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EXPONENTIAL KANTOROVICH-STANCU OPERATORS

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Abstract

In this paper we will obtain some Bernstein-Kantorovich operators modified in Stancu sense which preserve exponential function $e^{\mu x}$, where $\mu > 0$. Concerning these operators we prove they verify Korovkin's theorem conditions and also that they approximate functions from a weighted L^p space. Moreover, we will obtain a Voronovskaya theorem and some quantitative estimates of approximation using the first order modulus of continuity. Also, we will prove some estimates concerning the approximation of functions from a weighted L^p space using Peetre's K-functional. Finally, we will obtain an estimate which involves the first order modulus of continuity and the second order modulus of smoothness by using the equivalence relation between these moduli and the corresponding K-functionals.

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1 Introduction

In recent research, there have been studied approximation properties of some exponential variants of certain classical operators such as Bernstein operators, Szasz-Mirakjan operators, Kantorovich operators and many more.

Our aim is to introduce a new class of operators of exponential type, obtained as a modification in Stancu sense of Bernstein-Kantorovich exponential operators and prove some important approximation results concerning these operators.

In order to state our results, let us recall some of the most important operators in Approximation Theory related to the subject of the paper.

In order to provide a proof of Weierstrass's approximation theorem, S.N. Bernstein introduced the following operators (see [8]):

$$B_n f(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right),\ f \in C([0,1]),\ x \in [0,1],\ n \in \mathbb{N}, \tag{1}$$

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where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $k \in \{0, 1, \ldots, n\}$ and by $C([0,1])$ we mean the set of all continuous functions on $[0,1]$. These operators have been intensively studied since they have a lot of approximation properties.

After Bernstein operators were introduced, many generalizations of these operators arose. Among them we mention Bernstein-Stancu operators introduced in [12]:

$$B_n^{\alpha,\beta} f(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right), \ f \in C([0,1]), \ x \in [0,1], \ n \in \mathbb{N}, \tag{2}$$

where $0 \le \alpha \le \beta$, and $p_{n,k}$ are defined as above

Another generalization of operators B_n is due to L.V. Kantorovich who introduced in paper [11] the following operators:

$$\mathcal{K}_n f(x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\, dt, \ f \in L^1([0,1]), \ x \in [0,1], \ n \in \mathbb{N}, \tag{3}$$

where $p_{n,k}$ are defined as above and by $L^1([0,1])$ we mean the set of all integrable functions on $[0,1]$. These operators provide a very useful tool in approximation of integrable functions on $[0,1]$. Also, a Stancu variant of the operators above was introduced (see [7]).

In recent literature a lot of studies regarding approximation by positive linear operators which preserve one or two exponential functions, for example e^{2ax}, $a > 0$ (see [2], where a varianat of Szász-Mirakjan operators which preserve the said function is introduced). In this direction, there have been introduced exponential variants of Bernstein operators ([5]), of Szasz-Mirakjan operators ([1]) and also of Kantorovich operators ([6]). More studies related to approximation by operators which preserve the exponential functions mentioned above and their properties can be found in [3] and [9].

The aim of our paper is to provide a Stancu modification of the exponential variant of operators $\mathcal{K}_n$ introduced by Angeloni and Costarelli in [4]:

$$K_n f(x) = \sum_{k=0}^{n} e^{\mu x} p_{n,k}(a_{n+1}(x))(n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) e^{-\mu t} dt, \tag{4}$$

where $\mu > 0$, $f \in C([0,1])$ $x \in [0,1]$, $n \in \mathbb{N}$, and $a_n(x) := \frac{e^{\frac{\mu x}{n}} - 1}{e^{\frac{\mu}{n}} - 1}$ are increasing, continuous and convex functions on $[0,1]$ such that $a_n(0) = 0$ and $a_n(1) = 1$.

This paper is divided in four sections. In Section 2 we provide the definition of our operators, some convergence results for continuous functions on $[0,1]$ via Korovkin's theorem and a convergence result for functions in L^p spaces. In Section 3 we prove a Voronovskaya estimate of approximation by our operators and finally we provide some quantitative estimates using moduli of smoothness and K- functionals.

2 Definition and convergence results

Let us introduce the following operators for functions $f \in C([0,1])$:

$$K_n^{\alpha,\beta,\mu} f(x) = (n+\beta+1)e^{\mu x}\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} e^{-\mu t} f(t)dt, \qquad (5)$$

where $x \in [0,1]$ and $0 \le \alpha \le \beta$, $\mu > 0$, $a_n(x) = \frac{e^{\frac{\mu x}{n+\beta}}-1}{e^{\frac{\mu}{n+\beta}}-1}$, $n \in \mathbb{N}$. In order to obtain a Korovkin type approximation theorem for these operators we will check their convergence for test functions $e_0(x) = 1$, $exp_\mu(x) = e^{\mu x}$ and $exp_\mu^2(x) = e^{2\mu x}$, for $x \in [0,1]$, which form a Chebyshev set. First, it is obvious that

$$K_n^{\alpha,\beta,\mu} exp_\mu(x) = exp_\mu(x), \ x \in [0,1]. \qquad (6)$$

In order to obtain our approximation results we will need the following Lemma.

Lemma 1. *For $x \in [0,1]$ we have that:*

$$K_n^{\alpha,\beta,\mu} e_0(x) = \frac{n+\beta+1}{\mu} e^{\mu x}(1-e^{-\frac{\mu}{n+\beta+1}})e^{-\frac{\mu(\alpha+n)}{n+\beta+1}}(1-e^{\frac{\mu x}{n+\beta+1}}+e^{\frac{\mu}{n+\beta+1}})^n, \quad (7)$$

$$K_n^{\alpha,\beta,\mu} exp_\mu^2(x) = \frac{n+\beta+1}{\mu} e^{\mu x} e^{\frac{\mu\alpha}{n+\beta+1}}(e^{\frac{\mu}{n+\beta+1}}-1)e^{\frac{\mu n x}{n+\beta+1}}, \qquad (8)$$

$$K_n^{\alpha,\beta,\mu} exp_\mu^3(x) = \frac{n+\beta+1}{2\mu} e^{\mu x} e^{\frac{2\mu\alpha}{n+\beta+1}}(e^{\frac{2\mu}{n+\beta+1}}-1) \times \left(e^{\frac{\mu(x+1)}{n+\beta+1}} + e^{\frac{\mu x}{n+\beta+1}} - e^{\frac{\mu}{n+\beta+1}}\right)^n, \qquad (9)$$

$$K_n^{\alpha,\beta,\mu} exp_\mu^4(x) = \frac{n+\beta+1}{3\mu} e^{\mu x} e^{\frac{3\mu\alpha}{n+\beta+1}}(e^{\frac{3\mu}{n+\beta+1}}-1) \times \left(e^{\frac{\mu(x+2)}{n+\beta+1}} + e^{\frac{\mu(x+1)}{n+\beta+1}} + e^{\frac{\mu x}{n+\beta+1}} - e^{\frac{2\mu}{n+\beta+1}} - e^{\frac{\mu}{n+\beta+1}}\right)^n. \qquad (10)$$

Proof. In order to prove (7) we have that:

$$\begin{aligned} K_n^{\alpha,\beta,\mu} e_0(x) &= (n+\beta+1)e^{\mu x}\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} e^{-\mu t}dt \\ &= (n+\beta+1)e^{\mu x}\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))\frac{e^{-\mu t}}{-\mu}\Bigg|_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} \\ &= \frac{n+\beta+1}{\mu} e^{\mu x}\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))(e^{-\mu\frac{k+\alpha}{n+\beta+1}} - e^{-\mu\frac{k+\alpha+1}{n+\beta+1}}) \\ &= \frac{n+\beta+1}{\mu} e^{\mu x}(e^{-\mu\frac{\alpha}{n+\beta+1}} - e^{-\mu\frac{\alpha+1}{n+\beta+1}})\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))e^{-\mu\frac{k}{n+\beta+1}}, \end{aligned}$$

but,

$$\begin{aligned}\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))e^{-\mu\frac{k}{n+\beta+1}} &= \sum_{k=0}^{n}\binom{n}{k}(a_{n+1}(x)e^{\frac{-\mu}{n+\beta+1}})^k(1-a_{n+1}(x))^{n-k}\\ &= (a_{n+1}(x)e^{-\frac{\mu}{n+\beta+1}}+1-a_{n+1}(x))^n\\ &= \left(a_{n+1}(x)\frac{1-e^{\frac{\mu}{n+\beta+1}}}{e^{\frac{\mu}{n+\beta+1}}}+1\right)^n\\ &= (e^{-\frac{\mu}{n+\beta+1}}-e^{-\frac{\mu(x-1)}{n+\beta+1}}+1)^n\\ &= e^{-\frac{\mu n}{n+\beta+1}}(1-e^{\frac{\mu x}{n+\beta+1}}+e^{\frac{\mu}{n+\beta+1}})^n,\end{aligned}$$

therefore

$$\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))e^{-\mu\frac{k}{n+\beta+1}} = e^{-\frac{\mu n}{n+\beta+1}}(1-e^{\frac{\mu x}{n+\beta+1}}+e^{\frac{\mu}{n+\beta+1}})^n. \tag{11}$$

Now, replacing (11) in the above formula for $K_n^{\alpha,\beta,\mu}e_0(x)$ and after some calculations we get (7).

Next, we have that

$$\begin{aligned}K_n^{\alpha,\beta,\mu}exp_\mu^2(x) &= (n+\beta+1)e^{\mu x}\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} e^{\mu t}dt\\ &= (n+\beta+1)e^{\mu x}\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))\frac{e^{\mu t}}{\mu}\Big|_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}}\\ &= \frac{n+\beta+1}{\mu}e^{\mu x}\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))(e^{\mu\frac{k+\alpha+1}{n+\beta+1}}-e^{\mu\frac{k+\alpha}{n+\beta+1}})\\ &= \frac{n+\beta+1}{\mu}e^{\mu x}(e^{\mu\frac{\alpha+1}{n+\beta+1}}-e^{\mu\frac{\alpha}{n+\beta+1}})\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))e^{\mu\frac{k}{n+\beta+1}}.\end{aligned}$$

In the following, by proceeding as in (11) we can show that

$$\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))e^{\mu\frac{k}{n+\beta+1}} = e^{\mu\frac{nx}{n+\beta+1}},$$

which can be replaced in the expression for $K_n^{\alpha,\beta,\mu}exp_\mu^2$, which yields (8). Also, it is easy to see that:

$$K_n^{\alpha,\beta,\mu}exp_\mu^3(x) = \frac{n+\beta+1}{2\mu}e^{\mu x}(e^{\frac{2\mu(\alpha+1)}{n+\beta+1}}-e^{\frac{2\mu\alpha}{n+\beta+1}})\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))e^{2\mu\frac{k}{n+\beta+1}}, \tag{12}$$

and

$$K_n^{\alpha,\beta,\mu} exp_\mu^4(x) \tag{13}$$
$$= \frac{n+\beta+1}{3\mu} e^{\mu x}(e^{\frac{3\mu(\alpha+1)}{n+\beta+1}} - e^{\frac{3mu\alpha}{n+\beta+1}}) \sum_{k=0}^{n} p_{n,k}(a_{n+1}(x)) e^{2\mu \frac{k}{n+\beta+1}}.$$

Using a similar approach as in (11), we get

$$\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x)) e^{2\mu \frac{k}{n+\beta+1}} = \left(\frac{e^{\frac{\mu(x+2)}{n+\beta+1}} + e^{\frac{\mu}{n+\beta+1}} - e^{\frac{\mu x}{n+\beta+1}} - e^{\frac{2\mu}{n+\beta+1}}}{e^{\frac{\mu}{n+\beta+1}} - 1} \right)^n, \tag{14}$$

and

$$\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x)) e^{2\mu \frac{k}{n+\beta+1}} = \left(\frac{e^{\frac{\mu(x+3)}{n+\beta+1}} + e^{\frac{\mu}{n+\beta+1}} - e^{\frac{\mu x}{n+\beta+1}} - e^{\frac{3\mu}{n+\beta+1}}}{e^{\frac{\mu}{n+\beta+1}} - 1} \right)^n. \tag{15}$$

By replacing (14) and (15) in (12) and (13) we will get the last two identities from our Lemma. □

Having in mind Lemma 1 and the fact that e_0, exp_μ and exp_μ^2 form a Chebyshev set we can prove that operators $K_n^{\alpha,\beta,\mu}$ uniformly converge to functions $f \in C([0,1])$.

Theorem 1. *For $f \in C[0,1]$ we have that $K_n^{\alpha,\beta,\mu}$ converges uniformly to f on $[0,1]$.*

Proof. From Lemma 1 it follows that $K_n^{\alpha,\beta,\mu} e_0(x) \to 1$, $K_n^{\alpha,\beta,\mu} exp_\mu^2(x) \to exp_\mu^2(x)$ uniformly for $x \in [0,1]$ as $n \to \infty$, and $K_n^{\alpha,\beta} exp_\mu(x) = exp_\mu(x)$, $x \in [0,1]$. Since operators $K_n^{\alpha,\beta,\mu}$ are linear and positive and e_0, exp_μ and exp_μ^2 form a Chebyshev set, we have that our operators satisfy the conditions of the well-known Theorem of Korovkin, therefore we obtain that $K_n^{\alpha,\beta,\mu} f(x) \to f(x)$ uniformly $[0,1]$, $f \in C[0,1]$. □

Now, we will provide an approximation result for functions belonging to a weighted version of L^p spaces.

In the following, by the space $L_\mu^p([0,1])$ we mean the space of all functions f that satisfy:

$$\|f\|_{p,\mu} = \left\{ \int_0^1 |e^{-\mu x} f(x) dx|^p \right\}^{\frac{1}{p}} < \infty.$$

Also, it can be seen that if $f \in L_\mu^p([0,1])$, then $f \in L^p([0,1])$ and reciprocally.

Theorem 2. *For $f \in L_\mu^p([0,1])$ and $n \in \mathbb{N}$, we have that:*

$$\|K_n^{\alpha,\beta,\mu} f\|_{p,\mu} \le \left(\left(1 + \frac{\beta}{n+1}\right) \frac{e^{\frac{\mu}{n+\beta+1}} - 1}{\frac{\mu}{n+\beta+1}} \right)^{\frac{1}{p}} \|f\|_{p,\mu}, \tag{16}$$

and consequently,

$$\|K_n^{\alpha,\beta,\mu}f\|_{p,\mu} \le \Theta_{\mu,\beta}\|f\|_{p,\mu}, \tag{17}$$

where $\Theta_{\mu,\beta} = \left(\left(1+\frac{\beta}{2}\right)\frac{e^{\mu}-1}{\mu}\right)^{\frac{1}{p}}$. *Moreover,*

$$\|K_n^{\alpha,\beta,\mu}f - f\|_{p,\mu} \to 0, \quad \text{as } n \to \infty. \tag{18}$$

Proof. First, if we apply Jensen's inequality twice we get:

$$\begin{aligned}\|K_n^{\alpha,\beta,\mu}f\|_{p,\mu}^p &= \int_0^1 \left|(n+\beta+1)\sum_{k=0}^n p_{n,k}(a_{n+1}(x))\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} e^{-\mu t}f(t)dt\right|^p dx\\ &\le \int_0^1 \sum_{k=0}^n p_{n,k}(a_{n+1}(x))\left|(n+\beta+1)\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} e^{-\mu t}f(t)dt\right|^p dx\\ &\le \sum_{k=0}^n \int_0^1 p_{n,k}(a_{n+1}(x))(n+\beta+1)dx\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} e^{-\mu pt}|f(t)|^p dt.\end{aligned}$$

Next, we denote $I_{n,k,\beta} = \int_0^1 p_{n,k}(a_{n+1}(x))(n+\beta+1)dx$. After the change of variable:

$$t = \frac{e^{\frac{\mu x}{n+\beta+1}}-1}{e^{\frac{\mu}{n+\beta+1}}-1} \text{ and } x = \frac{n+\beta+1}{\mu}\ln\left[t(e^{\frac{\mu}{n+\beta+1}}-1)+1\right],$$

we get $dx = \frac{n+\beta+1}{\mu}\frac{e^{\frac{\mu}{n+\beta+1}}-1}{t(e^{\frac{\mu}{n+\beta+1}}-1)+1}dt$, the integral $I_{n,k,\beta}$ becomes:

$$\begin{aligned}I_{n,k,\beta} &= \frac{n+\beta+1}{\mu}\int_0^1 (n+\beta+1)p_{n,k}(t)\frac{e^{\frac{\mu}{n+\beta+1}}-1}{t(e^{\frac{\mu}{n+\beta+1}}-1)+1}dt\\ &\le \frac{e^{\frac{\mu}{n+\beta+1}}-1}{\frac{\mu}{n+\beta+1}}\int_0^1 (n+\beta+1)p_{n,k}(t)dt\\ &= \frac{e^{\frac{\mu}{n+\beta+1}}-1}{\frac{\mu}{n+\beta+1}}\binom{n}{k}(n+\beta+1)B(k+1,n-k+1)\\ &= \frac{e^{\frac{\mu}{n+\beta+1}}-1}{\frac{\mu}{n+\beta+1}}\cdot\frac{n+\beta+1}{n+1},\end{aligned}$$

where $B(x,y)$ is Euler's beta function.

We have:

$$\|K_n^{\alpha,\beta,\mu}f\|_{p,\mu}^{\le}\frac{e^{\frac{\mu}{n+\beta+1}}-1}{\frac{\mu}{n+\beta+1}}\cdot\frac{n+\beta+1}{n+1}\sum_{k=0}^n\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} e^{-\mu pt}|f(t)|^p dt.$$

Since $e^{-\mu pt}|f(t)|^p > 0$, after the summation of the integrals we obtain the inequality below:

$$\sum_{k=0}^n\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} e^{-\mu pt}|f(t)|^p dt \le \int_0^1 e^{-\mu pt}|f(t)|^p dt \le \|f\|_{p,\mu}^p,$$

which implies that:

$$\|K_n^{\alpha,\beta,\mu} f\|_{p,\mu}^p \le \frac{e^{\frac{\mu}{n+\beta+1}}-1}{\frac{\mu}{n+\beta+1}} \cdot \left(1+\frac{\beta}{n+1}\right)\|f\|_{p,\mu}^p.$$

Now, because $\frac{\mu}{n+\beta+1} < \mu$, $n \in \mathbb{N}$, and $u(t) = \frac{e^t-1}{t}$ is an increasing function for $t \in [0,1]$ and $u_n = \frac{\beta}{n+1}$, $n \in \mathbb{N}$, is a decreasing sequence we have that

$$\|K_n^{\alpha,\beta,\mu} f\|_{p,\mu}^p \le \left(1+\frac{\beta}{2}\right)\frac{e^{\mu}-1}{\mu}\|f\|_{p,\mu}^p.$$

Next, in order to prove relation (18), let us take $g \in C([0,1])$ and write:

$$\|K_n^{\alpha,\beta,\mu} f - f\|_{p,\mu} \le \|K_n^{\alpha,\beta,\mu} f - K_n^{\alpha,\beta,\mu} g\|_{p,\mu} + \|K_n^{\alpha,\beta,\mu} g - g\|_{p,\mu} + \|g-f\|_{p,\mu}. \quad (19)$$

Now, let $\varepsilon > 0$ be fixed. From Theorem 1 there exists $n_\varepsilon \in \mathbb{N}$, such that for all $n > n_\varepsilon$, we have that

$$\|K_n^{\alpha,\beta,\mu} g - g\|_{p,\mu} \le \|K_n^{\alpha,\beta,\mu} g - g\|_p \le \|K_n^{\alpha,\beta,\mu} g - g\|_\infty \le \frac{\varepsilon}{2}.$$

Also, because the set $C([0,1])$ is dense in $L_\mu^p([0,1])$ we can choose $g \in C([0,1])$ such that

$$\|f-g\|_{p,\mu} \le \frac{\varepsilon}{2(\Theta_{\mu,\beta}+1)}.$$

Using (17) and the linearity of $K_n^{\alpha,\beta,\mu}$ we have that

$$\|K_n^{\alpha,\beta,\mu} f - K_n^{\alpha,\beta,\mu} g\|_{p,\mu} \le \Theta_{\mu,\beta}\|f-g\|_{p,\mu}.$$

Hence, going back to (19), we obtain:

$$\begin{aligned}\|K_n^{\alpha,\beta,\mu} f - f\|_{p,\mu} &\le \Theta_{\mu,\beta}\|f-g\|_{p,\mu} + \|g-f\|_{p,\mu} + \frac{\varepsilon}{2}\\ &\le (\Theta_{\mu,\beta}+1)\|f-g\|_{p,\mu} + \frac{\varepsilon}{2} \le \varepsilon,\end{aligned}$$

which proves (18). □

Remark 1. *The inequalities in Theorem 2 can be rewritten using norm $\|\cdot\|_p$, and also we have that:*

$$\|K_n^{\alpha,\beta,\mu}\|_p \le e^{\mu}\Theta_{\mu,\beta}. \quad (20)$$

3 Voronovskaya Theorem

In this section we will prove a Voronovskaya type theorem in order to get the rate of approximation by our operators. In what follows, because the Chebyshev set considered is $\{e_0, exp_\mu, exp_\mu^2\}$, we will write our function $f \in C^2([0,1])$ as $f(x) = (f \circ \ln_\mu)(exp_\mu)$, $x \in [0,1]$ where $\ln_\mu(x) = \log_{e^\mu}(x)$ is the inverse of $exp_\mu(x)$. Here, by $C^2([0,1])$ we mean the set of all continuously differentiable functions whose second derivative is also continuous on $[0,1]$.

For such functions, the following Voronovskaya formula holds.

Theorem 3. *Let $f \in C^2[0,1]$. The following limit*

$$\lim_{n\to\infty} n(K_n^{\alpha,\beta,\mu} f - f)(x) = \left[-\frac{1}{2} - \alpha + (1+\beta+\mu)x - \mu x^2\right](\mu f(x) - f'(x)) \quad (21)$$
$$+\frac{x(1-x)}{2}(f''(x) + \mu f(x)),$$

holds uniformly for $x \in [0,1]$.

Proof. Let $f \in C^2[0,1]$ and $f(x) = (f \circ \ln_\mu)(exp_\mu),\ x \in [0,1]$. Let us write the Taylor polynomial with Peano remainder associated to this function:

$$f(t) = (f\circ \ln_\mu)(exp_\mu)(t) = (f\circ \ln_\mu)(e^{\mu x}) + (f\circ \ln_\mu)'(e^{\mu x})[e^{\mu t} - e^{\mu x}]$$
$$+\frac{1}{2}(f\circ \ln_\mu)''(e^{\mu x})[e^{\mu t} - e^{\mu x}]^2 + h_x(t)[e^{\mu t} - e^{\mu x}]^2,$$

where $h_x(t) = h(t-x)$ is a globally continuous function on $[0,1]\times[0,1]$, hence $h_x(t) \to 0$ uniformly as $t \to x$.

Further, we have:

$$f(t) = f(x) + f'(x)\mu^{-1}e^{-\mu x}[e^{\mu t} - e^{\mu x}]$$
$$+\frac{1}{2}e^{-2\mu x}(\mu^{-2}f''(x) - \mu^{-1}f'(x))[e^{\mu t} - e^{\mu x}]^2 + h_x(t)[e^{\mu t} - e^{\mu x}]^2.$$

Now, applying operator $K_n^{\alpha,\beta,\mu}$ to the formula above we obtain:

$$K_n^{\alpha,\beta,\mu} f(t) = f(x)K_n^{\alpha,\beta,\mu}e_0(x) + f'(x)\mu^{-1}e^{-\mu x}[K_n^{\alpha,\beta,\mu}exp_\mu(x) - e^{\mu x}K_n^{\alpha,\beta,\mu}e_0(x)]$$
$$+\frac{1}{2}e^{-2\mu x}(\mu^{-2}f''(x) - \mu^{-1}f'(x))[K_n^{\alpha,\beta,\mu}exp_\mu^2(x) - 2e^{\mu x}K_n^{\alpha,\beta,\mu}exp_\mu(x)$$
$$+e^{2\mu x}K_n^{\alpha,\beta,\mu}e_0(x)] + K_n^{\alpha,\beta,\mu}\left(h_x(exp_\mu - e^{\mu x})^2(x)\right),$$

which, after taking into account that $K_n^{\alpha,\beta,\mu}exp_\mu(x) = exp_\mu(x)$ becomes:

$$K_n^{\alpha,\beta,\mu} f(t) = f(x)K_n^{\alpha,\beta,\mu}e_0(x) + f'(x)\mu^{-1}[1 - K_n^{\alpha,\beta,\mu}e_0(x)]$$
$$+\frac{1}{2}e^{-2\mu x}(\mu^{-2}f''(x) - \mu^{-1}f'(x))[K_n^{\alpha,\beta,\mu}exp_\mu^2(x) - 2e^{2\mu x} + e^{2\mu x}K_n^{\alpha,\beta,\mu}e_0(x)]$$
$$+K_n^{\alpha,\beta,\mu}\left(h_x(exp_\mu - e^{\mu x})^2(x)\right).$$

Therefore, we get:

$$\lim_{n\to\infty} n(K_n^{\alpha,\beta,\mu} f - f)(t) = \quad (22)$$
$$\lim_{n\to\infty} n\left\{f(x)[K_n^{\alpha,\beta,\mu}e_0(x) - 1] + f'(x)\mu^{-1}[1 - K_n^{\alpha,\beta,\mu}e_0(x)]\right.$$
$$+\frac{1}{2}e^{-2\mu x}(\mu^{-2}f''(x) + \mu^{-1}f'(x))[K_n^{\alpha,\beta,\mu}exp_\mu^2(x) - 2e^{2\mu x}$$
$$\left.+e^{2\mu x}K_n^{\alpha,\beta,\mu}e_0(x)] + K_n^{\alpha,\beta,\mu}\left(h_x(exp_\mu - e^{\mu x})^2(x)\right)\right\}.$$

Next, using the software Wolfram Mathematica we obtained:

$$\lim_{n\to\infty} n[K_n^{\alpha,\beta,\mu} e_0(x) - 1] = \mu\left[-\frac{1}{2} - \alpha + (1+\beta+\mu)x - \mu x^2\right],$$

and

$$\lim_{n\to\infty} n[K_n^{\alpha,\beta,\mu} exp_\mu^2(x) - 2e^{2\mu x} + e^{2\mu x} K_n^{\alpha,\beta,\mu} e_0(x)] = x(1-x)\mu^2 e^{2\mu x}.$$

Now, we will prove prove $\lim_{n\to\infty} nK_n^{\alpha,\beta,\mu}\left(h_x(exp_\mu - e^{\mu x})^2(x)\right) = 0$. Using Cauchy - Schwarz inequality we have:

$$nK_n^{\alpha,\beta,\mu}\left(h_x(exp_\mu - e^{\mu x})^2\right) \le \sqrt{K_n^{\alpha,\beta,\mu} h_x^2(x)}\sqrt{n^2 K_n^{\alpha,\beta,\mu}(exp_\mu - e^{\mu x})^4(x)},$$

where, by using the approximation result proved in Theorem 1, we have that $K_n^{\alpha,\beta,\mu} h_x^2(x) \to h_x^2(x) \to 0$. Using Wolfram Mathematica we have that:

$$\begin{aligned}&\lim_{n\to\infty} \sqrt{n^2 K_n^{\alpha,\beta,\mu}(exp_\mu - e^{\mu x})^4(x)}\\ &= \lim_{n\to\infty} n\Big\{K_n^{\alpha,\beta,\mu} exp_\mu^4(x) - 4e^{\mu x} K_n^{\alpha,\beta,\mu} exp_\mu^3(x)\\ &+6e^{2\mu x} K_n^{\alpha,\beta,\mu} exp_\mu^2(x) - 4e^{3\mu x} K_n^{\alpha,\beta,\mu} exp_\mu(x) + e^{4\mu x} K_n^{\alpha,\beta,\mu} e_0(x)\Big\}^{\frac{1}{2}}\\ &= \sqrt{3} e^{2\mu x}\mu^2 x(1-x),\end{aligned}$$

which means that $nK_n^{\alpha,\beta,\mu}\left(h_x(exp_\mu - e^{\mu x})^2\right) \to 0$ uniformly on $[0,1]$, as $n \to \infty$. Returning to (22) we obtain the asymptotic result in (21). □

4 Quantitative estimates

In what follows we will provide some characterizations of the rate of convergence of our operators to functions from $C([0,1])$. The results are obtained in terms of certain K-functionals, first order modulus of continuity and second order modulus of smoothness which will be defined along the way. Some of these results are obtained using the equivalence between K-functionals and the moduli presented.

To obtain the estimates mentioned in the beginning of this section we will need the following auxiliary result.

Lemma 2. *For $y \in [0,1]$ we have that*

$$\sum_{k=0}^{n} p_{n,k}(y)\left|\frac{k+\alpha}{n+\beta+1} - y\right| \le \Omega_{n,\beta}, \tag{23}$$

where $\Omega_{n,\beta} = \frac{\sqrt{(\beta+1)^2 + n/4}}{n+\beta+1}$.

Proof. Using Cauchy-Schwarz inequality we have that:

$$\sum_{k=0}^{n} p_{n,k}(y)\left|\frac{k+\alpha}{n+\beta+1}-y\right| \le \sqrt{\sum_{k=0}^{n} p_{n,k}(y)\left[\left(\frac{k+\alpha}{n+\beta+1}\right)^2-2y\frac{k+\alpha}{n+\beta+1}+y^2\right]}.$$

Now, let

$$\begin{aligned} A(y) &:= \left(\frac{k+\alpha}{n+\beta+1}\right)^2-2y\frac{k+\alpha}{n+\beta+1}+y^2 \\ &= \frac{k(k-1)}{(n+\beta+1)^2}+k\left(\frac{2\alpha+1}{(n+\beta+1)^2}-\frac{2y}{n+\beta+1}\right)+y^2 \\ &\quad -2y\frac{\alpha}{n+\beta+1}+\frac{\alpha^2}{(n+\beta+1)^2}. \end{aligned}$$

Then, we have:

$$\begin{aligned} \sum_{k=0}^{n} p_{n,k}(y)A(y) &= y^2\left(\frac{n(n-1)}{(n+\beta+1)^2}-\frac{2n}{n+\beta+1}+1\right) \\ &\quad +y\left(\frac{(2\alpha+1)n}{(n+\beta+1)^2}-\frac{2\alpha}{n+\beta+1}\right)+\frac{\alpha^2}{(n+\beta+1)^2}, \end{aligned}$$

which, after some computations becomes:

$$\begin{aligned} \sum_{k=0}^{n} p_{n,k}(y)A(y) &= \frac{1}{(n+\beta+1)^2}\{y^2[(\beta+1)^2-n]+y[n-2\alpha(\beta+1)]+\alpha^2\} \\ &= \frac{\left[y((\beta+1)^2-\alpha)\right]^2+ny(1-y)}{n+\beta+1} \\ &\le \frac{(\max\{\alpha,\beta+1-\alpha\})^2+ny(1-y)}{n+\beta+1} \\ &\le \frac{(\beta+1)^2+n/4}{n+\beta+1}, \end{aligned}$$

hence, we obtain (23). □

Now, we can state our first quantitative result which involves the first order modulus of continuity defined as:

$$\omega_1(f,\delta)=\sup\{|f(t)-f(x)|,\ t,x\in[0,1],\ |t-x|<\delta\},\ f\in C([0,1]),\ \delta>0. \quad (24)$$

To this purpose, the following theorem holds.

Theorem 4. *Let $f \in C([0,1])$. Then, for $n \in \mathbb{N}$ we have that:*

$$\begin{aligned} |K_n^{\alpha,\beta,\mu}f(x)-f(x)| &\le |f(x)|\frac{C^1_{\alpha,\beta,\mu}}{n}+e^{\mu}\omega_1(exp_\mu^{-1}f,\tau_n) \quad (25) \\ &+e^{\mu}\omega_1\left(exp_\mu^{-1}f,\frac{1}{\sqrt{n+\beta+1}}\right)\left\{1+\frac{1}{2\sqrt{n+\beta+1}}+\Omega_{n,\beta}\right\}, \end{aligned}$$

where

$$\tau_n = \max_{x\in[0,1]} |a_{n+1}(x) - x|, \tag{26}$$

and $C^1_{\alpha,\beta,\mu}$ *is a constant depending on* α, β, μ.

Proof. First we will prove that sequence τ_n, $n \in \mathbb{N}$, converges to 0.

Note, that functions $|a_{n+1}(x) - x|$ achieve their maximum at point $x_n = \frac{n+\beta+1}{\mu} \ln\left[\frac{n+\beta+1}{\mu}(e^{\frac{\mu}{n+\beta+1}} - 1)\right]$. Hence, if we replace x from (26) by x_n and pass to the limit, then it immediately follows that $\lim_{n\to\infty} \tau_n = 0$.

Now, we can see that:

$$\begin{aligned} K_n^{\alpha,\beta,\mu} f(x) - f(x) &= K_n^{\alpha,\beta,\mu}(f - f(x)e_0)(x) + f(x)(K_n^{\alpha,\beta,\mu} e_0(x) - 1) \\ &= (n+\beta+1)e^{\mu x} \sum_{k=0}^{n} p_{n,k}(a_{n+1}(x)) \int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} [e^{-\mu t} f(t) - e^{-\mu x} f(x)] dt \\ &\quad + f(x)(K_n^{\alpha,\beta,\mu} e_0(x) - 1), \end{aligned} \tag{27}$$

for every fixed $x \in [0,1]$. Hence,

$$\begin{aligned} &|K_n^{\alpha,\beta,\mu} f(x) - f(x)| \\ &\le (n+\beta+1)e^{\mu} \sum_{k=0}^{n} p_{n,k}(a_{n+1}(x)) \int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} \omega_1(exp_\mu^{-1} f, |t-x|) dt \\ &\quad + |f(x)| \cdot |K_n^{\alpha,\beta,\mu} e_0(x) - e_0(x)|. \end{aligned}$$

However,

$$|t - x| \le |t - a_{n+1}(x)| + |a_{n+1}(x) - x| \le |t - a_{n+1}(x)| + \tau_n,$$

therefore, we have

$$\begin{aligned} &|K_n^{\alpha,\beta,\mu} f(x) - f(x)| \\ &\le (n+\beta+1)e^{\mu} \sum_{k=0}^{n} p_{n,k}(a_{n+1}(x)) \int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} \omega_1(exp_\mu^{-1} f, |t - a_{n+1}(x)|) dt \\ &\quad + (n+\beta+1)e^{\mu} \sum_{k=0}^{n} p_{n,k}(a_{n+1}(x)) \int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} \omega_1(exp_\mu^{-1} f, \tau_n) dt \\ &\quad + |f(x)| \cdot |K_n^{\alpha,\beta,\mu} e_0(x) - e_0(x)|. \end{aligned} \tag{28}$$

Now, for simplicity, we denote

$$I_1 := (n+\beta+1)e^{\mu} \sum_{k=0}^{n} p_{n,k}(a_{n+1}(x)) \int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} \omega_1(exp_\mu^{-1} f, |t - a_{n+1}(x)|) dt,$$

and

$$I_2 := (n+\beta+1)e^{\mu} \sum_{k=0}^{n} p_{n,k}(a_{n+1}(x)) \int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} \omega_1(exp_\mu^{-1} f, \tau_n) dt.$$

It is immediate that

$$I_2 \le \omega_1(exp_\mu^{-1} f, \tau_n) e^\mu. \tag{29}$$

Regarding I_1, using the well-known property of the modulus of continuity: $\omega_1(\lambda\delta) \le (1+\lambda)\omega_1(\delta)$, with $\lambda, \delta > 0$, we have:

$$I_1 \le e^\mu \omega_1(exp_\mu^{-1} f, \delta)\left(1 + \frac{n+\beta+1}{\delta}\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x)) \int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} |t - a_{n+1}(x)| dt\right).$$

But,

$$\begin{aligned}
&\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} |t - a_{n+1}(x)| dt \\
&\le \int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} \left(t - \frac{k+\alpha}{n+\beta+1}\right) dt + \int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha}{n+\beta+1}} \left|\frac{k+\alpha}{n+\beta+1} - a_{n+1}(x)\right| dt \\
&= \frac{1}{2(n+\beta+1)^2} + \frac{1}{n+\beta+1}\left|\frac{k+\alpha}{n+\beta+1} - a_{n+1}(x)\right|.
\end{aligned}$$

Now,

$$\begin{aligned}
I_1 &\le e^\mu \omega_1(exp_\mu^{-1} f, \delta)\left[1 + \frac{n+\beta+1}{\delta}\left(\frac{1}{2(n+\beta+1)^2}\right.\right. && (30)\\
&\quad \left.\left. + \frac{1}{n+\beta+1}\sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))\left|\frac{k+\alpha}{n+\beta+1} - a_{n+1}(x)\right|\right)\right] \\
&= e^\mu \omega_1(exp_\mu^{-1} f, \delta)\left[1 + \frac{1}{\delta}\left(\frac{1}{2(n+\beta+1)} + \right.\right. && (31)\\
&\quad \left.\left. + \sum_{k=0}^{n} p_{n,k}(a_{n+1}(x))\left|\frac{k+\alpha}{n+\beta+1} - a_{n+1}(x)\right|\right)\right].
\end{aligned}$$

Setting $\delta = \frac{1}{\sqrt{n+\beta+1}}$ if we go back to (30) and use (23), we obtain:

$$I_1 \le e^\mu \omega_1\left(exp_\mu^{-1} f, \frac{1}{\sqrt{n+\beta+1}}\right)\left\{1 + \frac{1}{2\sqrt{n+\beta+1}} + \Omega_{n,\beta}\right\}. \tag{32}$$

Next, we will need an estimate for $|K_n^{\alpha,\beta,\mu} e_0(x) - e_0(x)|$.

Let us denote $g_n(x) = n[K_n^{\alpha,\beta,\mu} e_0(x) - e_0(x)]$, $x \in [0,1]$ and also $C_{\alpha,\beta,\mu} = \mu \cdot \left(\max_{x\in[0,1]} \left|-\frac{1}{2} - \alpha + (1+\beta+\mu)x - \mu x^2\right| + \frac{1}{8}\right)$. From Theorem 3 we have the uniform limit:

$$\lim_{n\to\infty} g_n(x) = \mu\left[-\frac{1}{2} - \alpha + (1+\beta+\mu)x - \mu x^2 + \frac{x(1-x)}{2}\right], \quad x \in [0,1],$$

therefore

$$\lim_{n\to\infty} \|g_n\|_\infty = C_{\alpha,\beta,\mu},$$

so, there exists a constant $C^1_{\alpha,\beta,\mu} > C_{\alpha,\beta,\mu}$ such that $\|g_n\|_\infty \le C^1_{\alpha,\beta,\mu}$, for any $n \in \mathbb{N}$, hence

$$|K_n^{\alpha,\beta,\mu}e_0(x) - e_0(x)| \le \|K_n^{\alpha,\beta,\mu}e_0 - e_0\|_\infty \le \frac{C^1_{\alpha,\beta,\mu}}{n}. \tag{33}$$

Now, returning to (28), using (29), (32) and (33), we obtain (25). □

Further, we will provide an estimate for the approximation of functions $f \in L^p([0,1])$, $1 \le p < \infty$ using Peetre's K-functional:

$$\mathcal{K}_1(f,\delta)_p = \inf_{g\in C^1[0,1]}\{\|f-g\|_p + \delta\|g'\|_\infty\},\ \delta > 0,\ 1 \le p < \infty. \tag{34}$$

Theorem 5. *Let $f \in L^p([0,1])$, $1 \le p < \infty$. Then:*

$$\|K_n^{\alpha,\beta,\mu}f - f\|_p \le \frac{C^1_{\alpha,\beta,\mu}}{n}\|f\|_\infty + e^\mu(\Theta_{\beta,\mu}+1)\mathcal{K}_1\left(f, \frac{\delta_n^{\alpha,\beta}}{\Theta_{\beta,\mu}+1}\right)_p, \tag{35}$$

for every $n \in \mathbb{N}$ sufficiently large, where $\delta_n^{\alpha,\beta} = \frac{1}{2(n+\beta+1)} + \Omega_{n,\beta} + \tau_n$ and $C^1_{\alpha,\beta,\mu}$ is a constant depending on α, β and μ.

Proof. We have that:

$$\begin{aligned}\|K_n^{\alpha,\beta,\mu}f - f\|_p &\le \|f\|_\infty\|K_n^{\alpha,\beta,\mu}e_0 - e_0\|_p + \|K_n^{\alpha,\beta,\mu}(f - f(\cdot)e_0)\|_p \\ &\le \|f\|_\infty\|K_n^{\alpha,\beta,\mu}e_0 - e_0\|_\infty + \|K_n^{\alpha,\beta,\mu}(f - f(\cdot)e_0)\|_p.\end{aligned} \tag{36}$$

Now, let $g \in C^1([0,1])$. It follows that:

$$\begin{gathered}\|K_n^{\alpha,\beta,\mu}(f - f(\cdot)e_0)\|_p \le \|K_n^{\alpha,\beta,\mu}(f-g)\|_p + \|K_n^{\alpha,\beta,\mu}(g - g(\cdot)e_0)\|_p \\ +\|g-f\|_p\|K_n^{\alpha,\beta,\mu}e_0\|_p \le \|f-g\|_p\|K_n^{\alpha,\beta,\mu}\|_p + \|K_n^{\alpha,\beta,\mu}(g - g(\cdot)e_0)\|_p \\ +\|g-f\|_p\|K_n^{\alpha,\beta,\mu}e_0\|_\infty \le \|f-g\|_p\left(\|K_n^{\alpha,\beta,\mu}\|_p + \|K_n^{\alpha,\beta,\mu}e_0\|_\infty\right) \\ +\|g'\|_\infty\|K_n^{\alpha,\beta,\mu}\left(|e_1 - e_1(\cdot)e_0|\right)\|_p.\end{gathered}$$

Next, from the definition of our operators, we have:

$$\|K_n^{\alpha,\beta,\mu}\|_\infty \le e^\mu. \tag{37}$$

Further, using (37) and (20), we get

$$\|K_n^{\alpha,\beta,\mu}f - f(\cdot)e_0\|_p \le e^\mu(\Theta_{\beta,\mu}+1)\|f-g\|_p + \|g'\|_\infty\|K_n^{\alpha,\beta,\mu}\left(|e_1 - e_1(\cdot)e_0|\right)\|_p,$$

But,

$$\begin{aligned}&K_n^{\alpha,\beta,\mu}\left(|e_1 - e_1(\cdot)e_0|\right) = (n+\beta+1)e^{\mu x}\sum_{k=0}^n p_{n,k}(a_{n+1}(x))\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}} e^{-\mu t}|t-x|dt \\ &\le (n+\beta+1)e^\mu\sum_{k=0}^n p_{n,k}(a_{n+1}(x))\int_{\frac{k+\alpha}{n+\beta+1}}^{\frac{k+\alpha+1}{n+\beta+1}}[|t-a_{n+1}(x)| - |a_{n+1}(x)-x|]dt \\ &\le e^\mu\sum_{k=0}^n p_{n,k}(a_{n+1}(x))\left[\frac{1}{2(n+\beta+1)} + \left|\frac{k+\alpha}{n+\beta+1} - a_{n+1}(x)\right| + \tau_n\right],\end{aligned}$$

so, after using (23) we obtain

$$K_n^{\alpha,\beta,\mu}\left(|e_0 - e_0(\cdot)|\right) \leq e^{\mu}\left[\frac{1}{2(n+\beta+1)} + \Omega_{n,\beta} + \tau_n\right],$$

which means that

$$\begin{aligned}\|K_n^{\alpha,\beta,\mu} f - f(\cdot)e_0\|_p &\leq e^{\mu}(\Theta_{\beta,\mu}+1)\|f-g\|_p \\ &+\|g'\|_\infty e^{\mu}\left[\frac{1}{2(n+\beta+1)} + \Omega_{n,\beta} + \tau_n\right].\end{aligned} \tag{38}$$

Going back to (36) and using (33) and (38) we get:

$$\begin{aligned}\|K_n^{\alpha,\beta,\mu} f - f\|_p &\leq \frac{C^1_{\alpha,\beta,\mu}}{n}\|f\|_\infty + e^{\mu}(\Theta_{\beta,\mu}+1)\|g-f\|_p \\ &+e^{\mu}\|g'\|_\infty\left[\frac{1}{2(n+\beta+1)} + \Omega_{n,\beta} + \tau_n\right].\end{aligned}$$

In order to simplify our notation let us denote $\delta_n^{\alpha,\beta} = \frac{1}{2(n+\beta+1)} + \Omega_{n,\beta} + \tau_n$. Then:

$$\|K_n^{\alpha,\beta,\mu} f - f\|_p \leq \frac{C^1_{\alpha,\beta,\mu}}{n}\|f\|_\infty + e^{\mu}(\Theta_{\beta,\mu}+1)\left(\|f-g\|_p + \frac{\delta_n^{\alpha,\beta}}{\Theta_{\beta,\mu}+1}\|g'\|_\infty\right),$$

where passing to the infimum yields (35). □

Now, to proceed with our last result we will need the following definitions of K-functionals and of the second order smoothness modulus ω_2:

$$\mathcal{K}_j(f,\delta) = \inf_{g\in C^j[0,1]}\{\|f-g\|_\infty + \delta^j\|g^{(j)}\|_\infty\},\ f\in C([0,1]),\ \delta>0,\ j=1,2,$$

and

$$\omega_2(f,\delta) = \sup_{h\in[0,\delta]}\ \sup_{x\in[0,1-\frac{h}{2}]}\left|\Delta_h^2(f,x)\right|,$$

where $\Delta_h^2(f,x) = f(x) - 2f(x+h) + f(x+2h)$.

It is well-known that between these K-functionals and ω_1 and ω_2 the following relations exist (see [10]): $\mathcal{K}_j(f,\delta) \leq C_j\omega_j(f,\delta),\ f\in C([0,1]),\ \delta>0,\ j=1,2,$ where C_j are constants depending only on j.

Theorem 6. *Let $f \in C([0,1])$. Then:*

$$\|K_n^{\alpha,\beta,\mu} f - f\|_\infty \leq 2\frac{C^1_{\alpha,\beta,\mu}}{n}\|f\|_\infty + C_1^*\omega_1\left(f,\frac{1}{n}\right) + C_2^*\omega_2\left(f,\frac{1}{\sqrt{n}}\right), \tag{39}$$

for every $n\in\mathbb{N}$ sufficiently large, where C_1^* and C_2^* are constants depending on α, β, μ.

Proof. We have that:

$$\|K_n^{\alpha,\beta,\mu}f - f\|_\infty \leq \|f\|_\infty \|K_n^{\alpha,\beta,\mu}e_0 - e_0\|_\infty + \|K_n^{\alpha,\beta,\mu}(f - f(\cdot)e_0)\|_\infty. \tag{40}$$

Now, let $g \in C^2([0,1])$. It follows that:

$$\begin{aligned} &\|K_n^{\alpha,\beta,\mu}(f - f(\cdot)e_0)\|_\infty \\ &\leq \|K_n^{\alpha,\beta,\mu}(f-g)\|_\infty + \|K_n^{\alpha,\beta,\mu}(g - g(\cdot)e_0)\|_\infty + \|g-f\|_\infty \|K_n^{\alpha,\beta,\mu}e_0\|_\infty \\ &\leq \|f-g\|_\infty \left(\|K_n^{\alpha,\beta,\mu}\|_\infty + \|K_n^{\alpha,\beta,\mu}e_0\|_\infty\right) + \|K_n^{\alpha,\beta,\mu}(g - g(\cdot)e_0)\|_\infty. \end{aligned} \tag{41}$$

Next, to evaluate the last term in (41) we use Taylor's polynomial with integral remainder up to order 2 in a point $x \in [0,1]$ of the function g. For $t \in [0,1]$ we have:

$$\begin{aligned} g(t) &= (g \circ \ln_\mu)(e^{\mu t}) \\ &= (g \circ \ln_\mu)(e^{\mu x}) + (g \circ \ln_\mu)'(e^{\mu x})[e^{\mu t} - e^{\mu x}] + \int_{e^{\mu x}}^{e^{\mu t}} (g \circ \ln_\mu)''(u)(e^{\mu t} - u)du \\ &= g(x) + \frac{e^{-\mu x}}{\mu} g'(x)[e^{\mu t} - e^{\mu x}] + \int_{e^{\mu x}}^{e^{\mu t}} \frac{g''(\ln_\mu u) - \mu g'(\ln_\mu u)}{\mu^2} \cdot \frac{e^{\mu t} - u}{u^2} du. \end{aligned}$$

First, let us denote $R(t,x) = \int_{e^{\mu x}}^{e^{\mu t}} \frac{g''(\ln_\mu u) - \mu g'(\ln_\mu u)}{\mu^2} \cdot \frac{e^{\mu t} - u}{u^2} du$. Then by applying $K_n^{\alpha,\beta,\mu}$ to the identity above, we get:

$$\begin{aligned} K_n^{\alpha,\beta,\mu}(g - g(x)e_0)(x) &= g(x)\left(K_n^{\alpha,\beta,\mu}e_0(x) - e_0(x)\right) \\ &+ \frac{e^{-\mu x}}{\mu} g'(x)[K_n^{\alpha,\beta,\mu} exp_\mu(x) - e^{\mu x} K_n^{\alpha,\beta,\mu} e_0](x) + K_n^{\alpha,\beta,\mu} R(\cdot, x)(x), \end{aligned}$$

so

$$\begin{aligned} |K_n^{\alpha,\beta,\mu}(g - g(x)e_0)(x)| &\leq |g(x)| \cdot \left|\left(K_n^{\alpha,\beta,\mu}e_0(x) - e_0(x)\right)\right| \\ &+ \frac{e^{-\mu x}}{\mu}|g'(x)| \cdot |K_n^{\alpha,\beta,\mu} exp_\mu - e^{\mu x} K_n^{\alpha,\beta,\mu} e_0(x)| + K_n^{\alpha,\beta,\mu}(|R(\cdot, x)|)(x) \\ &=: T_1 + T_2 + T_3. \end{aligned}$$

Here, from (33) we have that

$$T_1 \leq \frac{C^1_{\alpha,\beta,\mu}}{n} \|g\|_\infty. \tag{42}$$

Next, regarding T_2 we have:

$$|K_n^{\alpha,\beta,\mu} exp_\mu(x) - e^{\mu x} K_n^{\alpha,\beta,\mu} e_0(x)| \leq |K_n^{\alpha,\beta,\mu} exp_\mu(x) - e^{\mu x}| + e^{\mu x}|K_n^{\alpha,\beta,\mu} e_0(x) - e_0(x)|.$$

Proceeding in a similar manner as we did in order to obtain (33) we can see there exists a constant $C^2_{\alpha,\beta,\mu}$ such that:

$$|K_n^{\alpha,\beta,\mu} exp_\mu(x) - e^{\mu x}| \leq \frac{C^2_{\alpha,\beta,\mu}}{n}. \tag{43}$$

Now, using (42) and (43) we have that

$$T_2 \le \frac{C^3_{\alpha,\beta,\mu}}{n}\|g'\|_\infty, \tag{44}$$

where $C^3_{\alpha,\beta,\mu}$ is a constant depending on α, β, μ.

Now, going further, we can see that:

$$T_3 \le \frac{1}{2}\left(\frac{\|g''\|_\infty}{\mu^2} + \frac{\|g'\|_\infty}{\mu}\right)|K_n^{\alpha,\beta,\mu} exp^2_\mu(x) - 2e^{2\mu x} + e^{2\mu x}K_n^{\alpha,\beta,\mu}e_0(x)|.$$

In order to estimate T_3 we have

$$\begin{gathered}|K_n^{\alpha,\beta,\mu} exp^2_\mu(x) - 2exp^2_\mu(x) + exp^2_\mu(x)K_n^{\alpha,\beta,\mu}e_0(x)| \\ \le |K_n^{\alpha,\beta,\mu} exp^2_\mu(x) - exp^2_\mu(x)| + exp^2_\mu(x)|K_n^{\alpha,\beta,\mu}e_0(x) - e_0(x)|.\end{gathered}$$

Now, using the uniform limit from Theorem 7, proceeding in a similar fashion as we did for the estimate in (33), we can find a constant $C^4_{\alpha,\beta,\mu} > 0$ depending on α, β and μ such that:

$$\|K_n^{\alpha,\beta,\mu} exp^2_\mu - exp^2_\mu\|_\infty \le \frac{C^4_{\alpha,\beta,\mu}}{n},$$

so, we have that

$$|K_n^{\alpha,\beta,\mu} exp^2_\mu(x) - 2e^{2\mu x} + e^{2\mu x}K_n^{\alpha,\beta,\mu}e_0(x)| \le \frac{1}{n}(C^4_{\alpha,\beta,\mu} + e^{2\mu}C^1_{\alpha,\beta,\mu}), \tag{45}$$

which implies

$$T_3 \le \frac{1}{2n}(C^4_{\alpha,\beta,\mu} + e^{2\mu}C^1_{\alpha,\beta,\mu})\left(\frac{\|g''\|_\infty}{\mu^2} + \frac{\|g'\|_\infty}{\mu}\right),$$

therefore, there exists a constant $C^5_{\alpha,\beta,\mu}$ depending on α, β and μ such that:

$$T_3 \le \frac{1}{n}C^5_{\alpha,\beta,\mu}(\|g''\|_\infty + \|g'\|_\infty). \tag{46}$$

Hence,

$$\begin{gathered}\|K_n^{\alpha,\beta,\mu}(g - g(x)e_0)\|_\infty \\ \le \frac{1}{n}\left[C^1_{\alpha,\beta,\mu}\|g\|_\infty + \left(C^3_{\alpha,\beta,\mu} + C^5_{\alpha,\beta,\mu}\right)\|g'\|_\infty + C^5_{\alpha,\beta,\mu}\|g''\|_\infty\right].\end{gathered}$$

Therefore,

$$\begin{gathered}\|K_n^{\alpha,\beta,\mu}(f - f(x)e_0)\|_p \le \Theta_{\mu,\beta}(e^\mu + 1)\|f - g\|_\infty \\ + \frac{1}{n}\left[C^1_{\alpha,\beta,\mu}\|g\|_\infty + \left(C^3_{\alpha,\beta,\mu} + C^5_{\alpha,\beta,\mu}\right)\|g'\|_\infty + C^5_{\alpha,\beta,\mu}\|g''\|_\infty\right].\end{gathered} \tag{47}$$

Further, by using the inequality $\|g\|_\infty \le \|f\|_\infty + \|f-g\|_\infty$ in the relation from above, we get

$$\begin{aligned} &\|K_n^{\alpha,\beta,\mu}(f - f(x)e_0)\|_\infty \\ &\le \frac{C^1_{\alpha,\beta,\mu}}{n}\|f\|_\infty + \left(\Theta_{\mu,\beta}(e^\mu+1) + \frac{C^1_{\alpha,\beta,\mu}}{n}\right)\|f-g\|_\infty \\ &+\frac{1}{n}\left[\left(C^3_{\alpha,\beta,\mu} + C^5_{\alpha,\beta,\mu}\right)\|g'\|_\infty + C^5_{\alpha,\beta,\mu}\|g''\|_\infty\right]. \end{aligned}$$

Now, using (40) and (33), we obtain:

$$\begin{aligned} \|K_n^{\alpha,\beta,\mu}f - f\|_\infty &\le 2\frac{C^1_{\alpha,\beta,\mu}}{n}\|f\|_\infty + \left(\Theta_{\mu,\beta}(e^\mu+1) + \frac{C^1_{\alpha,\beta,\mu}}{n}\right)\|f-g\|_\infty \\ &+\frac{1}{n}\left[\left(C^3_{\alpha,\beta,\mu} + C^5_{\alpha,\beta,\mu}\right)\|g'\|_\infty + C^5_{\alpha,\beta,\mu}\|g''\|_\infty\right], \end{aligned}$$

where, for the simplicity of notation, we denote $\Gamma_{\alpha,\beta,\mu} = \left(\Theta_{\mu,\beta}(e^\mu+1) + \frac{C^1_{\alpha,\beta,\mu}}{n}\right)$, $K_1 = C^3_{\alpha,\beta,\mu} + C^5_{\alpha,\beta,\mu}$ and $K_2 = C^5_{\alpha,\beta,\mu}$.

Further, we have

$$\begin{aligned} &\|K_n^{\alpha,\beta,\mu}f - f\|_\infty \\ &\le 2\frac{C^1_{\alpha,\beta,\mu}}{n}\|f\|_\infty + \Gamma_{\alpha,\beta,\mu}\left(\|f-g\|_\infty + \frac{1}{n}\cdot\frac{K_1}{\Gamma_{\alpha,\beta,\mu}}\|g'\|_\infty + \frac{1}{n}\cdot\frac{K_2}{\Gamma_{\alpha,\beta,\mu}}\|g''\|_\infty\right) \\ &= 2\frac{C^1_{\alpha,\beta,\mu}}{n}\|f\|_\infty + \frac{\Gamma_{\alpha,\beta,\mu}K_1}{K_1+K_2}\left(\|f-g\|_\infty + \frac{1}{n}\cdot\frac{K_1+K_2}{\Gamma_{\alpha,\beta,\mu}}\|g'\|_\infty\right) \\ &+\frac{\Gamma_{\alpha,\beta,\mu}K_2}{K_1+K_2}\left(\|f-g\|_\infty + \frac{1}{n}\cdot\frac{K_1+K_2}{\Gamma_{\alpha,\beta,\mu}}\|g''\|_\infty\right) \\ &\le 2\frac{C^1_{\alpha,\beta,\mu}}{n}\|f\|_\infty + \frac{\Gamma_{\alpha,\beta,\mu}K_1}{K_1+K_2}\mathcal{K}_1\left(f, \frac{K_1+K_2}{\Gamma_{\alpha,\beta,\mu}}\cdot\frac{1}{n}\right) \\ &+\frac{\Gamma_{\alpha,\beta,\mu}K_2}{K_1+K_2}\mathcal{K}_2\left(f, \sqrt{\frac{K_1+K_2}{\Gamma_{\alpha,\beta,\mu}}\cdot\frac{1}{n}}\right) \\ &\le 2\frac{C^1_{\alpha,\beta,\mu}}{n}\|f\|_\infty + \frac{\Gamma_{\alpha,\beta,\mu}K_1}{K_1+K_2}C_1\omega_1\left(f, \frac{K_1+K_2}{\Gamma_{\alpha,\beta,\mu}}\cdot\frac{1}{n}\right) \\ &+\frac{\Gamma_{\alpha,\beta,\mu}K_2}{K_1+K_2}C_2\omega_2\left(f, \sqrt{\frac{K_1+K_2}{\Gamma_{\alpha,\beta,\mu}}\cdot\frac{1}{n}}\right). \end{aligned}$$

Now, using the inequalities:

$$\omega_1\left(f, \frac{K_1+K_2}{\Gamma_{\alpha,\beta,\mu}}\cdot\frac{1}{n}\right) \le \left(1 + \frac{K_1+K_2}{\Gamma_{\alpha,\beta,\mu}}\right)\omega_1\left(f, \frac{1}{n}\right),$$

and:

$$\omega_2\left(f, \sqrt{\frac{K_1+K_2}{\Gamma_{\alpha,\beta,\mu}}\cdot\frac{1}{n}}\right) \le \left(1 + \sqrt{\frac{K_1+K_2}{\Gamma_{\alpha,\beta,\mu}}}\right)^2\omega_2\left(f, \frac{1}{\sqrt{n}}\right),$$

we get:

$$\|K_n^{\alpha,\beta,\mu} f - f\|_\infty \leq 2\frac{C^1_{\alpha,\beta,\mu}}{n}\|f\|_\infty + C_1^*\omega_1\left(f, \frac{1}{n}\right) + C_2^*\omega_2\left(f, \frac{1}{\sqrt{n}}\right),$$

where $C_1^* = \left(\frac{\Gamma_{\alpha,\beta,\mu}K_1}{K_1+K_2} + K_1\right)C_1$ and $C_2^* = \frac{\Gamma_{\alpha,\beta,\mu}K_2}{K_1+K_2}\left(1 + \sqrt{\frac{K_1+K_2}{\Gamma_{\alpha,\beta,\mu}}}\right)^2 C_2$, which is our result. □

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