Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 5(67), No. 2 - 2025, 101-114
https://doi.org/10.31926/but.mif.2025.5.67.2.8

FINSLERIAN HYPERSURFACES OF A FINSLER SPACE WITH DEFORMED DOUGLAS INFINITE SERIES METRIC

V. K. CHAUBEY[*,1] and Brijesh Kumar TRIPATHI[2]

Abstract

In present paper we studied the geometrical properties of Finslerian hypersurfaces and its reducibility of Cartan $C-$ tensor in various forms for a Finsler space F^n equipped with deformed Infinite series metric. Further we obtained the value of main scalar I for the hypersurface in a two-dimensional case.

2020 Mathematics Subject Classification: 53B40, 53C60.
Key words: Finslerian hypersurface, infinite series metric, Matsumoto Douglas type metric, C2-like Finsler space, main scalar.

1 Introduction

Matsumoto [7] studied a Finsler metric of two variable α and β on a n-dimensional manifold M^n and summerised all the results for the Finsler space F^n which is equipped with $(\alpha, \beta)-$metric. Further in 1998 he [9] introduced a special (α, β)-metric which was defined as

$$L = \alpha + \frac{\beta^2}{\alpha} \tag{1}$$

and named as Douglas type metric and the space equiped with this metric was known as Finsler space with Douglas type metric. Since the Douglas space was a generalization of Berwald space, so this metric was very important in the development of Finsler geometry.

In 2004 Lee and Park [6] introduced a r-th series (α, β)-metric

$$L(\alpha, \beta) = \beta \sum_{k=0}^{r} (\frac{\alpha}{\beta})^k, \qquad \alpha < \beta \tag{2}$$

[1*] *Corresponding author*, Department of Mathematics, North-Eastern Hill University, Shillong-793022, India, e-mail: vkcoct@gmail.com
[2]Department of Mathematics, L. D. College of Engineering, Navrangpura, Ahmedabad-380015, India e-mail: brijeshkumartripathi4@gmail.com

where α is a Riemannian metric and β is one form. The above equation is reduces in special and important form of an (α, β)-metric for the various values of r. e.g.

1. If $r = 1$ then r-th series metric reduces in a special and important form which is known as Randers metric which is widely used in the field of physics.

2. If $r = 2$ then above metric reduces in $L = \alpha + \beta + \frac{\alpha^2}{\beta}$ which is a combination of Randers metric and Kropina metric.

3. If $r = \infty$ then above metric is expressed as $L(\alpha, \beta) = \frac{\beta^2}{\beta - \alpha}$ which is an remarkable form of an (α, β) that represent the difference of Randers and Matsumoto metric.

Matsumoto introduced the concept of Finslerian hypersurface [8] and studied its geometrical properties in various forms. Further many authors studied hypersurface properties in a Finsler space [1, 2, 3, 4, 5, 10, 11] for various Finsler metrics and obtained very interesting results in the field of Finsler geometry.

In the present paper we combine douglas type Finsler metric $L = \alpha + \dfrac{\beta^2}{\alpha}$ and Infinite series metric $L = \frac{\beta^2}{\beta - \alpha}$ and introduced a deformed douglas Infinite series metric and studied some intrinsic properties of the hypersurfaces and its reducibility of Cartan C-tensor in various forms for a Finsler space with deformed douglas Infinite series metric. In the last article of present paper we obtain the the value of main scalar I in a two-dimensional case for the hypersurfcae of a Finsler space with deformed douglas Infinite series metric.

2 Preliminaries

The concept of Finslerian hypersurfaces introduced by [8] and studied in detail by[4] in a n-dimensional Finsler space $F^n = (M^n, L)$ with respect to cartan connection $C\Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$ and obtained the expression for fundamental metric tensor g_{ij} and C-tensor C_{ijk} which can be written as

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}, \qquad C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}, \tag{3}$$

Further Matsumoto [8] and Kitayama [4] considered the parametric form of an hypersurface of an n-dimensional manifold M^n is given by

$$x^i = x^i(u^\alpha)$$

where u^α $\{\alpha = 1, 2, 3...(n-1)\}$, are the Gaussian coordinates and the hypersurface of M^n is denoted by M^{n-1}.

Again the supporting line element y^i which is tangential to the hypersurface M^{n-1} at the point (u^α) is given by

$$y^i = B^i_\alpha(u)v^\alpha, \tag{4}$$

where the projection factors $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$ represent a tangent space at a point of the hypersurface and matrix corresponding to it is of rank $n-1$ whereas v^α is the line element of support of the hypersurface M^{n-1} at the point (u^α).

Since the metric function $L(u,v) = L\{x(u), y(u,v)\}$ satisfied the conditions of a Finsler metric in the hypersurface M^{n-1}, so we have an $(n-1)$ dimensional Finsler space $F^{n-1} = \{M^{n-1}, L\}$ equipped with the metric $L(u,v)$.

Now the expression for metric tensor, hv-tensor, a unit normal vector, angular metric tensor and the relation between projection factors and its inverse for a Finslerain hypersurface F^{n-1} [8] at a point (u^α) are given by

$$g_{\alpha\beta} = g_{ij}B^i_\alpha B^j_\beta, \quad C_{\alpha\beta\gamma} = C_{ijk}B^i_\alpha B^j_\beta B^k_\gamma,$$

$$g_{ij}\{x(u,v), y(u,v)\}B^i_\alpha N^j = 0, \quad g_{ij}\{x(u,v), y(u,v)\}N^i N^j = 1.$$

$$h_{\alpha\beta} = h_{ij}B^i_\alpha B^j_\beta, \quad h_{ij}B^i_\alpha N^j = 0, \quad h_{ij}N^i N^j = 1. \tag{5}$$

$$B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta, \quad B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad B^\alpha_i N^i = 0, \quad B^i_\alpha N_i = 0.$$

$$N_i = g_{ij}N^j, \quad B^k_i = g^{kj}B_{ji}, \quad B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j.$$

The cartan connection $IC\Gamma = (\Gamma^{*\alpha}_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ for the Finslerian hypersurface F^{n-1} are given by

$$\Gamma^{*\alpha}_{\beta\gamma} = B^\alpha_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}B^j_\beta B^k_\gamma) + M^\alpha_\beta H_\gamma.$$

$$G^\alpha_\beta = B^\alpha_i(B^i_{0\beta} + \Gamma^{*i}_{0j}B^j_\beta), \quad C^\alpha_{\beta\gamma} = B^\alpha_i C^i_{jk}B^j_\beta B^k_\gamma,$$

where

$$M_{\beta\gamma} = N_i C^i_{jk}B^j_\beta B^k_\gamma, \quad M^\alpha_\beta = g^{\alpha\gamma}M_{\beta\gamma}, \quad H_\beta = N_i(B^i_{0\beta} + \Gamma^{*i}_{oj}B^j_\beta),$$

and

$$B^i_{\beta\gamma} = \frac{\partial B^i_\beta}{\partial u^\gamma}, \qquad B^i_{0\beta} = B^i_{\alpha\beta}v^\alpha.$$

Note: "The tensorial form $M_{\alpha\beta}$ and H_α are known as the second fundamental v-tensor and normal curvature vector respectively".

Further the second fundamental h-tensor $H_{\beta\gamma}$ can be expressed as [8]

$$H_{\beta\gamma} = N_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}B^j_\beta B^k_\gamma) + M_\beta H_\gamma, \tag{6}$$

where, $M_\beta = N_i C^i_{jk} B^j_\beta N^k$.
Since form above it is clear that the tensorial quantity $H_{\beta\gamma}$ is not symmetric so we have

$$H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta. \tag{7}$$

Now the h and v-covariant derivatives of the projection factor B^i_α with respect to $IC\Gamma$ can be expressed as

$$B^i_{\alpha|\beta} = H_{\alpha\beta} N^i, \qquad B^i_\alpha|_\beta = M_{\alpha\beta} N^i.$$

Now when we contracting $H_{\beta\gamma}$ and $H_{\gamma\beta}$ by v^β we get

$$H_{0\gamma} = H_\gamma, \quad H_{\gamma 0} = H_\gamma + M_\gamma H_0, \tag{8}$$

Thus following important results for the Finslerian hypersurface [8] we shall use in our present paper

Lemma 1. *The normal curvature tensor vanishes identically iff the normal curvature vector vanishes in a Finslerian hypersurface $F^{(n-1)}$.*

Lemma 2. *If F^n be a Finsler space and $F^{(n-1)}$ be its hypersurface then the hypersurfave $F^{(n-1)}$ will be a hyperplane of the first kind iff normal curvature vector vanishes identically.*

Lemma 3. *If F^n be a Finsler space and $F^{(n-1)}$ be its hypersurface then the hypersurfave $F^{(n-1)}$ will be a hyperplane of the second kind iff normal curvature vector and second fundamental h-tensor vanishes identically.*

Lemma 4. *If F^n be a Finsler space and $F^{(n-1)}$ be its hypersurface then the hypersurfave $F^{(n-1)}$ will be a hyperplane of the third kind iff normal curvature vector, second fundamental h and v tensor vanishes identically.*

3 Finsler space with deformed Douglas infinite series metric

The deformed Douglas Infinite series metric is a combination of Matsumoto Douglas type and Infinite series metric which we defined as

Definition 1. *Let F^n be an n-dimensional Finsler space consisting of an n-dimensional differentiable manifold M^n equipped with a fundamental function L defined as*

$$L(\alpha, \beta) = \alpha + \frac{\beta^2}{\alpha} + \frac{\beta^2}{(\beta - \alpha)} \tag{9}$$

then the metric L is known as deformed Infinite series and the Finsler space $F^n = \{M^n, L(\alpha, \beta)\}$ equiped with this metric is known as deformed infinite series Finsler space.

Now differentiating equation (9) partially with respect to α and β we have following important identities

$$L_\alpha = \frac{\alpha^4-\beta^4-2\alpha^3\beta-2\alpha\beta^3+\alpha^2\beta^2}{\alpha^2(\beta-\alpha)^2}, \qquad L_\beta = \frac{2\beta^3-3\alpha\beta^2}{\alpha(\beta-\alpha)^2} \qquad L_{\alpha\alpha} = \frac{2\{\alpha^3+(\beta-\alpha)^3\}\beta^2}{\alpha^3(\beta-\alpha)^2}$$

$$L_{\beta\beta} = \frac{2\{\alpha^3+(\beta-\alpha)^3\}}{\alpha(\beta-\alpha)^3}, \qquad L_{\alpha\beta} = \frac{-2\beta\{\alpha^3+(\beta-\alpha)^3\}\beta}{\alpha^2(\beta-\alpha)^3}$$

where $L_\alpha = \frac{\partial L}{\partial\alpha}$, $L_\beta = \frac{\partial L}{\partial\beta}$, $L_{\alpha\alpha} = \frac{\partial L\alpha}{\partial\alpha}$, $L_{\beta\beta} = \frac{\partial L_\beta}{\partial\beta}$, $L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial\beta}$.

In Finsler space $F^n = \{M^n, L(\alpha,\beta)\}$ the normalized element of support $l_i = \dot{\partial}_i L$ and angular metric tensor h_{ij} are given by:

$$l_i = \alpha^{-1}L_\alpha Y_i + L_\beta b_i \tag{10}$$

$$h_{ij} = pa_{ij} + q_0 b_i b_j + q_{-1}(b_i Y_j + b_j Y_i) + q_{-2} Y_i Y_j \tag{11}$$

where $Y_i = a_{ij}y^j$ and the scalars p, q_0, q_{-1} and q_{-2} are constants and its values are given below

$$\begin{aligned} p &= \frac{(\beta^3-\alpha^3+\alpha^2\beta)(\alpha^4-\beta^4-2\alpha\beta^3-2\alpha^3\beta+\alpha^2\beta^2)}{\alpha^4(\beta-\alpha)^3}, \\ q_0 &= \frac{2(\beta^3-\alpha^3+\alpha^2\beta)\{\alpha^3+(\beta-\alpha)^3\}}{\alpha^2(\beta-\alpha)^4}, \\ q_{-1} &= \frac{-2\beta(\beta^3-\alpha^3+\alpha^2\beta)\{\alpha^3+(\beta-\alpha)^3\}}{\alpha^4(\beta-\alpha)^4}, \\ q_{-2} &= \frac{(\beta^3-\alpha^3+\alpha^2\beta)(\alpha^5+3\beta^5+3\alpha^3\beta^2+7\alpha^2\beta^3-9\alpha\beta^4)}{\alpha^6(\beta-\alpha)^4}. \end{aligned} \tag{12}$$

Note: $0, -1, -2$ in the subscript represents homoginity of the respective terms.

Fundamental metric tensor $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$ and its reciprocal tensor g^{ij} for $L = L(\alpha,\beta)$ are given by [8]

$$g_{ij} = pa_{ij} + p_0 b_i b_j + p_{-1}(b_i Y_j + b_j Y_i) + p_{-2} Y_i Y_j \tag{13}$$

$$g^{ij} = p^{-1}a^{ij} - s_0 b^i b^j - s_{-1}(b^i y^j + b^j y^i) - s_{-2} y^i y^j \tag{14}$$

where $b^i = a^{ij}b_j$ *and* $b^2 = a_{ij}b^i b^j$

$$p_0 = q_0 + L_\beta^2, \qquad p_{-1} = q_{-1} + L^{-1}pL_\beta, \qquad p_{-2} = q_{-2} + p^2 L^{-2} \tag{15}$$

$$\begin{aligned} s_0 &= \frac{1}{\tau p}\{pp_0 + (p_0 p_{-2} - p_{-1}^2)\alpha^2\}, \\ s_{-1} &= \frac{1}{\tau p}\{pp_{-1} + (p_0 p_{-2} - p_{-1}^2)\beta\}, \\ s_{-2} &= \frac{1}{\tau p}\{pp_{-2} + (p_0 p_{-2} - p_{-1}^2)b^2\}, \\ \tau &= p(p + p_0 b^2 + p_{-1}\beta) + (p_0 p_{-2} - p_{-1}^2)(\alpha^2 b^2 - \beta^2) \end{aligned} \tag{16}$$

The hv-torsion tensor $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ is given by

$$2pC_{ijk} = p_{-1}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_i m_j m_k \tag{17}$$

where,

$$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_{-1}q_0, \qquad m_i = b_i - \alpha^{-2}\beta Y_i \tag{18}$$

Here m_i is a non-vanishing covariant vector orthogonal to the element of support y^i. Thus we have

Proposition 1. *The normalised supporting element l_i and angular metric tensor h_{ij} of an n-dimensional Finsler space F^n equipped with a deformed Douglas Infinite series metric L are given by (10) and (11) respectively.*

Proposition 2. *The fundamentiacl metric tensor g_{ij} and its reciprocal tensor g^{ij} of an n-dimensional Finsler space F^n equipped with a deformed Douglas Infinite series metric L are given by (13) and (14) respectively.*

Proposition 3. *The Cartan hv-torsion tensor of an n-dimensional Finsler space F^n equipped with a deformed Douglas Infinite series metric L is given by (17).*

Let $\{^{\,i}_{jk}\}$ be the component of christoffel symbols of the associated Riemannian space R^n and $\bigtriangledown_k$ be the covariant derivative with respect to x^k relative to this christoffel symbol. Now we define,

$$2E_{ij} = b_{ij} + b_{ji}, \qquad 2F_{ij} = b_{ij} - b_{ji} \tag{19}$$

where $b_{ij} = \bigtriangledown_j b_i$.

Let $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, \Gamma^{i}_{jk})$ be the cartan connection of F^n. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{^{\,i}_{jk}\}$ of the special Finsler space F^n is given by

$$\begin{aligned} D^i_{jk} = B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} \\ -C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} + \lambda^s (C^i_{jm} C^m_{sk} + \\ C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}) \end{aligned} \tag{20}$$

where

$$\begin{aligned} B_k = p_0 b_k + p_{-1} Y_k, \quad B^i = g^{ij} B_j, \quad F^k_i = g^{kj} F_{ji} \\ B_{ij} = \frac{\{p_{-1}(a_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial p_0}{\partial \beta} m_i m_j\}}{2}, \quad B^k_i = g^{kj} B_{ji} \\ A^m_k = B^m_k E_{00} + B^m E_{k0} + B_k F^m_0 + B_0 F^m_k \\ \lambda^m = B^m E_{00} + 2B_0 F^m_0, \quad B_0 = B_i y^i \end{aligned} \tag{21}$$

where $'0'$ denote contraction with y^i except for the quantities p_0, q_0 and s_o. Thus

Proposition 4. *The difference tensor D^i_{jk} of the Cartan connection $C\Gamma$ for the n-dimensional Finsler space F^n equipped with a deformed Douglas Infinite series metric L is given by (20).*

4 Finslerian hypersurface $F^{(n-1)}(c)$ for a Finsler space with deformed Douglas infinite series metric

Let us consider a Finsler space with the deformed Infinite series metric $L(\alpha, \beta) = \alpha + \frac{\beta^2}{\alpha} + \frac{\beta^2}{(\beta-\alpha)}$, where, $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and vector field $b_i(x) = \frac{\partial b}{\partial x^i}$ is a gradient of some scalar function $b(x)$. Now we consider a hypersurface $F^{(n-1)}(c)$ given by equation $b(x) = c$, a constant [11].

From the parametric equation $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$, we get

$$\frac{\partial b(x)}{\partial u^\alpha} = 0 \qquad \frac{\partial b(x)}{\partial x^i}\frac{\partial x^i}{\partial u^\alpha} = 0, \qquad b_i B^i_\alpha = 0$$

Above shows that $b_i(x)$ are covarient component of a normal vector field of hypersurface $F^{n-1}(c)$. Further, we have

$$b_i B^i_\alpha = 0 \quad and \quad b_i y^i = 0 \quad i.e \quad \beta = 0 \tag{22}$$

and induced matric $L(u, v)$ of $F^{n-1}(c)$ is given by

$$L(u, v) = a_{\alpha\beta} v^\alpha v^\beta, a_{\alpha\beta} = a_{ij} B^i_\alpha B^j_\beta \tag{23}$$

which is a Riemannian metric.

Writing $\beta = 0$ in the equations (10), (11) and (13) we get

$$\begin{aligned} &p = 1, \quad q_0 = 0, \quad q_{-1} = 0 \quad q_{-2} = -\alpha^{-2} \quad p_0 = 0 \quad p_{-1} = 0 \\ &p_{-2} = 0 \quad \tau = 1, \quad s_0 = 0 \quad s_{-1} = 0 \quad s_{-2} = 0 \end{aligned} \tag{24}$$

from (12) we get,

$$g^{ij} = a^{ij} \tag{25}$$

thus along $F^{n-1}(c)$, (25) and (22) leads to

$$g^{ij} b_i b_j = b^2$$

So we get

$$b_i(x(u)) = bN_i, \qquad b^2 = a^{ij} b_i b_j \tag{26}$$

where b is the length of the vector b^i

Again from (25) and (26), we get

$$b^i = a^{ij} b_j = N^i \tag{27}$$

thus we have,

Proposition 5. *The induced Riemannian metric in a Finslerian hypersurface $F^{(n-1)}(c)$ of the Finsler space F^n equipped with a deformed Douglas Infinite series metric is given by (23) and its scalar function $b(x)$ is given by (26) and (27).*

Now the angular metric tensor h_{ij} and metric tensor g_{ij} of F^n are given by

$$h_{ij} = a_{ij} - \frac{1}{\alpha^2} Y_i Y_j \quad and \quad g_{ij} = a_{ij} \tag{28}$$

From equation (22), (28) and (5) it follows that if $h^{(a)}_{\alpha\beta}$ denote the angular metric tensor of the Riemannian $a_{ij}(x)$ then we have along $F^{n-1}_{(c)}$, $h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$.

thus along $F^{n-1}_{(c)}$, $\frac{\partial p_0}{\partial \beta} = \frac{-6}{\alpha}$

from equation (16) we get

$$r_1 = \frac{-6}{\alpha}, \qquad m_i = b_i$$

then hv-torsion tensor becomes

$$C_{ijk} = -\frac{3}{\alpha} b_i b_j b_k \tag{29}$$

in the deformed Douglas Infinite series Finsler hypersurface $F^{(n-1)}_{(c)}$. Due to fact from (5), (6), (8), (22) and (29) we have

$$M_{\alpha\beta} = 0 \quad and \quad M_\alpha = 0 \tag{30}$$

Therefore from equation (6) it follows that $H_{\alpha\beta}$ is symmetric. Thus we have

Proposition 6. *The second fundamental v-tensor in a Finslerian hypersurface $F^{(n-1)}(c)$ of the Finsler space F^n equipped with a deformed Douglas Infinite series metric is given by (30) and the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric.*

Now from (22) we have $b_i B^i_\alpha = 0$. Then we have

$$b_{i|\beta} B^i_\alpha + b_i B^i_{\alpha|\beta} = 0$$

Therefore, from (8) and using $b_{i|\beta} = b_{i|j} B^j_\beta + b_i|_j N^j H_\beta$, we have

$$b_{i|j} B^i_\alpha B^j_\beta + b_{i|j} B^i_\alpha N^j H_\beta + b_i H_{\alpha\beta} N^i = 0 \tag{31}$$

since $b_i|_j = -b_h C^h_{ij}$, we get

$$b_{i|j} B^i_\alpha N^j = 0$$

Therefore from equation (31) we have,

$$b H_{\alpha\beta} + b_{i|j} B^i_\alpha B^j_\beta = 0 \tag{32}$$

because $b_{i|j}$ is symmetric. Now contracting (32) with v^β and using (4) we get

$$b H_\alpha + b_{i|j} B^i_\alpha y^j = 0 \tag{33}$$

Again contracting by v^α equation (33) and using (4), we have

$$bH_0 + b_{i|j}y^iy^j = 0 \tag{34}$$

Now by using Lemmas 1, 2 and equation (34), it is clear that hypersuface $F^{(n-1)}_{(c)}$ of Finsler space with deformed Douglas Infinite series is a hyperplane of first kind if and only if $H_0 = 0$.

Since $b_{i|j}$ represent the covariant derivative with respect to $C\Gamma$ in the Finsler space F^n defined on y^i, but $b_{ij} = \bigtriangledown_j b_i$ is the covariant derivative with respect to Riemannian connection $\{^{\,i}_{jk}\}$. Hence b_{ij} does not depend on y^i. So we shall consider the difference $b_{i|j} - b_{ij}$ where $b_{ij} = \bigtriangledown_j b_i$. Since b_i is a gradient vector, then from (19) we have

$$E_{ij} = b_{ij} \qquad F_{ij} = 0 \qquad \textit{and} \qquad F^i_j = 0$$

Thus using above fact the in equation (20), the difference tensor can be rewritten as

$$\begin{aligned} D^i_{jk} = B^ib_{jk} + B^i_jb_{0k} + B^i_kb_{0j} - b_{0m}g^{im}B_{jk} - C^i_{jm}A^m_k - C^i_{km}A^m_j \\ +C_{jkm}A^m_sg^{is} + \lambda^s(C^i_{jm}C^m_{sk} + C^i_{km}C^m_{sj} - C^m_{jk}C^i_{ms}) \end{aligned} \tag{35}$$

where

$$\begin{aligned} &B_i = 0, \quad B^i = 0, \quad B_iB^i = 0, \quad \lambda^m = B^mb_{00}, \quad B_{ij} = -\frac{3}{\alpha}b_ib_j, \\ &B^i_j = -\frac{3}{\alpha}b^2, \quad A^m_k = B^m_kb_{00} + B^mb_{k0} \end{aligned} \tag{36}$$

In view of (24) and (25), the relation in (19) becomes to by virture of (36) we have $B^i_0 = 0,\ B_{i0} = 0$ which leads $A^m_0 = B^mb_{00}$.

Now contracting (35) by y^k we get

$$D^i_{j0} = B^ib_{j0} + B^i_jb_{00} - B^mC^i_{jm}b_{00}$$

Again contracting the above equation with respect to y^j we have

$$D^i_{00} = 0$$

Paying attention to (22), along $F^{(n-1)}_{(c)}$, we get

$$b_iD^i_{j0} = -\frac{3}{\alpha}b^2b_ib_{00} \tag{37}$$

Now we contract (37) by y^j we have

$$b_iD^i_{00} = 0 \tag{38}$$

From (26), (27), (30) and $M_\alpha = 0$, we have

$$b_i b^m C^i_{jm} B^j_\alpha = b^2 M_\alpha = 0$$

Thus the relation $b_{i|j} = b_{ij} - b_r D^r_{ij}$ the equation (37) and (38) gives

$$b_{i|j} y^i y^j = b_{00} - b_r D^r_{00} = b_{00}$$

Consequently (33) and (34) may be written as

$$bH_\alpha + b_{i0} B^i_\alpha = 0, \qquad \textit{and} \qquad bH_0 + b_{00} = 0 \tag{39}$$

Thus the condition $H_0 = 0$ is equivalent to $b_{00} = 0$. Using the fact $\beta = b_i y^i = 0$ the condition $b_{00} = 0$ can be written as $b_{ij} y^i y^j = b_i y^i b_j y^j$ for some $c_j(x)$. Thus we can write,

$$2b_{ij} = b_i c_j + b_j c_i \tag{40}$$

Now from (22) and (40) we get

$$b_{00} = 0, \quad b_{ij} B^i_\alpha B^j_\beta = 0, \quad b_{ij} B^i_\alpha y^j = 0$$

Hence using the above condition and equation (39) we have

$$H_\alpha = 0 \tag{41}$$

Thus using Lemma 2 and above condition we have

Theorem 1. *The necessary and sufficient condition for a Finslerian hypersurface $F^{(n-1)}(c)$ of the Finsler space F^n equipped with a deformed Douglas Infinite series metric to be a hyperplane of first kind is (40).*

again from (40) and (36) we get

$$b_{i0} b^i = \frac{c_0 b^2}{2}, \quad \lambda^m = 0, \quad A^i_j B^j_\beta = 0 \quad \textit{and} \quad B_{ij} B^i_\alpha B^j_\beta = -\frac{3}{\alpha} b_i b_j B^i_\alpha B^j_\beta = 0.$$

Now we use equation (25), (26), (27), (30) and (35) then we have

$$b_r D^r_{ij} B^i_\alpha B^j_\beta = 0 \tag{42}$$

Thus the equation (32) reduces to

$$H_{\alpha\beta} = 0 \tag{43}$$

Hence the hypersurface $F^{n-1}_{(c)}$ is umbilic.

Now from Lemma 3, $F^{(n-1)}_{(c)}$ is a hyperplane of second kind if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$. Thus from (42), we get

$$c_0 = c_i(x) y^i = 0$$

Therefore there exist a function $\psi(x)$ such that

$$c_i(x) = \psi(x) b_i(x)$$

Therefore (40) we get

$$2b_{ij} = b_i(x)\psi(x)b_j(x) + b_j(x)\psi(x)b_i(x)$$

This can also be written as

$$b_{ij} = \psi(x)b_ib_j \tag{44}$$

Thus using the condition (41) and (43) we have

Theorem 2. *The necessary and sufficient condition for in a Finslerian hypersurface $F^{(n-1)}(c)$ of the Finsler space F^n equipped with a deformed Douglas Infinite series metric to be a hyperplane of second kind is (44).*

Again Lemma 4, together with (30) and $M_\alpha = 0$ shows that $F^{n-1}_{(c)}$ become a hyperplane of third kind.

Theorem 3. *The Finslerian hypersurface $F^{(n-1)}(c)$ of the Finsler space F^n equipped with a deformed Douglas Infinite series metric is a hyperplane of the third kind.*

5 Some important result of hypersurface $F^{(n-1)}(c)$ of a Finsler space $F^n(c)$ with deformed Douglas infinite series metric

The hv-torsion tensor C_{ijk} of $F^{(n-1)}(c)$ with deformed infinite series metric written in equation (29) as

$$C_{ijk} = \tfrac{-3}{\alpha}b_ib_jb_k$$

Contracting by g^{jk}, we have

$$C_i = C_{ijk}g^{jk} = \tfrac{-3b^2}{\alpha}b_i$$

This implies that

$$b_i = \tfrac{-\alpha}{3b^2}C_i$$

Therefore equation (29) becomes

$$C_{ijk} = \frac{\alpha^2}{9b^6}C_iC_jC_k \tag{45}$$

Definition 2. *A Finsler space F^n is called C2-like,if the (h) hv-tortion tensor C_{ijk} is written in the form*

$$C_{ijk} = \tfrac{1}{C^2}C_iC_jC_k$$

Thus using the above dfinition and equation (45) we have

$$C = \frac{3b^3}{\alpha} \tag{46}$$

Thus

Proposition 7. *The Finslerian hypersurface $F^{(n-1)}(c)$ of the Finsler space F^n eqquipped with a deformed Douglas Infinite series metric is always be a C2-like Finsler space if equation (46) is satisified.*

Since the main scalar of two dimensional Finsler space is defined as

$$LC_{ijk} = Im_i m_j m_k$$

From equation (16) $m_i = b_i$ we have

$$LC_{ijk} = Ib_i b_j b_k$$

Contracting g^{jk} we have

$$LC_i = Ib^2 b_i$$

which gives

$$b_i = \tfrac{L}{b^2} C_i$$

Now the main scalar of two dimensional Finsler

$$LC_{ijk} = \frac{IL^3}{b^6} C_i C_j C_k \tag{47}$$

From equation (45) and (47), we have

$$I = \frac{\alpha^2}{9L^2} \tag{48}$$

Proposition 8. *The main scalar I of a Finslerian hypersurface $F^{(n-1)}(c)$ for the Finsler space F^n eqquipped with a deformed Douglas Infinite series metric in a two dimensional case is given by (48).*

6 Conclusion

In the present paper, we obtained the conditions for a Finslerian hypersurface $F^{(n-1)}(c)$ of a Finsler space F^n with deformed Douglas Infinite series metric to be a hyperplane of first, second, and third kind in the Theorems 1, 2, and 3, respectively. Further for application point of view we also obtained the Proposition 7 which states the condition under which hypersurface $F^{n-1}(c)$ will be reduces in C2-like Finsler space and in Propostion 8 gives the value of the main scalar I for the hypersurface in two-dimensional case.

References

[1] Chaubey, V. K. and Mishra A., *Hypersurface of a Finsler space with special (α, β)-metric*, Journal of Contemporary Mathematical Analysis **52** (1) (2017), 1-7.

[2] Chaubey, V. K. and Tripathi, Brijesh Kumar, *Hypersurfaces of a Finsler space with deformed berwald-matsumoto metric*, Bulletin of the Transilvania University of Brasov **11**(60) no. 1 (2018), 37-48.

[3] Chaubey, V. K. and Tripathi, Brijesh Kumar, *Finslerian hypersurface of a finsler spaces with special* (γ, β)*-metric*, Journal of Dynamical System and Geometric Theories **12**(1) (2014), 19-27.

[4] Kitayama, M., *On Finslerian hypersurfaces given by* $\beta-$ *change*, Balkan Journal of Geometry and Its Applications **7**2 (2002), 49-55.

[5] Lee, I. Y., Park, H. Y. and Lee, Y. D., *On a hypersurface of a special Finsler spaces with a metric* $(\alpha + \frac{\beta^2}{\alpha})$, Korean J. Math. Sciences **8** (2001), 93-101.

[6] Lee, I. Y. and Park, H. Y., *Finsler spaces with infinite series* (α, β)*- metric*, J.Korean Math. Society **41**No. 3 (2004), 567-589.

[7] Matsumoto, M., *Theory of Finsler spaces with* (α, β)*-metric*, Rep. on Math. Phys. **31** (1992), 43-83.

[8] Matsumoto, M., *The induced and intrinsic Finsler connections of a hypersurface and finslerian projective geometry*, J. Math. Kyoto Univ. **25** (1985), 107-144.

[9] Matsumoto, M., *Finsler spaces with* (α, β)*-metric of douglas type*, Tensor N.S. **60** (1998), 123-134.

[10] Pandey, T. N. and Tripathi, B. K., *On a hypersurface of a Finsler space with special* (α, β)*-metric*, Tensor, N. S. **68** (2007), 158-166.

[11] Singh, U. P. and Kumari, Bindu, *On a hypersurface of a Matsumoto space*, Indian J. pure appl. Math. **32** (2001), 521-531.