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COEFFICIENT BOUNDS FOR A FAMILY OF BI-UNIVALENT FUNCTIONS INVOLVING TELEPHONE NUMBERS

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Abstract

In the present paper, we introduce and investigate a new family $\mathbb{F}_{\Sigma}(\gamma, \lambda, \delta; \vartheta)$ of holomorphic and bi-univalent functions by using the generalized telephone numbers which defined in the open unit disk Λ. We find upper bounds for the initial Taylor-Maclaurin coefficients and Fekete-Szegö inequality for functions in this family. We also indicate certain special cases and consequences for our results.

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1 Introduction

Indicate by $\mathcal{A}$ the family of holomorphic functions in the open unit disk $\Lambda = \{\xi \in \mathbb{C} : |\xi| < 1\}$, of the form

$$f(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n. \tag{1}$$

We denote by S the subfamily of $\mathcal{A}$ consisting of functions which are also univalent in Λ.

We say that $f \in S$ is called starlike of order $\gamma (0 \leqq \gamma < 1)$ if

$$\Re\left(\frac{\xi f'(\xi)}{f(\xi)}\right) > \gamma, \qquad (\xi \in \Lambda)$$

and a function $f \in S$ is called convex of order $\gamma (0 \leqq \gamma < 1)$ if

$$\Re\left(\frac{\xi f''(\xi)}{f'(\xi)} + 1\right) > \gamma, \qquad (\xi \in \Lambda).$$

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We know that $\mathcal{S}^*(\gamma)$ and $\mathcal{C}(\gamma)$ are the families of functions that starlike of order γ and convex of order γ in the unit disk Λ, respectively.

The image of Λ under each univalent function $f \in \mathcal{A}$ contain a disk of radius $\frac{1}{4}$, see the Koebe one-quarter theorem [13] and each function $f \in S$ has an inverse f^{-1} defined by $f^{-1}(f(\xi)) = \xi$ and

$$f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f), r_0(f) \geqq \frac{1}{4}\right)$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots .$$

A say that $f \in \mathcal{A}$ is named bi-univalent function in Λ if both f and f^{-1} are univalent functions in Λ. The family of all bi-univalent functions in Λ denoted by Σ.

Very large number of works related to the bi-univalent functions have been presented in the papers (see [1, 2, 4, 5, 7, 8, 10, 15, 16, 17, 18, 19, 20, 21, 25, 27, 28, 29, 30]). We recall some examples of functions in the family Σ, from the work of Srivastava et al. [26],

$$\frac{\xi}{1-\xi}, \quad -\log(1-\xi) \qquad \text{and} \qquad \frac{1}{2}\log\left(\frac{1+\xi}{1-\xi}\right).$$

The Fekete-Szegö problem $\left|a_3 - \eta a_2^2\right|$ for $f \in S$ is well known for its rich history in the field of Geometric Function Theory and its origin was in the disproof by Fekete and Szegö (see [14]) of the Littlewood-Paley conjecture that the coefficients of odd univalent functions are bounded by unity. Many authors obtained Fekete-Szegö inequalities for different families of functions. This topic has become of considerable interest among researchers in Geometric Function Theory (see, for example, [1, 3, 9, 12, 32, 22, 23, 24, 26, 33]).

The conventional telephone numbers are quantified by the recurrence relation

$$T(k) = T(k-1) + (k-1)T(k-2) \qquad k \geqq 2,$$

with initial conditions

$$T(0) = T(1) = 1.$$

For integers $k \geqq 0$ and $\tau \geqq 1$, Wloch and Wolowiec-Musial [31] defined generalized telephone numbers $T(\tau, k)$ by the recurrence relation:

$$T(\tau, k) = \tau T(\tau, k-1) + (k-1)T(\tau, k-2),$$

with initial conditions

$$T(\tau, 0) = 1 \quad \text{and} \quad T(\tau, 1) = \tau.$$

Recently, Bednarz and Wolowiec-Musial [6] considered accessible generalization of telephone numbers by

$$T\tau(k) = T\tau(k-1) + \tau(k-1)T\tau(k-2),$$

where $k \geqq 2$ and $\tau \geqq 1$ with initial conditions

$$T\tau(0) = T\tau(1) = 1.$$

Very recently, Deniz [12] investigated the exponential generating function for $T\tau(k)$ as follows:

$$e^{\left(r+\tau\frac{r}{2}\right)} = \sum_{k=0}^{\infty} T\tau(k)\frac{r^k}{k!}.$$

Clearly, when $\tau = 1$, we have $T\tau(k) \equiv T(k)$ classical telephone numbers.

Now, we study the function

$$\vartheta(\xi) = e^{\left(\xi+\tau\frac{\xi^2}{2}\right)} = 1 + \xi + \frac{\xi^2}{2} + \frac{1+\tau}{6}\xi^3 + \frac{1+3\tau}{24}\xi^4 + \cdots \tag{2}$$

with its domain of definition as the open unit disk Λ. We note that $\vartheta(\xi)$ is holomorphic function in Λ, with positive real part, where $\vartheta(0) = 1, \quad \vartheta'(0) > 0$ and where ϑ maps Λ onto a region starlike with respect to 1 and symmetric with respect to the real axis.

Lemma 1. *([13], p.41) Let $h \in P$ be given by the following series:*

$$h(\xi) = 1 + c_1\xi + c_2\xi^2 + \cdots \quad , \quad \textit{where} \quad \xi \in \Lambda.$$

The sharp estimate is given by

$$|c_n| \leqq 2 \quad , \textit{where} \quad n \in \mathbb{N}$$

holds true.

2 Main results

We now provide, using the generalized telephone numbers, the following subfamily of holomorphic and bi-univalent functions.

Definition 1. *The family $\mathbb{F}_\Sigma(\gamma, \lambda, \delta; \vartheta)$ contains all the functions $f \in \Sigma$ if it fulfills the next subordinations:*

$$\left(\frac{\xi f'(\xi)}{f(\xi)}\right)^{\gamma}\left[(1-\delta)\frac{\xi f'(\xi)}{f(\xi)} + \delta\left(1 + \frac{\xi f''(\xi)}{f'(\xi)}\right)\right]^{\lambda} \prec e^{\left(\xi+\tau\frac{\xi^2}{2}\right)} =: \vartheta(\xi)$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\gamma}\left[(1-\delta)\frac{wg'(w)}{g(w)} + \delta\left(1 + \frac{wg''(w)}{g'(w)}\right)\right]^{\lambda} \prec e^{\left(w+\tau\frac{w^2}{2}\right)} =: \vartheta(w),$$

where $0 \leqq \gamma \leqq 1$, $0 \leqq \lambda \leqq 1$, $0 \leqq \delta \leqq 1$, $1 \leqq \tau < 2$ and $g(w) = f^{-1}(w)$.

Remark 1. *1. If we take $\lambda = 0$ and $\gamma = 1$ in Definition 1, the family $\mathbb{F}_\Sigma(\gamma, \lambda, \delta; \vartheta)$ reduce to the family $\mathcal{S}^*_\Sigma(\vartheta)$ which was studied recently by Cotîrlă and Wanas (see [11]).*

2. If we take $\gamma = 0$ and $\lambda = \delta = 1$ in Definition 1, the family $\mathbb{F}_\Sigma(\gamma, \lambda, \delta; \vartheta)$ reduce to the family $\mathcal{C}_\Sigma(\vartheta)$ which was introduced recently by Cotîrlă and Wanas (see [11]).

Theorem 1. *If f given by (1) is in the family $\mathbb{F}_\Sigma(\gamma, \lambda, \delta; \vartheta)$ $(0 \leqq \gamma \leqq 1, 0 \leqq \lambda \leqq 1, 0 \leqq \delta \leqq 1)$, then*

$$|a_2| \leqq \min\left\{ \frac{1}{\gamma + \lambda(\delta+1)}, \sqrt{\frac{2}{\left|\left[\gamma(\gamma+1) + \lambda(\delta+1)\left(2(\gamma+1) + (\lambda-1)(\delta+1)\right)\right] + (1-\tau)\left(\gamma + \lambda(\delta+1)\right)^2\right|}} \right\}$$

and

$$|a_3| \leqq \min\left\{ \frac{1}{4\left(\gamma + \lambda(2\delta+1)\right)} + \frac{\tau+1}{\gamma(\gamma+1) + \lambda(\delta+1)\left(2(\gamma+1) + (\lambda-1)(\delta+1)\right)}, \frac{1}{\left(\gamma + \lambda(\delta+1)\right)^2} + \frac{1}{2\left(\gamma + \lambda(2\delta+1)\right)} \right\}.$$

Proof. Let $f \in \mathbb{F}_\Sigma(\gamma, \lambda, \delta; \vartheta)$ and $f^{-1} = g$. There are the functions $\Phi, \Psi : \Lambda \longrightarrow \Lambda$ holomorphic, with $\Phi(0) = \Psi(0) = 0$, fulfills the following conditions:

$$\left(\frac{\xi f'(\xi)}{f(\xi)}\right)^\gamma \left[(1-\delta)\frac{\xi f'(\xi)}{f(\xi)} + \delta\left(1 + \frac{\xi f''(\xi)}{f'(\xi)}\right)\right]^\lambda = \vartheta(\Phi(\xi)), \quad \xi \in \Lambda \tag{3}$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^\gamma \left[(1-\delta)\frac{wg'(w)}{g(w)} + \delta\left(1 + \frac{wg''(w)}{g'(w)}\right)\right]^\lambda = \vartheta(\Psi(w)), \quad w \in \Lambda. \tag{4}$$

Define the functions x and y by

$$x(\xi) = \frac{1 + \Phi(\xi)}{1 - \Phi(\xi)} = 1 + x_1\xi + x_2\xi^2 + \cdots$$

and

$$y(\xi) = \frac{1 + \Psi(\xi)}{1 - \Psi(\xi)} = 1 + y_1\xi + y_2\xi^2 + \cdots .$$

It follows that x, y are analytic functions in Λ, where $x(0) = 1 = y(0)$. Then, we get $\Phi, \Psi : \Lambda \longrightarrow \Lambda$, where x and y are the functions with a positive real part in

Λ.
But, we have

$$\Phi(\xi) = -\frac{1-x(\xi)}{x(\xi)+1} = \frac{1}{2}\left[x_1\xi + \left(x_2 - \frac{x_1^2}{2}\right)\xi^2\right] + \cdots \quad ,\xi \in \Lambda \tag{5}$$

and

$$\Psi(\xi) = -\frac{1-y(\xi)}{y(\xi)+1} = \frac{1}{2}\left[y_1\xi + \left(y_2 - \frac{y_1^2}{2}\right)\xi^2\right] + \cdots \quad ,\xi \in \Lambda. \tag{6}$$

By substituting (5) and (6) into (3) and (4) and applying (2), we get

$$\begin{aligned}&\left(\frac{\xi f'(\xi)}{f(\xi)}\right)^{\gamma}\left[(1-\delta)\frac{\xi f'(\xi)}{f(\xi)} + \delta\left(1 + \frac{\xi f''(\xi)}{f'(\xi)}\right)\right]^{\lambda}\\ &= \vartheta\left(\Phi(\xi)\right) = e^{\left(\frac{x(\xi)-1}{x(\xi)+1} + \tau\frac{\left(\frac{x(\xi)-1}{x(\xi)+1}\right)^2}{2}\right)} = 1 + \frac{1}{2}x_1\xi + \left[\frac{x_2}{2} + \frac{(\tau-1)x_1^2}{8}\right]\xi^2 + \cdots\end{aligned} \tag{7}$$

and

$$\begin{aligned}&\left(\frac{wg'(w)}{g(w)}\right)^{\gamma}\left[(1-\delta)\frac{wg'(w)}{g(w)} + \delta\left(1 + \frac{wg''(w)}{g'(w)}\right)\right]^{\lambda}\\ &= \vartheta\left(\Psi(w)\right) = e^{\left(\frac{y(w)-1}{1+y(w)} + \tau\frac{\left(\frac{y(w)-1}{y(w)+1}\right)^2}{2}\right)} = 1 + \frac{1}{2}y_1 w + \left[\frac{y_2}{2} + \frac{(\tau-1)y_1^2}{8}\right]w^2 + \cdots\end{aligned} \tag{8}$$

Equating the coefficients in (7) and (8), yields

$$\left(\gamma + \lambda(\delta+1)\right)a_2 = \frac{1}{2}x_1, \tag{9}$$

$$\begin{aligned}&\frac{1}{2}[\gamma(\gamma-1) + \lambda(\delta+1)(2\gamma + (\lambda-1)(\delta+1)) - 2(\gamma + \lambda(3\delta+1))]a_2^2\\ &+2(\gamma + \lambda(2\delta+1))a_3 = \frac{x_2}{2} + \frac{(\tau-1)x_1^2}{8},\end{aligned} \tag{10}$$

$$-\left(\gamma + \lambda(\delta+1)\right)a_2 = \frac{1}{2}y_1 \tag{11}$$

and

$$\begin{aligned}&\frac{1}{2}\left[\gamma(\gamma-1) + \lambda(\delta+1)\left(2\gamma + (\lambda-1)(\delta+1)\right) - 2\left(\gamma + \lambda(3\delta+1)\right)\right]a_2^2\\ &+2\left(\gamma + \lambda(2\delta+1)\right)\left(2a_2^2 - a_3\right) = \frac{y_2}{2} + \frac{(\tau-1)y_1^2}{8}.\end{aligned} \tag{12}$$

From (9) and (11), we have

$$x_1 = -y_1 \tag{13}$$

and

$$2\left(\gamma+\lambda(\delta+1)\right)^2 a_2^2=\frac{1}{4}(x_1^2+y_1^2). \tag{14}$$

If we add (10) to (12), we obtain

$$\left[\gamma(\gamma+1)+\lambda(\delta+1)\left(2(\gamma+1)+(\lambda-1)(\delta+1)\right)\right]a_2^2=\frac{1}{2}(x_2+y_2)+\frac{1}{8}(\tau-1)(x_1^2+y_1^2). \tag{15}$$

Substituting from (14) the value of $x_1^2+y_1^2$ in the relation (15), we get

$$a_2^2=\frac{x_2+y_2}{2\left[\left[\gamma(\gamma+1)+\lambda(\delta+1)\left(2(\gamma+1)+(\lambda-1)(\delta+1)\right)\right]+(1-\tau)\left(\gamma+\lambda(\delta+1)\right)^2\right]}. \tag{16}$$

Applying Lemma 1 for the coefficients x_1, x_2, y_1, y_2 in (14) and (16), we get

$$|a_2|\leqq\frac{1}{\gamma+\lambda(\delta+1)}$$

and

$$|a_2|\leqq\sqrt{\frac{2}{\left|\left[\gamma(\gamma+1)+\lambda(\delta+1)\left(2(\gamma+1)+(\lambda-1)(\delta+1)\right)\right]+(1-\tau)\left(\gamma+\lambda(\delta+1)\right)^2\right|}}.$$

In order to find the bound on $|a_3|$, from (10) we subtract (12) and applying (13), we get $x_1^2=y_1^2$, hence

$$4\left(\gamma+\lambda(2\delta+1)\right)(a_3-a_2^2)=\frac{1}{2}(x_2-y_2), \tag{17}$$

then by substituting of the value of a_2^2 from (14) into (17), we obtain

$$a_3=\frac{x_1^2+y_1^2}{8\left(\gamma+\lambda(\delta+1)\right)^2}+\frac{x_2-y_2}{8\left(\gamma+\lambda(2\delta+1)\right)}.$$

So we have

$$|a_3|\leqq\frac{1}{\left(\gamma+\lambda(\delta+1)\right)^2}+\frac{1}{2\left(\gamma+\lambda(2\delta+1)\right)}.$$

Also, substituting the value of a_2^2 from (15) into (17), we get

$$a_3=\frac{x_2-y_2}{8\left(\gamma+\lambda(2\delta+1)\right)}+\frac{x_2+y_2}{2\left[\gamma(\gamma+1)+\lambda(\delta+1)\left(2(\gamma+1)+(\lambda-1)(\delta+1)\right)\right]}+\frac{(\tau-1)(x_1^2+y_1^2)}{8\left[\gamma(\gamma+1)+\lambda(\delta+1)\left(2(\gamma+1)+(\lambda-1)(\delta+1)\right)\right]}$$

and we have

$$|a_3|\leqq\frac{1}{4\left(\gamma+\lambda(2\delta+1)\right)}+\frac{\tau+1}{\gamma(\gamma+1)+\lambda(\delta+1)\left(2(\gamma+1)+(\lambda-1)(\delta+1)\right)}.$$

□

When $\lambda = 0$ and $\gamma = 1$, the Theorem 1 reduced to the corresponding results of Cotîrlă and Wanas (see [11]).

Corollary 1. *[11] If f given by* (1) *is in the family $S^*_\Sigma(\vartheta)$, then*

$$|a_2| \leqq \min\left\{1, \sqrt{\frac{2}{|3-\tau|}}\right\}$$

and

$$|a_3| \leqq \min\left\{\frac{2+\tau}{2}, \frac{3}{2}\right\}.$$

If we put $\gamma = 0$ and $\lambda = \delta = 1$ in Theorem 1, the results reduced to the corresponding results of Cotîrlă and Wanas (see [11]).

Corollary 2. *[11] Let f given by* (1) *be in the family $C_\Sigma(\vartheta)$. Then*

$$|a_2| \leqq \min\left\{\frac{1}{4}, \sqrt{\frac{1}{2\,|2-\tau|}}\right\}$$

and

$$|a_3| \leqq \min\left\{\frac{3\tau+5}{12}, \frac{5}{12}\right\}.$$

In the next theorem, We provide the Fekete-Szegö problem for the family $\mathcal{F}_\Sigma(\gamma, \lambda, \delta; \vartheta)$.

Theorem 2. *For $0 \leqq \gamma \leqq 1$, $0 \leqq \lambda \leqq 1$, $0 \leqq \delta \leqq 1$ and $\eta \in \mathbb{R}$, let $f \in \mathbb{F}_\Sigma(\gamma, \lambda, \delta; \vartheta)$ be of the form* (1)*. Then*

$$\begin{aligned}
&\left|a_3 - \eta a_2^2\right| \leq \\
&\leq \begin{cases} \frac{1}{2(\gamma+\lambda(2\delta+1))}; & |\eta-1| \leqq \Xi\left(\gamma,\lambda,\tau,\delta\right), \\ \frac{2|\eta-1|}{\left|\left[\gamma(\gamma+1)+\lambda(\delta+1)(2(\gamma+1)+(\lambda-1)(\delta+1))\right]+(1-\tau)(\gamma+\lambda(\delta+1))^2\right|}; & |\eta-1| \geqq \Xi\left(\gamma,\lambda,\tau,\delta\right). \end{cases}
\end{aligned}$$

where

$$\begin{aligned}
&\Xi\left(\gamma,\lambda,\tau,\delta\right) \qquad (18)\\
&= \frac{\left|\left[\gamma(\gamma+1)+\lambda(\delta+1)\left(2(\gamma+1)+(\lambda-1)(\delta+1)\right)\right]+(1-\tau)\left(\gamma+\lambda(\delta+1)\right)^2\right|}{4\left(\gamma+\lambda(2\delta+1)\right)}.
\end{aligned}$$

Proof. It follows from (16) and (17) that

$$a_3 - \eta a_2^2 = \frac{x_2 - y_2}{8\left(\gamma + \lambda(2\delta+1)\right)} + (1-\eta)\, a_2^2$$

$$= \frac{x_2 - y_2}{8\left(\gamma + \lambda(2\delta+1)\right)}$$

$$+ \frac{(x_2 + y_2)(1-\eta)}{2\left[\left[\gamma(\gamma+1) + \lambda(\delta+1)\left(2(\gamma+1) + (\lambda-1)(\delta+1)\right)\right] + (1-\tau)\left(\gamma + \lambda(\delta+1)\right)^2\right]}$$

$$= \frac{1}{2}\left[\left(\psi(\eta,\tau) + \frac{1}{4\left(\gamma+\lambda(2\delta+1)\right)}\right) x_2 + \left(\psi(\eta,\tau) - \frac{1}{4\left(\gamma+\lambda(2\delta+1)\right)}\right) y_2\right],$$

where

$$\psi(\eta,\tau) = \frac{1-\eta}{\left[\gamma(\gamma+1) + \lambda(\delta+1)\left(2(\gamma+1) + (\lambda-1)(\delta+1)\right)\right] + (1-\tau)\left(\gamma+\lambda(\delta+1)\right)^2}.$$

According to Lemma 1 and (2), we find that

$$\left|a_3 - \eta a_2^2\right| \leqq \begin{cases} \frac{1}{2(\gamma+\lambda(2\delta+1))}, & 0 \leqq |\psi(\eta,\tau)| \leqq \frac{1}{4(\gamma+\lambda(2\delta+1))}, \\ 2\left|\psi(\eta,\tau)\right|, & |\psi(\eta,\tau)| \geqq \frac{1}{4(\gamma+\lambda(2\delta+1))}. \end{cases}$$

After some computations, we obtain

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{1}{2(\gamma+\lambda(2\delta+1))}; & |\eta - 1| \leqq \Xi(\gamma,\lambda,\tau,\delta), \\ \frac{2|\eta-1|}{\left|\left[\gamma(\gamma+1)+\lambda(\delta+1)(2(\gamma+1)+(\lambda-1)(\delta+1))\right]+(1-\tau)(\gamma+\lambda(\delta+1))^2\right|}; & |\eta-1| \geqq \Xi(\gamma,\lambda,\tau,\delta). \end{cases}$$

where $\Xi(\gamma,\lambda,\tau,\delta)$ as given in (18). □

For $\lambda = 0$ and $\gamma = 1$ Theorem 2 gives the results of Cotîrlă and Wanas (see [11]).

Corollary 3. *[11] For $\eta \in \mathbb{R}$, let $f \in S_E^*(\vartheta)$ be of the form* (1)*. Then*

$$\left|a_3 - \eta a_2^2\right| \leqq \begin{cases} \frac{1}{2}; & |\eta-1| \leqq \frac{|3-\tau|}{4}, \\ \frac{2|\eta-1|}{|3-\tau|}; & |\eta-1| \geqq \frac{|3-\tau|}{4}. \end{cases}$$

When $\gamma = 0$ and $\lambda = \delta = 1$ Theorem 2 leads to the known result on Cotîrlă and Wanas for the family $\mathcal{C}_E(\vartheta)$ (see [11]).

Corollary 4. *[11] For $\eta \in \mathbb{R}$, let $f \in C_E(\vartheta)$ be of the form* (1)*. Then*

$$\left|a_3 - \eta a_2^2\right| \leqq \begin{cases} \frac{1}{6}; & |\eta-1| \leqq \frac{|2-\tau|}{3}, \\ \frac{|\eta-1|}{2|2-\tau|}; & |\eta-1| \geqq \frac{|2-\tau|}{3}. \end{cases}$$

If we take $\eta = 1$ in Theorem 2, we get the next result:

Corollary 5. *If $f \in \mathbb{F}_\Sigma(\gamma, \lambda, \delta; \vartheta)$ be of the form* (1)*, then we have that*

$$\left|a_3 - a_2^2\right| \leqq \frac{1}{2\left(\gamma + \lambda(2\delta + 1)\right)}.$$

3 Conclusion

Studies of bi-univalent functions which are defined by generalized telephone numbers is relatively new, and only a small number of papers have been written on this subject so far. In the present investigation, we create a certain family $\mathbb{F}_\Sigma(\gamma, \lambda, \delta; \vartheta)$ of holormorphic and bi-univalent functions which are defined by generalized telephone numbers. We generated Taylor-Maclaurin coefficient inequalities for functions belonging to this family and viewed the famous Fekete-Szegö problem and indicated certain special cases and consequences for our results.

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