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COMMON LOCAL SPECTRAL PROPERTIES FOR THE OPERATORS AC AND BD

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Abstract

In [6], Chen and Abdolyousefi studied the common spectral properties of operators AC and BD satisfying

$$(AC)^2A = ACDBA = DBACA = (DB)^2A;$$
$$(AC)^2D = ACDBD = DBACD = (DB)^2D.$$

In this note, we continue studying the common local spectral properties of these operators. We show that AC and BD shared the single-valued extension property, the Bishop property (β), the property (β_ϵ), the decomposition property (δ), and decomposability. Furthermore, we investigate the closedness of the analytic core and the quasinilpotent part. We also examine the Dunford's property (C) and the property (Q).

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1 Introduction

For Banach bounded linear operators A and B, Jacobson's Lemma asserts the equality

$$\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\} \tag{1}$$

where $\sigma(\cdot)$ represents the ordinary spectrum. This equality (1) has been a subject of extensive study aimed at demonstrating that AB and BA also share various local spectral properties see [2, 3, 22, 10, 12, 20, 18, 16, 15, 5] and the references therein. In their work, Benhida and Zerouali [5] established the sharing of properties such as the Single-Valued Extension Property (SVEP), Bishop property (β), the property (β_ϵ) , the decomposition property (δ) and decomposability for

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operators AB and BA. Additionally, Zeng and Zhong in their work [20], further extended the shared local spectral properties for operators for AC and BA where A, B and C are bounded linear operators under the equality

$$ABA = ACA. \tag{2}$$

For operators A, B, C and D satisfying

$$ACD = DBD \text{ and } BDA = ACA, \tag{3}$$

Yan and Fang showed the local spectral properties for operators AC and BD in their study [18]. More recently, Yan et al. introduced conditions in their work [19]

$$BAC = BDB \text{ and } CDB = CAC. \tag{4}$$

to further investigate the common local spectral theory for operators $I - AC$ and $I - BD$ under these conditions.

[6] presented and studied common spectral properties for operators BD and AC under following condition

$$\begin{aligned} (AC)^2A = ACDBA = DBACA = (DB)^2A; \\ (AC)^2D = ACDBD = DBACD = (DB)^2D. \end{aligned} \tag{5}$$

In this study, we focus on operators AC and BD under assumption (5) and explore their shared local spectral properties. Our findings reveal that AC and BD have the Single-Valued Extension Property, Bishop property (β), property (β_ϵ) , decomposition property (δ) and decomposability. Additionally, we delve the closedness of the analytic core and the quasinilpotent part of these operators.

2 Background from local spectral theory

Throughout this paper $\mathcal{B}(X)$ denotes the set of all bounded linear operators acting on a complex Banach space X. For $T \in \mathcal{B}(X)$, the *local resolvent* set $\rho_T(x)$ of T at a vector x in X is the union of all open subsets $U \subset \mathbb{C}$ for which there exists an analytic function $f : U \to X$ such that

$$(T - \mu)f(\mu) = x, \text{ for all } \mu \in U.$$

The *local spectrum* $\sigma_T(x)$ is defined by $\sigma_T(x) = \mathbb{C} \backslash \rho_T(x)$. The local spectrum $\sigma_T(x)$ is a closed, possibly empty, subset of $\sigma(T)$.

The operator $T \in \mathcal{B}(X)$ is said to *have the single valued extension property* (SVEP, for short) at $\mu \in \mathbb{C}$ provided that there exists an open disc U_μ centered at μ such that for every open subset $V \subset U_\mu$, the constant function $f \equiv 0$ is the only analytic solution of the equation

$$(T - \mu)f(\mu) = 0 \quad \forall \mu \in V.$$

We denote by $\sigma_{SVEP}(T)$ the set where T fails to have the SVEP and we say that T has the SVEP if $\sigma_{SVEP}(T) = \emptyset$. In the case where T has the SVEP, $\sigma_T(x) = \emptyset$ if and only if $x = 0$. The *local spectral radius* of T at x is given by

$$r_T(x) = \lim \sup_{n \to +\infty} \|T^n x\|^{\frac{1}{n}}$$

and if We assume that $\sigma_T(x) \neq \emptyset$ then $\max\{\mu \in \sigma_T(x)\} \leq r_T(x)$, moveover the last inequality becomes equality when T has the SVEP [13, Proposition 3.3.13].

For a subset $F \subseteq \mathbb{C}$, let $X_T(F)$ denote the *local spectral subspace* defined by

$$X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}.$$

Clearly, $X_T(F)$ is a linear (not necessarily closed) subspace of X.

For an open set U of $\mathbb{C}$, let $\mathcal{O}(U, X)$ be the Fréchet space of all X-valued analytic function on U endowed with the topology defined by uniform convergence on every compact subset of U. An operator $T \in \mathcal{B}(X)$ is said to satisfy the *Bishop's property* (β) on an open set $U \subseteq C$ provided that for every open subset V of U and for any sequence $(f_n)_n$ of analytic X-valued functions on V

$$(T - \mu) f_n(\mu) \to 0 \text{ in } \mathcal{O}(V, X) \Longrightarrow f_n(\mu) \to 0 \text{ in } \mathcal{O}(V, X).$$

Let $\rho_\beta(T)$ be the largest open set on which T has the property (β). Its complement $\sigma_\beta(T) = \mathbb{C} \backslash \rho_\beta(T)$ is a closed, possibly empty, subset of $\sigma(T)$. Then T is said to satisfy the Bishop's property (β) , precisely when $\sigma_\beta(T) = \emptyset$, [16]. It is well known that the following implications hold

$$\text{Bishop's property } (\beta) \Rightarrow \text{ SVEP}.$$

For a closed subset F in $\mathbb{C}$, the *glocal spectral analytic space* $\mathcal{X}_T(F)$ is the set of vectors $x \in X$ for which there exists an analytic function $f : \mathbb{C} \backslash F \to X$ such that

$$(T - \mu) f(\mu) = x, \text{ for all } \mu \in \mathbb{C} \backslash F.$$

$\mathcal{X}_T(F)$ is a linear subspace contained in $X_T(F)$. Moreover, the equality $\mathcal{X}_T(F) = X_T(F)$ holds for all closed sets $F \subseteq \mathbb{C}$ precisely when T has the SVEP [13, Proposition 3.3.2].

An operator $T \in \mathcal{B}(X)$ is said to have the *decomposition property* (δ) on U provided that for all open sets $V, W \subseteq \mathbb{C}$ for which $\mathbb{C} \backslash U \subseteq V \subseteq \overline{V} \subseteq W$,

$$\mathcal{X}_T(\mathbb{C} \backslash V) + \mathcal{X}_T(\overline{W}) = X.$$

Let $\rho_\delta(T)$ be the largest open set on which the operator T has the property (δ). Its complement $\sigma_\delta(T) = \mathbb{C} \backslash \rho_\delta(T)$ is a closed, possibly empty, subset of $\sigma(T)$ [16, Corollary 17]. Then T has the decomposition property (δ) if $\sigma_\delta(T) = \emptyset$. Properties (β) and (δ) are known to be dual to each other in the sense that T has (δ) on U if and only if T^* satisfies (β) on U [13, 16]. Moreover

$$\sigma_\beta(T) = \sigma_\delta(T^*) \text{ and } \sigma_\delta(T) = \sigma_\beta(T^*)$$

The operator $T \in \mathcal{B}(X)$ is said to be *decomposable on* U provided that for every finite open cover $\{U_1, \ldots, U_n\}$ of $\mathbb{C}$, with $\sigma(T) \backslash U \subseteq U_1$, there exists $X_1, \ldots, X_n$ closed T-invariant subspaces of X for which

$$\sigma\left(T \mid X_i\right) \subseteq U_i \text{ for } i=1, \ldots, n \text{ and } X_1+\cdots+X_n=X$$

Let $\rho_{\mathrm{dec}}(T)$ be the largest open set $U \subseteq \mathbb{C}$ on which T is decomposable. Its complement $\sigma_{\mathrm{dec}}(T)=\mathbb{C} \backslash \rho_{\mathrm{dec}}(T)$ is a closed, possibly empty, subset of $\sigma(T)$. We say that T is decomposable if $\sigma_{dec}(T)=\emptyset$. The class of decomposable operators contains all normal operators and more generally all spectral operators. Operators with totally disconnected spectrum are decomposable by the Riesz functional calculus. In particular, compact and algebraic operators are decomposable. It is also known that T has (β) on U precisely when it is similar to the restriction to a closed invariant subspace of an operator that is decomposable on U, while T has the property (δ) on U precisely when it is similar to the quotient of a decompsable operator on U by a closed invariant subspace [13]. And we have

$$\sigma_{dec}(T)=\sigma_\beta(T) \cup \sigma_\delta(T)=\sigma_\beta(T) \cup \sigma_\beta\left(T^*\right)=\sigma_{dec}\left(T^*\right).$$

3 Common decomposability properties

This section commences with the following result, illustrating that for condition (5), we have AC and BD share the property (β).

Theorem 1. *If $A, B, C, D \in \mathcal{L}(X)$ satisfy (5), then the following statements are equivalent:*
i) AC possesses the property (β) on an open set W of $\mathbb{C}$.
ii) BD possesses the property (β) on an open set W of $\mathbb{C}$.
In particular,

$$\sigma_\beta(AC)=\sigma_\beta(BD).$$

Proof. $i) \Rightarrow ii)$: Suppose that BD satisfies the property (β) on W. Let V be an open subset of W and let $\left(f_n\right)_n$ be a sequence of X-valued analytic functions such that

$$(AC-\mu)\, f_n(\mu) \rightarrow 0 \quad \text{in } \mathcal{O}(W, X). \tag{6}$$

Hence

$$(CA-\mu)\, C f_n(\mu) \rightarrow 0 \quad \text{in } \mathcal{O}(W, X).$$

Then

$$BDBA\,(CA-\mu)\, C f_n(\mu) \rightarrow 0 \quad \text{in } \mathcal{O}(W, X).$$

Using the fact, $DBACA=(DB)^2 A$, we obtain

$$\left((BD)^2 BA-\mu BDBA\right) C f_n(\mu) \rightarrow 0 \quad \text{in } \mathcal{O}(W, X).$$

$$(BD-\mu) BDBAC f_n(\mu) \rightarrow 0 \quad \text{in } \mathcal{O}(W, X).$$

Since BD satisfies property (β) on W and $BDBACf_n(\mu)$ is analytic on W, then

$$BDBACf_n(\mu) \to 0 \,\text{in}\, \mathcal{O}(W, X).$$

Then

$$(DB)^2 ACf_n(\mu) = DBDBACf_n(\mu) \to 0 \,\text{in}\, \mathcal{O}(W, X).$$

By equalities (5) we get

$$(AC)^3 f_n(\mu) \to 0 \,\text{in}\, \mathcal{O}(W, X).$$

Thus, using equation (1.3), we can conclude that $\mu f_n(\mu) \to 0$ in $\mathcal{O}(W, X)$. Since f_n are analytic on W, the maximum modulus principle implies that the sequence $(f_n)_n$ converges to zero on compact sets in V. Consequently, $f_n(\mu) \to 0$ in $\mathcal{O}(V, X)$, which establishes that BD satisfies (β).

The converse implication is similar. □

Because T satisfies property (β) on W if and only if T^* satisfies property (δ) on W, this duality relationship leads us to the following result.

Theorem 2. *If $A, B, C, D \in \mathcal{L}(X)$ satisfy (5), then the following statements are equivalent:*

i) AC possesses the property (δ) on an open set W of $\mathbb{C}$.

ii) BD possesses the property (δ) on an open set W of $\mathbb{C}$.

In particular,

$$\sigma_\delta(AC) = \sigma_\delta(BD).$$

A weaker version of property (δ) was introduced in [21]: an operator U is said to possess the *weak spectral property* (δ_w) if, for any finite open cover $\{V_1, \dots, V_n\}$ of $\mathbb{C}$, we have

$$\mathcal{X}_U(\overline{V_1}) + \dots + \mathcal{X}_U(\overline{V_n}) \text{ is dense in } X.$$

If U has the weak property (δ_w) then U^* has the SVEP. If G is an locally compact abelian group, then every convolution operator in $L^1(G)$ has the weak property (δ_w), [21].

Proposition 1. *If $A, B, C, D \in \mathcal{L}(X)$ satisfy (5), and if D and B have dense ranges, then the following are equivalent:*

i) AC possesses the property (δ_w).

ii) BD possesses the property (δ_w).

Proof. Suppose that BD has the property (δ_w). Let$\{V_1, V_2, ..., V_n\}$ be a finite open cover of $\mathbb{C}$. Then

$$\mathcal{X}_{BD}(\overline{V_1}) + \mathcal{X}_{BD}(\overline{V_2}) + ... + \mathcal{X}_{BD}(\overline{V_n}) \text{ is dense in } X.$$

It is easy to see that $DBD(\mathcal{X}_{BD}(\overline{V_i})) \subseteq \mathcal{X}_{AC}(\overline{V_i})$, $\forall i = 1, 2, ..., n$. Since B and D are dense range then DBD is dense in X. Then

$$\mathcal{X}_{AC}(\overline{V_1}) + \mathcal{X}_{AC}(\overline{V_2}) + ... + \mathcal{X}_{AC}(\overline{V_n}) \text{ is dense in } X.$$

Therefore, AC satisfies the property (δ_w).

Conversely, suppose that AC has the property (δ_w). Let $\{V_1, V_2, ..., V_n\}$ be a finite open cover of $\mathbb{C}$. Then

$$\mathcal{X}_{AC}(\overline{V_1}) + \mathcal{X}_{AC}(\overline{V_2}) + ... + \mathcal{X}_{AC}(\overline{V_n}) \text{ is dense in } X.$$

It is easy to see that $BDBAC(\mathcal{X}_{AC}(\overline{V_i})) \subseteq \mathcal{X}_{BD}(\overline{V_i})$, $\forall i = 1, 2, ..., n$. Since B and D is dense range then $BDBAC$ is also dense range, hence

$$\mathcal{X}_{BD}(\overline{V}_1) + \mathcal{X}_{BD}(\overline{V}_2) + ... + \mathcal{X}_{BD}(\overline{V}_n) \text{ is dense in } X$$

Therefore, BD satisfies the property (δ_w). □

Since decomposability is equivalent to both (β) and (δ), then :

Corollary 1. *If $A, B, C, D \in \mathcal{L}(X)$ satisfy (5), then the following statements are equivalent:*

i) AC is decomposable on an open set W of $\mathbb{C}$.

ii) BD is decomposable on an open set W of $\mathbb{C}$.

In particular,

$$\sigma_{dec}(AC) = \sigma_{dec}(BD).$$

The operator U is said to satisfy the *property* (β_ϵ) at $\mu \in \mathbb{C}$ provided that there exists an open disc W centered at μ such that for every open subset $V \subset W$ and for any sequence $(f_n)_n$ of infinitely differentiable X-valued functions on V, we have

$$(U - \mu)f_n(\mu) \to 0 \text{ in } \mathcal{E}(V, X) \Longrightarrow f_n(\mu) \to 0 \text{ in } \mathcal{E}(V, X);$$

where $\mathcal{E}(V, X)$ is the Fréchet space of all X-valued C^∞-functions on V. Let $\sigma_{\beta_\epsilon}(T)$ be the set of all points where U fails to satisfy the property (β_ϵ). Then U is said to satisfy the property (β_ϵ), precisely when $\sigma_{\beta_\varepsilon}(U) = \emptyset$ ([8]).

Theorem 3. *If $A, B, C, D \in \mathcal{L}(X)$ satisfy (5), then the following statements are equivalent:*

i) AC possesses the property (β_ϵ) at μ.

ii) BD possesses the property (β_ϵ) at μ.

In particular,

$$\sigma_{\beta_\epsilon}(AC) = \sigma_{\beta_\epsilon}(BD).$$

Proof. By the same argument as in the proof of Theorem 1 and using [5, Lemma 2.1]. □

An operator T satisfies (β_ϵ) if and only if T is subscalar, is the sense that it has a generalized scalar extension. An operator T is said to be generalized scalar if there exists a continuous homomorphism algebra $\Phi : \mathcal{E}(\mathbb{C}) \to \mathcal{B}(X)$ with $\Phi(1) = I$ and $\Phi(z) = T$ [8, 13].

Corollary 2. *If $A, B, C, D \in \mathcal{L}(X)$ satisfy (5), then the following statements are equivalent:*

i) AC is subscaler.

ii) BD is subscalar.

4 Local spectral subspaces

Theorem 4. *If $A, B, C, D \in \mathcal{L}(X)$ satisfy (5) and $\mu \in \mathbb{C}$ then the following statements are equivalent:*
i) AC has the SVEP at μ.
ii) BD has the SVEP at μ.
In particular,

$$\sigma_{SVEP}(AC) = \sigma_{SVEP}(BD).$$

Proof. It is similar to the proof of Theorem 1. □

Theorem 5. *If $A, B, C, D \in \mathcal{L}(X)$ satisfy (5), then for $x \in X$*
i) $\sigma_{AC}(Dx) \subseteq \sigma_{BD}(x) \subseteq \sigma_{AC}(Dx) \cup \{0\}$ for all $x \in X$;
ii) $\sigma_{BD}(BACy) \subseteq \sigma_{AC}(y) \subseteq \sigma_{BD}(BACy) \cup \{0\}$ for all $y \in Y$.

Proof. i) Let $\mu \notin \sigma_{BD}(x) \cup \{0\}$, Then there exists an open neighborhood U of μ and an X-valued analytic function f on U such that

$$(BD - \mu)f(\mu) = x \quad \forall \mu \in U.$$

Then

$$(DB - \mu)Df(\mu) = Dx \quad \forall \mu \in U.$$

Then $\left((DB)^2 - \mu DB\right) Df(\mu) = DBDx$, $\forall \mu \in U$. From the equalities (5) above it follows that $(AC - \mu)\, DBDf(\mu) = DBDx$, $\forall \mu \in U$. Since $DBDf(\mu)$ is analytic on U, then $\mu \notin \sigma_{AC}\,(DBDx) \cup \{0\}$ then $\mu \notin \sigma_{CA}\,(CDBDx) \cup \{0\}$ by [5, Proposition 3.1]. From the equalities above it follows that $\mu \notin \sigma_{CA}\,(CACDx) \cup \{0\}$. Hence by [5, Proposition 3.1] three times, we get $\mu \notin \sigma_{AC}\,(Dx) \cup \{0\}$. Conversely, let $\mu \notin \sigma_{AC}(Dx) \cup \{0\}$, then $\mu \notin \sigma_{AC}((AC)^2 Dx) \cup \{0\} = \sigma_{AC}(ACDBDx) \cup \{0\}$, thus $\mu \notin \sigma_{AC}(DBDx) \cup \{0\}$ by [5, Proposition 3.1]. Let $g : \mathcal{V}_\mu \to Y$ be an analytic function defined on some neighborhood $\mathcal{V}_\mu$ of μ such that

$$(AC - \mu)g(\mu) = DBDx \quad \text{ for all } \mu \in \mathcal{V}_\mu.$$

Therefore, $g(\mu) = \frac{ACg(\mu) - DBDx}{\mu}$ and we have

$$\begin{aligned}
x &= \frac{[(BD)^3 x - ((BD)^3 - \mu^3)x]}{\mu^3} \\
&= \frac{[BDB(AC - \mu)g(\mu) - ((BD)^3 - \mu^3)x]}{\mu^3} \\
&= \frac{[(BDBAC - \mu BDB)g(\mu) - ((BD)^3 - \mu^3)x]}{\mu^3} \\
&= \frac{\left[(BDBAC - \mu BDB)\frac{ACg(\mu) - DBDx}{\mu} - ((BD)^3 - \mu^3)x\right]}{\mu^3}
\end{aligned}$$

$$
\begin{aligned}
&= \mu^{-3}\left[BDBACAC\frac{g(\mu)}{\mu} - BDBACDBD\frac{x}{\mu} - \mu BDBAC\frac{g(\mu)}{\mu}\right.\\
&\qquad \left. +\mu BDBDBD\frac{x}{\mu} - ((BD)^3-\mu^3)x\right]\\
&= \mu^{-3}\left[B(DB)^2AC\frac{g(\mu)}{\mu} - (BD)^4\frac{x}{\mu} - \mu BDBAC\frac{g(\mu)}{\mu} + \mu BDBDBD\frac{x}{\mu}\right.\\
&\qquad \left. -((BD)^3-\mu^3)x\right]\\
&= \frac{(BD-\mu)\left(BDBAC\frac{g(\mu)}{\mu} - BDBDBD\frac{x}{\mu} - (BD+(BD)^2+\mu)x\right)}{\mu^3}\\
&= \frac{(BD-\mu)(BDBg(\mu) - (BD+(BD)^2+\mu)x)}{\mu}.
\end{aligned}
$$

Define

$$h(\mu) = \frac{BDBg(\mu) - (BD+(BD)^2+\mu)x}{\mu} \qquad \text{for all } \mu \in \mathcal{V}_\mu.$$

Evidently, $h : \mathcal{V}_\mu \to X$ is analytic and $(BD-\mu)h(\mu) = x$, so $\mu \notin \sigma_{BD}(x)$.

ii) By [5, Proposition 3.1], we have

$$\sigma_{CA}(Cy) \subseteq \sigma_{AC}(y) \subseteq \sigma_{CA}(Cy) \cup \{0\}.$$

According to i) and interchanging B with C and D with A, we obtain $\sigma_{DB}(ACy) \subseteq \sigma_{CA}(y) \subseteq \sigma_{DB}(ACy) \cup \{0\}$. By [5, Proposition 3.1], we have from these inclusions, we conclude that $\sigma_{BD}(BACy) \subseteq \sigma_{AC}(y) \subseteq \sigma_{BD}(BACy) \cup \{0\}$. □

Remark 1. Let $x \in X$.

i) If D is injective, then $\sigma_{AC}(Dx) = \sigma_{BD}(x)$.

ii) If AC is injective, then $\sigma_{AC}(x) = \sigma_{BD}(BACx)$.

We recall that $\sigma_s(T) = \bigcup_{x\in X}\sigma_T(x)$, where $\sigma_s(T)$ is the *surjectivity spectrum* of T ([13]). Then we have

$$\sigma_s(AC) \cup \{0\} = \sigma_s(BD) \cup \{0\}.$$

and since $\sigma(T) = \sigma_s(T) \cup \sigma_{SVEP}(T)$ then by Theorem 4 we retrieve equality (1) of Jacobson's lemma

$$\sigma(AC) \cup \{0\} = \sigma(BD) \cup \{0\}.$$

The operator T is said to possess the *Dunford's property* $(\mathcal{C})$ if $X_T(F)$ is closed for every closed subset F of $\mathbb{C}$. The following implications are well known ([13]):

Bishop's property $(\beta) \Rightarrow$ Dunford's property $(\mathcal{C}) \Rightarrow$ SVEP.

Theorem 6. *Let $A, B, C, D \in \mathcal{L}(X)$ satisfy (5), then the following statements are equivalent:*

i) AC possesses Dunford's property $(\mathcal{C})$.

ii) BD possesses Dunford's property $(\mathcal{C})$.

The proof of Theorem 6 will be given by the following two lemmas.

Lemma 1. *Let $A, B, C, D \in \mathcal{L}(X)$ satisfy (5). Let F be a closed subset of $\mathbb{C}$ such that $0 \in F$, then the following statements are equivalent:*
i) $X_{AC}(F)$ is closed.
ii) $X_{BD}(F)$ is closed.

Proof. Suppose that $X_{AC}(F)$ is closed and let $(x_n)_n$ be a sequence in $X_{BD}(F)$ which converge to some x in X. Then $\sigma_{BD}(x_n) \subset F$ and $0 \in F$, hence $\sigma_{AC}(Dx_n) \cup 0 \subset F$. Thus by theorem 5 i), we have $\sigma_{AC}(Dx_n) \subset F$ and $Dx_n \in X_{AC}(F)$. Since $X_{AC}(F)$ is closed and Dx_n converge to Dx, then $Dx \in X_{AC}(F)$. Hence $\sigma_{AC}(Dx) \subset F$ and again by theorem 5 i) we have $\sigma_{BD}(x) \subset F$. Thus $x \in X_{BD}(F)$. Therefore $X_{BD}(F)$ is closed. The converse implication is similar. □

Lemma 2. *Let $A, B, C, D \in \mathcal{L}(X)$ satisfy (5). Let F be a closed subset of $\mathbb{C}$ such that $0 \notin F$ and BD has the SVEP.*
i) if $X_{AC}(F \cup \{0\})$ is closed, then $X_{BD}(F)$ is closed;
ii) if $X_{BD}(F \cup \{0\})$ is closed, then $X_{AC}(F)$ is closed.

Proof. Since BD has the SVEP then it follows from Theorem 4 that AC has the SVEP. According to [20, Lemma 2.5] the result follows at once from Lemma 1. □

The *analytical core* of T, introduced and studied in [14, 15], is the set $K(T)$ of all $x \in X$ such that there exist a constant $c > 0$ and a sequence $(x_n)_n \subset X$ such that

$$x_0 = x, Tx_n = x_{n-1} \text{ and } \|x_n\| \leq c^n \|x\| \text{ for all } n \in \mathbb{N}.$$

Recall that ([1, Theorem 2.18] or [13, Proposition 3.3.7])

$$K(T - \mu) = X_T(\mathbb{C} \backslash \{\mu\}) = \{x \in X : \mu \notin \sigma_T(x)\}.$$

In general, $K(T)$ is not need to be closed. For any non-invertible decomposable operator T, the point 0 is isolated in $\sigma(T)$ exactly when $K(T)$ is closed. In particular, if T is a compact operator, or more generally a Riesz operators, then $K(T)$ is closed precisely when T has finite spectrum, [17, Corollary 6].

Theorem 7. *Let $A, B, C, D \in \mathcal{L}(X)$ satisfy (5), then for all $0 \neq \mu \in \mathbb{C}$, the following statements are equivalent:*
i) $K(AC - \mu)$ is closed.
ii) $K(BD - \mu)$ is closed.

Proof. Suppose that $K(AC-\mu)$ is closed. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $K(BD-\mu)$ which converges to some $x \in X$. To show that $K(BD - \mu)$ is closed, it suffices to prove that $x \in K(BD - \mu)$. Since $x_n \in K(BD - \mu)$, $\mu \in \rho_{BD}(x_n)$ for all $n \in \mathbb{N}$. Hence by Theorem 5 i), $\mu \in \rho_{AC}(Dx_n)$, that is, $Dx_n \in K(AC - \mu)$. Since $Dx_n \to Dx$ and $K(AC - \mu)$ is closed, $Dx \in K(AC - \mu)$, that is, $\mu \in \rho_{AC}(Dx)$. By Theorem 5 i) again, $\mu \in \rho_{BD}(x)$, that is, $x \in K(BD - \mu)$.
The converse implication follows by using Theorem 5 ii) instead of Theorem 5 i). □

The *quasi-nilpotent part* $H_0(T)$ of T is defined by

$$H_0(T) := \left\{ x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0 \right\}.$$

Lemma 3. *Let $A, B, C, D \in \mathcal{L}(X)$ satisfy (5). Let F be a closed subset of $\mathbb{C}$ such that $0 \in F$, then the following statements are equivalent:*
i) $\mathcal{X}_{AC}(F)$ is closed.
ii) $\mathcal{X}_{BD}(F)$ is closed.

Proof. Suppose that $\mathcal{X}_{AC}(F)$ is closed. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{X}_{BD}(F)$ which converges to some $x \in X$. To show that $\mathcal{X}_{BD}(F)$ is closed, it suffices to prove that $x \in \mathcal{X}_{BD}(F)$. Since $x_n \in \mathcal{X}_{BD}(F)$, for every $n \in \mathbb{N}$ there exists an analytic function $f_n : \mathbb{C}\backslash F \to X$ such that

$$(BD - \mu) f_n(\mu) = x_n \quad \text{ for all } \mu \in \mathbb{C}\backslash F.$$

Thus, $DBD(BD-\mu)f_n(\mu) = (AC-\mu)DBDf_n(\mu) = DBDx_n$ and hence $DBDx_n \in \mathcal{X}_{AC}(F)$. Since $DBDx_n \to DBDx$ and $\mathcal{X}_{AC}(F)$ is closed, $DBDx \in \mathcal{Y}_{AC}(F)$. Thus, there exists an analytic function $g : \mathbb{C}\backslash F \to Y$ such that

$$(AC - \mu)g(\mu) = DBDx \text{ for all } \mu \in \mathbb{C}\backslash F$$

As in the proof of Theorem 3 ii), we obtain $x = \frac{BDBg(\mu)-(BD+(BD)^2+\mu)x}{\mu}$ for all $\mu \in \mathbb{C}\backslash F$. Define

$$h(\mu) = \frac{BDBg(\mu) - (BD + (BD)^2 + \mu)x}{\mu} \text{ for all } \mu \in \mathbb{C}\backslash F.$$

Evidently, $h : \mathbb{C}\backslash F \to X$ is analytic and $(BD - \mu)h(\mu) = x$, so $x \in \mathcal{X}_{BD}(F)$.
The converse implication follows by using Theorem 5 i). □

In general $H_0(T)$ is not closed and it coincides with the glocal spectral subspace $\mathcal{X}_T(\{0\})$ (see [1, Theorem 2.20]). As an immediate consequence of Lemma 3, we obtain

Theorem 8. *If $A, B, C, D \in \mathcal{L}(X)$ satisfy (5), then the following statements are equivalent:*
i) $H_0(AC)$ is closed.
ii) $H_0(BD)$ is closed.

An operator $T \in \mathcal{B}(X)$ is said to have the *property* (Q) if $H_0(\mu I - T)$ is closed for every $\mu \in \mathbb{C}$. It is known that if $H_0(\mu I - T)$ is closed then T has SVEP at μ, thus

$$\text{property } (C) \Rightarrow \text{ property } (Q) \Rightarrow \text{ SVEP.}$$

Therefore, for operators T having property (Q) we have $H_0(\mu I - T) = X_T(\{\mu\})$.

In [20, Corollary 3.8] it was observed that if $R \in L(Y, X)$ and $S \in L(X, Y)$ are both injective then RS has property (Q) precisely when SR has property (Q).

Theorem 9. *If $A, B, C, D \in \mathcal{L}(X)$ satisfy (5), then the following statements are equivalent:*
i) AC has the property (Q).
ii) BD has the property (Q).

Proof. Suppose that BD has the property (Q). Then BD has SVEP and, by Theorem 4, also AC has SVEP. Consequently, the local and glocal spectral subspaces relative to the a closed set coincide for AC and BD. Let $\mu \in \mathbb{C}$. Then by assumption $H_0(\mu I - BD) = X_{BD}(\{\mu\})$ is closed. If $\mu = 0$, then by Lemma 1 $H_0(AC) = X_{AC}(\{0\})$ is closed. Now assume that $\mu \neq 0$. Since BD has the SVEP then $X_{BD}(\{\mu\} \cup \{0\}) = X_{BD}(\{\mu\}) \oplus X_{BD}(\{0\})$ by [1, Theorem 2.17]. Hence $X_{BD}(\{\mu\} \cup \{0\})$ is closed. Thus by Lemma 2 $X_{AC}(\{\mu\})$ is closed.

Conversely, it follows similarly. □

We conclude this note by an example to illustrate that the results obtained in this note are proper generalizations of the corresponding ones in [20, 19].

Example 1. *We have four linear operators, A, B, C, D, all acting on the direct sum of four vector space X and P is a non-trivial projection. These operators are represented as matrices in this space as follows:*

$$A = B = C = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$D = \begin{pmatrix} I & I & I & 2P \\ 0 & 0 & P & P \\ 0 & 0 & 0 & -I \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then A, C, D and B leading to the equations

$$(AC)^2 A = ACDBA = DBACA = (DB)^2 A;$$
$$(AC)^2 D = ACDBD = DBACD = (DB)^2 D,$$

while $(DB)^2 \neq (AC)^2$, $DBA \neq ACA$ and $CDB \neq CAC$. Therefore, it's established that the common local spectral properties of the operators AC and BD can only be derived directly from the results presented in the current context, rather than being deduced from corresponding results in [20, 19].

Example 2. *Let A, B, C and D be the operators defined on the separable Hilbert space $l_2(\mathbb{N})$, respectively, by*

$$A\,(x_1, x_2, x_3, x_4, \cdots) = (x_2, x_3, 0, 0, \cdots),$$
$$B\,(x_1, x_2, x_3, x_4, \cdots) = (0, x_2, 0, 0, 0, \cdots),$$
$$C\,(x_1, x_2, x_3, x_4, \cdots) = (0, x_2, x_3, 0, 0, \cdots),$$
$$D\,(x_1, x_2, x_3, x_4, \cdots) = (x_1, -x_1, 0, 0, \cdots).$$

Then

$$(AC)^2A = ACDBA = DBACA = (DB)^2A;$$
$$(AC)^2D = ACDBD = DBACD = (DB)^2D.$$

while $DBA \neq ACA$, $(AC)^2 \neq (DB)(AC)$, $(AC)^2A \neq A(BA)^2$ and $CDB \neq CAC$. Hence, the common local spectral properties of AC and CB can only be deduced directly from the results obtained in this note, but not from the corresponding ones in [20, 19].

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