac{1}{4\lambda}K_{\mathscr{H}_\beta,z}(z).$$

This gives (i).

On the other hand, from (4) and (5) we have

$$PF^*_{\lambda,P}(h)(z) = \sum_{n=1}^{\infty}\frac{\beta_n h_n}{\lambda n^2\beta_{n-1} + \beta_n} z^n = \langle h, \Phi_z\rangle_{\mathscr{H}_\beta}, \tag{6}$$

where

$$\Phi_z(w) = \sum_{n=1}^{\infty}\frac{(w\overline{z})^n}{\lambda n^2\beta_{n-1} + \beta_n}.$$

Then

$$|PF^*_{\lambda,P}(h)(z)| \le \|\Phi_z\|_{\mathscr{H}_\beta}\|h\|_{\mathscr{H}_\beta}.$$

And by (1) we deduce that

$$\|\Phi_z\|^2_{\mathscr{H}_\beta} = \sum_{n=1}^{\infty}\beta_n\left[\frac{|z|^n}{\lambda n^2\beta_{n-1} + \beta_n}\right]^2 \le \frac{1}{4\lambda}\sum_{n=1}^{\infty}\frac{|z|^{2n}}{n^2\beta_{n-1}} \le \frac{1}{4\lambda c}K_{\mathscr{H}_\beta,z}(z).$$

This gives (ii).

Finally, from (5) we have

$$\|F^*_{\lambda,P}(h)\|^2_{\mathscr{H}_\beta} = \sum_{n=0}^{\infty}\beta_n\left[\frac{(n+1)\beta_{n+1}|h_{n+1}|}{\lambda(n+1)^2\beta_n + \beta_{n+1}}\right]^2.$$

Then we obtain

$$\|F^*_{\lambda,P}(h)\|^2_{\mathscr{H}_\beta} \leq \frac{1}{4\lambda}\sum_{n=0}^{\infty}\beta_{n+1}|h_{n+1}|^2 \leq \frac{1}{4\lambda}\|h\|^2_{\mathscr{H}_\beta},$$

which gives (iii) and completes the proof of the lemma. □

We establish approximate inversion formulas for the operator P.

Theorem 2. *If $\lambda > 0$ and $h \in \mathscr{H}_\beta$, then*

(i) $\lim_{\lambda\to 0^+}\|PF^*_{\lambda,P}(h) - h_0\|^2_{\mathscr{H}_\beta} = 0$, *where* $h_0(z) = h(z) - h(0)$,

(ii) $\lim_{\lambda\to 0^+}\|F^*_{\lambda,P}(Ph) - h\|^2_{\mathscr{H}_\beta} = 0$.

Proof. Let $\lambda > 0$ and $h \in \mathscr{H}_\beta$ with $h(z) = \sum_{n=0}^{\infty} h_n z^n$. From (6) we have

$$PF^*_{\lambda,P}(h)(z) - h_0(z) = \sum_{n=1}^{\infty}\frac{-\lambda n^2\beta_{n-1}h_n}{\lambda n^2\beta_{n-1}+\beta_n}z^n.$$

Therefore

$$\|PF^*_{\lambda,P}(h) - h_0\|^2_{\mathscr{H}_\beta} = \sum_{n=1}^{\infty}\beta_n\left[\frac{\lambda n^2\beta_{n-1}|h_n|}{\lambda n^2\beta_{n-1}+\beta_n}\right]^2$$

By dominated convergence theorem and the fact that

$$\beta_n\left[\frac{\lambda n^2\beta_{n-1}|h_n|}{\lambda n^2\beta_{n-1}+\beta_n}\right]^2 \leq \beta_n|h_n|^2, \quad n \geq 1,$$

we deduce (i).

Finally, from (4) and (5) we have

$$F^*_{\lambda,P}(Ph)(z) - h(z) = \sum_{n=0}^{\infty}\frac{-\lambda(n+1)^2\beta_n h_n}{\lambda(n+1)^2\beta_n+\beta_{n+1}}z^n.$$

So, one has

$$\|F^*_{\lambda,P}(Ph) - h\|^2_{\mathscr{H}_\beta} = \sum_{n=0}^{\infty}\beta_n\left[\frac{\lambda(n+1)^2\beta_n|h_n|}{\lambda(n+1)^2\beta_n+\beta_{n+1}}\right]^2.$$

Using the dominated convergence theorem and the fact that

$$\beta_n\left[\frac{\lambda(n+1)^2\beta_n|h_n|}{\lambda(n+1)^2\beta_n+\beta_{n+1}}\right]^2 \leq \beta_n|h_n|^2,$$

we deduce (ii). □

Since $\mathscr{H}_\beta$ is a reproducing kernel Hilbert space, the evaluation function is always bounded on $\mathscr{H}_\beta$. Then, from Theorem 2, we deduce the following pointwise approximate inversion formulas for the operator P.

Corollary 1. *If $\lambda > 0$ and $h \in \mathscr{H}_\beta$, then*

(i) $\lim_{\lambda\to 0^+}PF^*_{\lambda,P}(h)(z) = h_0(z)$,

(ii) $\lim_{\lambda\to 0^+}F^*_{\lambda,P}(Ph)(z) = h(z)$.

5 Conclusion and perspective

We investigated the extremal function $F^*_{\lambda,P}(h)$ related to the weighted Hardy space $\mathscr{H}_\beta$, which is the solution of the Tikhonov regularization problem

$$\inf_{f\in\mathscr{H}_\beta}\left\{\lambda\|f\|^2_{\mathscr{H}_\beta}+\|Pf-h\|^2_{\mathscr{H}_\beta}\right\}. \tag{7}$$

Some results related to this function on $\mathbb{D}$ are proven, such as

$$F^*_{\lambda,P}(h)(z)=(\lambda I+P^*P)^{-1}P^*h(z),\quad z\in\mathbb{D}. \tag{8}$$

The function $F^*_{\lambda,P}$ also obeys to the following conclusions.

$\triangleright$ Let $\delta,\lambda>0$ and $h,h_\delta\in\mathscr{H}_\beta$ such that $\|h-h_\delta\|_{\mathscr{H}_\beta}\leq\delta$. Then by Lemma 1 (iii), we have $\|F^*_{\lambda,P}(h)-F^*_{\lambda,P}(h_\delta)\|_{\mathscr{H}_\beta}\leq\frac{\delta}{2\sqrt{\lambda}}$. When h, h_δ contain errors or noises, we need error estimates for (8).

$\triangleright$ Let $h\in\mathscr{H}_\beta$, in the limit case $\lambda\to0$, the problem (7) reduces to the Tikhonov regularization problem $\inf_{f\in\mathscr{H}_\beta}\left\{\|Pf-h\|^2_{\mathscr{H}_\beta}\right\}$, and it's extremal function is defined as $F^*_{0,P}(h)(z)=\lim_{\lambda\to0}F^*_{\lambda,P}(h)(z)$, $z\in\mathbb{D}$. And from Corollary 1 (i), for $h\in\mathscr{H}_\beta$, we obtain $PF^*_{0,P}(h)(z)=\lim_{\lambda\to0}PF^*_{\lambda,h}(z)=h_0(z)=h(z)-h(0)$.

In the future work, by using computers, we shall illustrate numerical experiments approximation formulas for the primitive operator P in the case of some examples of weighted Hardy spaces $\mathscr{H}_\beta$, when $\lambda\to0$.

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