Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 5(67), No. 2 - 2025, 89-100
https://doi.org/10.31926/but.mif.2025.5.67.2.7

# QUANTUM APPROACH ON CONVOLUTION OF HARMONIC UNIVALENT MAPPINGS

**Asena ÇETİNKAYA[1] Omendra MISHRA[*,2] and Saurabh PORWAL[3]**

**Abstract**

In this paper, we construct a new family of locally univalent and sense-preserving harmonic mappings $T_{c,q}[f]$ by using quantum approach in the open unit disk $\mathbb{D}$. We prove that the convolution of a harmonic convex mapping $T_{c,q}[f]$ with a right half-plane mapping is $q$-convex in the direction of the real axis provided that the convolution is locally univalent and sense-preserving. Further, we show that the convolution of $T_{c,q}[f]$ with a vertical strip mapping is also $q$-convex in the direction of the real axis. In particular, the results in this paper generalize or improve (in certain cases) the corresponding results obtained by recent researchers.

2000 *Mathematics Subject Classification:* 30C45, 30C55.
*Key words:* $q$-calculus, $q$-derivative operator, harmonic univalent functions, convolution.

## 1 Introduction

Quantum calculus is the calculus without use of the limits. The history of quantum calculus dates back to the studies of Leonhard Euler (1707-1783) and Carl Gustav Jacobi (1804-1851). In [13, 14], Jackson initiated a serious study in $q$-alculus, and developed the $q$-derivative and $q$-integral in a systematic way. Later, geometrical interpretation of the $q$-calculus has been applied in studies of quantum groups. The great interest to quantum calculus is due to its applications in various branches of mathematics and physics; as for example, in quantum mechanics, analytic number theory, Sobolev spaces, group representation theory,

[1]Department of Mathematics and Computer Science, İstanbul Kültür University, İstanbul, Türkiye, e-mail: asnfigen@hotmail.com
[2*] *Corresponding author*, Department of Mathematical and Statistical Sciences, Institute of Natural Sciences and Humanities, Shri Ramswaroop Memorial University, Lucknow 225003, India, e-mail:mishraomendra@gmail.com; https://orcid.org/0000-0001-9614-8656
[3]Department of Mathematics, Department of Mathematics, Ram Sahai Government Degree College, Bairi-Shivrajpur, Kanpur-209205, (Uttar Pradesh), India,, e-mail: saurabhjcb@rediffmail.com

theta functions, gamma functions, operator theory and several other areas. For more definitions and properties of $q$-calculus, one may refer to the books by Gasper and Rahman [9], and Kac and Cheung [15].

The $q$-derivative operator (or $q$-difference operator) of a function $h$, defined on a subset of $\mathbb{C}$, is given in [13] by

$$(D_q h)(z) = \begin{cases} \frac{h(z)-h(qz)}{(1-q)z}, & z \neq 0 \\ h'(0), & z = 0, \end{cases}$$

where $q \in (0,1)$. Note that $\lim_{q\to 1^-}(D_q h)(z) = h'(z)$ if $h$ is differentiable at $z$.

For a function $h(z) = z^n$ $(n \in \mathbb{N})$, we observe that

$$D_q z^n = [n]_q z^{n-1},$$

where

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + ...q^{n-1}$$

is the $q$-number of $n$. Clearly, $\lim_{q\to 1^-}[n]_q = n$.

Under the hypothesis of the definition of $q$-derivative operator, we have the following rules [15]:

(1) $D_q(ah_1(z) \pm bh_2(z)) = aD_q h_1(z) \pm bD_q h_2(z)$, where $a$ and $b$ are real or complex constants,

(2) $D_q(h_1(z).h_2(z)) = h_1(qz)D_q h_2(z) + h_2(z)D_q h_1(z)$,

(3) $D_q\left(\frac{h_1(z)}{h_2(z)}\right) = \frac{h_2(z)D_q h_1(z) - h_1(z)D_q h_2(z)}{h_2(z)h_2(qz)}, \quad h_2(z)h_2(qz) \neq 0.$

Jackson [14] also introduced the $q$-integral of any function $h$ by

$$\int_0^z h(t)d_q t = z(1-q)\sum_{n=0}^{\infty} q^n h(zq^n),$$

provided that the series on right hand side converges.

Let $\mathcal{H}$ denote the class of complex-valued functions $f = u + iv$ which are harmonic in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$, where $u$ and $v$ are real-valued harmonic functions in $\mathbb{D}$. Functions $f \in \mathcal{H}$ can also be expressed as $f = h + \overline{g}$, where $h$ the analytic and $g$ the co-analytic parts of $f$, respectively. According to the Lewy's Theorem [17], every harmonic function $f = h + \overline{g} \in \mathcal{H}$ is locally univalent and sense preserving in $\mathbb{D}$ if and only if the Jacobian of $f$, given by $J_f(z) = |h'(z)|^2 - |\ g'(z)|^2$, is positive in $\mathbb{D}$. This case is equivalent to the existence of an analytic function $\omega(z) = g'(z)/h'(z)$ in $\mathbb{D}$, which is called as the dilatation of $f$, such that

$$|\omega(z)| < 1 \quad \text{for all} \quad z \in \mathbb{D}.$$

Clunie and Sheil-Small [5] introduced the class of all univalent, sense-preserving harmonic functions $f = h + \overline{g}$, denoted by $\mathcal{S}_{\mathcal{H}}$, with the normalized conditions $h(0) = 0 = g(0)$ and $h'(0) = 1$ . If the function $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$, then

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}, \quad (|b_1| < 1, z \in \mathbb{D}). \tag{1}$$

A subclass of functions $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$ with the condition $g'(0) = 0$ is denoted by $\mathcal{S}_{\mathcal{H}}^0$. The subclass of functions $f$ in $\mathcal{S}_{\mathcal{H}}$ $\left(\mathcal{S}_{\mathcal{H}}^0\right)$, denoted by $\mathcal{K}_{\mathcal{H}}$ $\left(\mathcal{K}_{\mathcal{H}}^0\right)$, consists of functions $f$ that map the unit disk $\mathbb{D}$ onto a convex region, and the subclass of functions $f$ in $\mathcal{S}_{\mathcal{H}}$ $\left(\mathcal{S}_{\mathcal{H}}^0\right)$, denoted by $\mathcal{S}_{\mathcal{H}}^*$ $\left(\mathcal{S}_{\mathcal{H}}^{*0}\right)$, consists of functions $f$ that are starlike. We also note that for $g(z) \equiv 0$ in $\mathbb{D}$, the class $\mathcal{S}_{\mathcal{H}}$ reduces to the class $\mathcal{S}$ of univalent functions, which is a subclass of normalized analytic functions, denoted by $\mathcal{A}$. The class $\mathcal{K}_{\mathcal{H}}$ reduces to the class $\mathcal{K}$ of convex, and the class $\mathcal{S}_{\mathcal{H}}^*$ reduces to the class $\mathcal{S}^*$ of starlike functions.

A domain $E$ in $\mathbb{C}$ is said to be convex in the direction $\eta$ $(0 \leq \eta \leq \pi)$ if every line parallel to the line through 0 and $e^{i\eta}$ has an empty or connected intersection with $E$. Let $\mathcal{K}_{\mathcal{H}}(\eta)$ be the subclass $\mathcal{S}_{\mathcal{H}}$ whose members map $\mathbb{D}$ onto the domain convex in the direction of $\eta$ $(0 \leq \eta \leq \pi)$. Functions in $\mathcal{K}_{\mathcal{H}}(0)$ are said to be convex in the direction of the real axis or simply CHD functions. Similarly, functions in $\mathcal{K}_{\mathcal{H}}(\pi/2)$ are called as convex in the direction of the imaginary axis.

Connection of $q$-calculus with geometric function theory was first introduced by Ismail *et al.* [12]. These authors defined the class of $q$-starlike functions by

$$\mathcal{S}_q^* = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zD_q f(z)}{f(z)}\right) > 0,\ q \in (0,1),\ z \in \mathbb{D} \right\}.$$

Later, Ahuja *et al.* [1] defined and studied the classes of $q$-convex and $q$-close-to-convex functions defined, respectively, by

$$\mathcal{K}_q = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{D_q(zD_q f(z))}{D_q f(z)}\right) > 0,\ q \in (0,1),\ z \in \mathbb{D} \right\},$$

$$\mathcal{C}_q = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{D_q f(z)}{D_q h(z)}\right) > 0,\ h \in \mathcal{K}_q,\ q \in (0,1),\ z \in \mathbb{D} \right\}.$$

The classes $\mathcal{S}_q^*$, $\mathcal{K}_q$ and $\mathcal{C}_q$ reduces to the classes $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{C}$ as $q \to 1^-$. Here, $\mathcal{C}$ denotes the usual class of close-to-convex functions. Recently, $q$-calculus was studied in the theory of analytic functions by several researchers [10, 11, 22]. But research on $q$-calculus in connection with harmonic functions is fairly new and not much studied (see [16, 27]).

In [2], Ahuja *et al.* defined the class $\mathcal{H}_q$ consisting of $q$-harmonic functions in $\mathbb{D}$. A harmonic function $f = h + \overline{g}$ defined by (1) is said to be $q$-harmonic, locally univalent and sense-preserving in $\mathbb{D}$, denoted by $\mathcal{H}_q$, if and only if the second dilatation $w_q$ satisfies the condition

$$|w_q(z)| = \left| \frac{D_q g(z)}{D_q h(z)} \right| < 1, \tag{2}$$

where $q \in (0,1)$ and $z \in \mathbb{D}$. Denote by $\mathcal{S}_{\mathcal{H}_q}$ the class of all univalent, sense-preserving harmonic functions $f = h + \overline{g}$. If $g'(0) = 0$ in series (1), we get the class $\mathcal{S}^0_{\mathcal{H}_q}$ Note that as $q \to 1^-$, $\mathcal{H}_q$, $\mathcal{S}_{\mathcal{H}_q}$ and $\mathcal{S}^0_{\mathcal{H}_q}$ reduce to the families of $\mathcal{H}$, $\mathcal{S}_{\mathcal{H}}$ and $\mathcal{S}^0_{\mathcal{H}}$, respectively. Further, denote by $\mathcal{K}_{\mathcal{H}_q}$ ($\mathcal{K}^0_{\mathcal{H}_q}$), the subclass of $\mathcal{S}_{\mathcal{H}_q}$ ($\mathcal{S}^0_{\mathcal{H}_q}$), which consists of functions $f$ that map the unit disk $\mathbb{D}$ onto a convex region. The class $\mathcal{K}_{\mathcal{H}_q}$ ($\mathcal{K}^0_{\mathcal{H}_q}$) was studied by Ahuja *et al.* in [3].

The convolution or Hadamard product of analytic functions $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $F(z) = \sum_{n=1}^{\infty} A_n z^n$ in $\mathbb{D}$ is defined by $(f * F)(z) = \sum_{n=1}^{\infty} a_n A_n z^n$. The convolution of two harmonic functions $f = h + \overline{g}$ and $F(z) = H(z) + \overline{G(z)}$ is defined by

$$(f * F)(z) = (h * H)(z) + \overline{(g * G)(z)}.$$

Rusheweyh and Sheil-Small [26] showed that the convolution of two convex analytic functions is convex. But the convolution of two harmonic convex mappings need not to be convex, and it may even fail to be univalent. Therefore, it is interesting to study the convolution properties of two harmonic convex mappings. Convolution of harmonic functions has been studied in [4, 6, 7, 8, 18, 19, 20, 21, 24].

Let $f_\eta = h_\eta + \overline{g_\eta}$ be a subclass of harmonic mappings obtained by shearing of analytic vertical strip mapping

$$h_\eta(z) + g_\eta(z) = \frac{1}{2i\sin\eta}\log\left(\frac{1+ze^{i\eta}}{1+ze^{-i\eta}}\right), \quad (\pi/2 \le \eta < \pi). \tag{3}$$

Note that the function, defined by (3), is convex and univalent in $\mathbb{D}$. In addition, this function maps $\mathbb{D}$ onto $\Omega_\eta = \{w : \frac{\eta-\pi}{2\sin\eta} < \mathrm{Re}(w) < \frac{\eta}{2\sin\eta}\}$ or onto the convex hull of three points (one of which may be the point at infinity) on the boundary of $\Omega_\eta$. In other words, the image of $\mathbb{D}$ may be a vertical strip, a halfstrip, a trapezium, or a triangle.

In 1984, Clunie and Shiel-Small [5] introduced a technique, called as shearing technique, to construct new univalent harmonic functions on the open unit disk $\mathbb{D}$, and provided an interesting example of a univalent harmonic right half-plane mapping given by

$$T_0(z) = \frac{1}{2}\left[\frac{z}{1-z} + \frac{z}{(1-z)^2}\right] + \frac{1}{2}\overline{\left[\frac{z}{1-z} - \frac{z}{(1-z)^2}\right]}.$$

The mapping $T_0$ is the right half-plane mapping that maps the disk $\mathbb{D}$ onto the region $\{w : \mathrm{Re}(w) > -1/2\}$ (see [8]).

In 2012, Stacey Muir [23] considered a transformation $T_c[f]$, where $f$ is an analytic function such that $f(0) = 0$ and $f'(0) = 1$, to generate the new harmonic function

$$T_c[f](z) = \frac{1}{1+c}\left[f(z) + czf'(z)\right] + \frac{1}{1+c}\overline{[f(z) - czf'(z)]},$$

where $c > 0$ is some real number. Clearly, for $c = 1$ and $f(z) = z/(1-z)$ the transformation $T_c[f]$ becomes to the transformation given by $T_0$.

In this paper, motivated by the above discussed transformations and by using $q$-derivative operator, we generate a new family of locally univalent and sense-preserving functions. For an analytic function $f$ in $\mathbb{D}$ with $f(0) = f'(0) - 1 = 0$, we define the function

$$T_{c,q}[f](z) = \frac{1}{1+c}\left[f(z) + czD_q(f(z))\right] + \frac{1}{1+c}\overline{[f(z) - czD_q(f(z))]}, \qquad (4)$$

where $c > 0$, $q \in (0,1)$ and $z \in \mathbb{D}$. Then, if letting $f(z) = z/(1-z)$, we observe the generalized right half-plane mapping by

$$T_{c,q}[I](z) = \frac{1}{1+c}\left[\frac{z}{1-z} + \frac{cz}{(1-z)(1-qz)}\right] + \frac{1}{1+c}\overline{\left[\frac{z}{1-z} - \frac{cz}{(1-z)(1-qz)}\right]},$$

and for $c = 1$ and $f(z) = z/(1-z)$, we have

$$T_{1,q}[I](z) = \frac{1}{2}\left[\frac{z}{1-z} + \frac{z}{(1-z)(1-qz)}\right] + \frac{1}{2}\overline{\left[\frac{z}{1-z} - \frac{z}{(1-z)(1-qz)}\right]}. \qquad (5)$$

We note that in the limiting case $q \to 1^-$, the transformation $T_{c,q}[f]$ becomes $T_c[f]$. For $c = 1$, $f(z) = z/(1-z)$ and $q \to 1^-$, the transformation $T_{c,q}[f]$ becomes $T_0$. It is interesting to note that (5) provides the extremal function for $q$-convex function, where

$$|a_n| \le \frac{[n]_q + 1}{2}, \quad |b_n| \le \frac{[n]_q - 1}{2}. \qquad (6)$$

In this paper, we establish a necessary and sufficient condition for functions $T_{c,q}[f]$ to be locally univalent and sense preserving in $\mathbb{D}$. Further, we prove that the convolution of harmonic convex mappings $T_{c,q}[f]$, respectively, with a right half-plane mapping and with a vertical strip mapping are univalent and $q$-convex in the direction of the real axis provided that the convolution is sense-preserving.

## 2 Auxiliary lemmas

We need the following results to prove our main theorems in the next section.

**Lemma 1.** [5] *A locally univalent harmonic function $f = h + \overline{g}$ in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the direction $\eta$ if and only if $h - e^{2i\eta}g$ is an analytic univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of $\eta$.*

**Lemma 2.** [26] *Let $\phi$ and $G$ be analytic in $\mathbb{D}$ with $\phi(0) = G(0) = 0$. If $\phi$ is convex and $G$ is starlike, then for each function $F$ analytic in $D$ satisfying* $\operatorname{Re} F(z) > 0$, *we have*

$$\operatorname{Re}\frac{(\phi * FG)(z)}{(\phi * G(z))} > 0, \quad (z \in \mathbb{D}).$$

**Lemma 3.** [25] *Let $f$ be an analytic function in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) \neq 0$, and let*

$$\varphi(z) = \frac{z}{(1+ze^{i\theta})(1+ze^{-i\theta})}, \quad (\theta \in \mathbb{R}, z \in \mathbb{D}).$$

*If*

$$\mathrm{Re}\left(\frac{zf'(z)}{\varphi(z)}\right) > 0, \quad (z \in \mathbb{D})$$

*then $f$ is convex in the direction of the real axis.*

# 3 Main results

In this section, we first investigate necessary and sufficient conditions for $T_{c,q}[f]$ to be locally univalent and sense-preserving in the open unit disk $\mathbb{D}$.

**Lemma 4.** *The function $T_{c,q}[f]$, defined by* (4)*, is locally univalent if and only if $f$ is $q$-convex.*

*Proof.* Consider $T_{c,q}[f] = F = H + \overline{G}$. Since $F$ is locally univalent if and only if $|D_qG(z)| < |D_qH(z)|$ for all $z \in \mathbb{D}$. The function $F$ will be locally univalent if and only if

$$\left|\frac{(1-c)D_q(f(z)) - czD_q(D_q(f(z))}{(1+c)D_q(f(z)) + czD_q(D_q(f(z))}\right| < 1.$$

It can be seen that above inequality is equivalent to

$$\mathrm{Re}\left\{\frac{zD_q(zD_q(f(z)))}{zD_q(f(z))}\right\} = \mathrm{Re}\left\{1 + \frac{qzD_q^2(f(z))}{D_q(f(z))}\right\} > 0,$$

which is analytic condition for convexity. Thus $T_{c,q}[f]$ is locally univalent if and only if $f$ is $q$-convex. □

**Theorem 1.** *The function $T_{c,q}[f]$, defined by* (4)*, is $q$-convex in the direction of imaginary axis if and only if $f$ is $q$-convex.*

*Proof.* We have $T_{c,q}[f] = H + \overline{G}$. Where $H + G = \frac{2z}{(1+c)(1-z)}$, from Lemma 1 and Lemma 4, the proof is obvious. □

**Theorem 2.** *Let $\phi \in \mathcal{K}_q$ and the function $T_{c,q}[f]$, given by* (4)*, be locally univalent and sense-preserving in $\mathbb{D}$. Then $\phi * T_{c,q}[f] \in \mathcal{S}_{\mathcal{H}_q}$ and is $q$-convex in the direction of the imaginary axis provided $f \in \mathcal{K}_q$.*

*Proof.* By convolution, the functions $\phi$ and $T_{c,q}[f] = h + \overline{g}$ can be written as

$$\phi * T_{c,q}[f] = \frac{1}{1+c}\left[\phi * f + cD_q(\phi * f)\right] + \frac{1}{1+c}\overline{\left[\phi * f - cD_q(\phi * f)\right]}. \qquad (7)$$

The mapping $\phi * T_{c,q}[f] = \phi * h + \phi * \overline{g}$ is locally univalent and sense-preserving because $|\phi * D_qg(z)| < |\phi * D_qh(z)|$ for all $z \in \mathbb{D}$; thus, by Lemma 4

$$\left|\frac{(1-c)D_q(\phi * f(z)) - czD_q(D_q(\phi * f(z))}{(1+c)D_q(\phi * f(z)) + czD_q(D_q(\phi * f(z))}\right| < 1.$$

Hence, the mapping in (7) is locally univalent and sense-preserving if and only if $f$ is $q$-convex, and equivalently shown by

$$\begin{aligned}\operatorname{Re}\left\{\frac{zD_q(zD_q(\phi * f(z)))}{zD_q(\phi * f(z))}\right\} &= \operatorname{Re}\left\{\frac{zD_q(\phi * zD_q(f(z)))}{\phi * zD_q(f(z))}\right\}\\ &= \operatorname{Re}\left\{\frac{\phi * zD_q(zD_q(f(z)))}{\phi * zD_q(f(z))}\right\} > 0,\end{aligned}$$

where we have used the fact

$$D_q(f_1(z) * f_2(z)) = f_1(z) * zD_q f_2(z).$$

It is noted that, the above expression reduces to $(f_1(z)*f_2(z))' = f_1(z)*zf_2'(z)$ when $q$ approaches 1. In view of Lemma 4 and Theorem 1, we conclude that $\phi * f$ is $q$-convex in the direction of the imaginary axis. □

Next, we investigate convolutions of $T_{c,q}[f]$ with a half-plane mapping and with a vertical strip mapping are $q$-convex in the direction of the real axis. We first generate the $q$-case of Lemma 3 for proving the next theorems.

**Lemma 5.** *Let $f$ be an analytic function in $\mathbb{D}$ with $f(0) = 0$ and $(D_q f)(0) \neq 0$. Suppose that $\varphi$ is given by*

$$\varphi(z) = z(D_q\psi)(z), \quad \text{where} \quad \lim_{q\to 1^-} \varphi(z) = \frac{z}{(1+ze^{i\eta})(1+ze^{-i\eta})}, \tag{8}$$

*for $\eta \in \mathbb{R}, z \in \mathbb{D}$. If*

$$\operatorname{Re}\left(\frac{z(D_q f)(z)}{\varphi(z)}\right) > 0, \quad (z \in \mathbb{D})$$

*then $f$ is $q$-convex in the direction of the real axis.*

*Proof.* By Ahuja *et al.* [1], an analytic function $f$ is said to be $q$-close-to-convex if there exists a $q$-convex function $\psi$ such that

$$\operatorname{Re}\left(\frac{(D_q f)(z)}{(D_q\psi)(z)}\right) > 0, \tag{9}$$

where $q \in (0,1)$ and $z \in \mathbb{D}$. Since $\psi \in \mathcal{K}_q$, then $\varphi = z(D_q\psi) \in \mathcal{S}^*{}_q$. Thus, the $q$-close-to-convex function in (9) can be rewritten as

$$\operatorname{Re}\left(\frac{z(D_q f)(z)}{\varphi(z)}\right) > 0, \tag{10}$$

where $\varphi$ is a $q$-starlike function.

Now, let consider the function which is an analytic function that maps $\mathbb{D}$ onto a convex hull. Taking $q$-derivative of the function $\psi$, we get

$$\varphi(z) = z(D_q\psi)(z). \tag{11}$$

Since $\psi$ is $q$-convex in one direction, it maps $|z| = r < 1$ for every $r$ near 1 onto a contour $C$ that is cut by every straight-line parallel to this direction in two, and only two points $z_1 = re^{i\eta_1}$ and $z_2 = re^{i\eta_2}$ for $\eta \in \mathbb{R}$. Hence, the function $\varphi = z(D_q\psi)$ given by (11) is $q$-starlike in one direction. Due to Pommerenke [25], we conclude that the function $f$ in (10) is $q$-convex in one direction of the real axis if the $q$-starlike function $\varphi$ in (10) has the form given by (11). □

**Theorem 3.** *Let the function $T_{c,q}[f_1]$, where $f_1 = h_1 + g_1 = \frac{2z}{(1+c)(1-z)}$ with $\omega_{q_1} = -z$, and let the function $f_2 = h_2 + \overline{g_2}$, where $h_2 + g_2 = \frac{z}{1-z}$ with $\omega_{q_2} = -z$ be two harmonic convex mappings . Then $T_{c,q}[f_1] * f_2 \in \mathcal{S}^0_{\mathcal{H}_q}$ and is $q$-convex in the direction of the real axis provided $T_{c,q}[f_1] * f_2$ is locally univalent and sense preserving.*

*Proof.* Let

$$F_1 = (h_1 - g_1) * (h_2 + g_2) = h_1 * h_2 + h_1 * g_2 - h_2 * g_1 - g_1 * g_2 \tag{12}$$

and

$$F_2 = (h_1 + g_1) * (h_2 - g_2) = h_1 * h_2 + h_2 * g_1 - h_1 * g_2 - g_1 * g_2, \tag{13}$$

where

$$F_1 = (h_1(z) - g_1(z)) * \frac{z}{1-z} = (h_1(z) - g_1(z)) \tag{14}$$

and

$$F_2 = (h_2(z) - g_2(z)) * \frac{2z}{(1+c)(1-z)} = \frac{2}{(1+c)} (h_2(z) - g_2(z)). \tag{15}$$

Then from (12) and (13), we observe that

$$\frac{1}{2}(F_1 + F_2) = h_1 * h_2 - g_1 * g_2.$$

We only need to prove that $F_1 + F_2$ is $q$-convex in the direction of the real axis. From (14) and (15), we get

$$(F_1 + F_2)(z) = (h_1(z) - g_1(z)) + \frac{2}{(1+c)} (h_2(z) - g_2(z))$$

and

$$\begin{aligned} &zD_q (F_1 + F_2)(z) \\ =&z (D_q h_1(z) + D_q g_1(z)) p_1(z) + \frac{2}{(1+c)} z (D_q h_2(z) + D_q g_2(z)) p_2(z) \\ =&\frac{2z}{(1+c)(1-z)(1-qz)} p_1(z) + \frac{2z}{(1+c)(1-z)(1-qz)} p_2(z), \end{aligned} \tag{16}$$

where for each $j = 1, 2$,

$$p_i(z) = \frac{D_q h_i(z) - D_q g_i(z)}{D_q h_i(z) + D_q g_i(z)},$$

and where we have used the fact

$$z(D_q f)(z) = \frac{z}{(1-z)(1-qz)} * f(z). \tag{17}$$

Since $f_i \in \mathcal{K}^0_{\mathcal{H}_q}$, the dilatation $\omega_{q_i} = D_q g_i / D_q h_i$ satisfy $|\omega_{q_i}| < 1$, and hence

$$\operatorname{Re}(p_i(z)) = \operatorname{Re}\left(\frac{1-\omega_{q_i}(z)}{1+\omega_{q_i}(z)}\right) > 0.$$

From (16), we get

$$zD_q\left(F_1 + F_2\right)(z) = \frac{2z}{(1+c)(1-z)(1-qz)}\left(p_1(z) + p_2(z)\right),$$

and, by letting the $q-$starlike function $\varphi(z) = \frac{z}{(1-z)(1-qz)}$, we get

$$\operatorname{Re}\left(\frac{zD_q\left(F_1+F_2\right)(z)}{\varphi(z)}\right) = \operatorname{Re}\left(\frac{2}{(1+c)}\left(p_1(z)+p_2(z)\right)\right) > 0$$

for all $z \in \mathbb{D}$. By Lemma 5, it follows that $F_1 + F_2$ is $q$-convex in the direction of real axis. □

**Remark 1.** *Letting $c = 1$ and $q \to 1^-$, we get the result by Dorff and Rolf [8, Theorem 4.126] as particular case of Theorem 3.*

**Theorem 4.** *Let the function $T_{c,q}[f_1] = h_1 + g_1 \in \mathcal{K}^0_{\mathcal{H}_q}$ be a mapping with $h_1 + g_1 = \frac{2z}{(1+c)(1-z)}$, and let the function $f_\eta = h_\eta + \overline{g_\eta} \in \mathcal{K}^0_{\mathcal{H}_q}$ be a vertical strip mapping given by (3). If $\operatorname{Re}\left(\frac{zD_q(F_3+F_4)(z)}{\varphi(z)}\right) > 0$, and $T_{c,q}[f_1] * f_\eta$ is locally univalent and sense preserving, then $T_{c,q}[f_1] * f_\eta \in \mathcal{S}^0_{\mathcal{H}_q}$ and is $q$-convex in the direction of the real axis.*

*Proof.* Let

$$F_3 = (h_1 - g_1) * (h_\eta + g_\eta) = h_1 * h_\eta + h_1 * g_\eta - h_\eta * g_1 - g_1 * g_\eta$$

and

$$F_4 = (h_1 + g_1) * (h_\eta - g_\eta) = h_1 * h_\eta + h_\eta * g_1 - h_1 * g_\eta - g_1 * g_\eta,$$

where

$$\frac{1}{2}\left(F_3 + F_4\right) = h_1 * h_\eta - g_1 * g_\eta.$$

As previous theorem, we want to show that $F_3 + F_4$ is $q$-convex in the direction of the real axis. First, we observe

$$\begin{aligned} z\left(D_q F_3\right)(z) &= zD_q\big[(h_1 - g_1)(z) * (h_\eta + g_\eta)(z)\big] \\ &= z(D_q h_1(z) - D_q g_1(z)) * (h_\eta + g_\eta)(z) \\ &= \left[z(D_q h_1(z) + D_q g_1(z))\frac{D_q h_1(z) - D_q g_1(z)}{D_q h_1(z) + D_q g_1(z)}\right] * (h_\eta + g_\eta)(z) \\ &= \frac{2z}{(1+c)(1-z)(1-qz)} p_1(z) * (h_\eta + g_\eta)(z), \end{aligned}$$

where $\mathrm{Re}(p_1(z)) > 0$ for every $z \in \mathbb{D}$. Since $\psi = h_\eta + g_\eta$ is $q$-convex, then $\varphi = zD_q(h_\eta + g_\eta)$ is $q$-starlike, thus by applying Lemma 2 and the fact (17), we get

$$\begin{aligned}\mathrm{Re}\left(\frac{z(D_qF_3)(z)}{\varphi(z)}\right) &= \mathrm{Re}\left(\frac{(h_\eta + g_\eta)(z) * p_1(z)\frac{2z}{(1+c)(1-z)(1-qz)}}{zD_q(h_\eta + g_\eta)(z)}\right)\\ &= \mathrm{Re}\left(\frac{(h_\eta + g_\eta)(z) * \frac{2p_1(z)}{(1+c)}\frac{z}{(1-z)(1-qz)}}{(h_\eta + g_\eta)(z) * \frac{z}{(1-z)(1-qz)}}\right) > 0,\ (z \in \mathbb{D}). \quad (18)\end{aligned}$$

Similarly, for $F_4$, we get

$$\begin{aligned}z\,(D_qF_4)\,(z) &= zD_q\big[(h_1 + g_1)(z) * (h_\eta - g_\eta)(z)\big]\\ &= (h_1 + g_1)(z) * z(D_qh_\eta(z) - D_qg_\eta(z))\\ &= (h_1 + g_1)(z) * \left[z(D_qh_\eta(z) + D_qg_\eta(z))\frac{D_qh_\eta(z) - D_qg_\eta(z)}{D_qh_\eta(z) + D_qg_\eta(z)}\right]\\ &= \frac{2z}{(1+c)(1-z)} * zD_q(h_\eta + g_\eta)(z)p_2(z)\\ &= \frac{2}{(1+c)}zD_q(h_\eta + g_\eta)(z)p_2(z), \quad (19)\end{aligned}$$

where $\mathrm{Re}(p_2(z)) > 0$ for all $z \in \mathbb{D}$. Then, from (19), we have

$$\mathrm{Re}\left(\frac{z(D_qF_4)(z)}{\varphi(z)}\right) > 0, \quad (z \in \mathbb{D}). \quad (20)$$

Considering (18) and (20) together, we arrive at

$$\mathrm{Re}\left(\frac{zD_q(F_3 + F_4)(z)}{\varphi(z)}\right) > 0, \quad (z \in \mathbb{D}).$$

Hence, by Lemma 5, we observe that $F_3 + F_4$ is $q$-convex in the direction of real axis. □

**Remark 2.** *Letting $c = 1$ and $q \to 1^-$, we get the result by Dorff [6, Theorem 7] as particular case of Theorem 4.*

Finally, we shall use shearing technique to enforce our idea for providing an alternative approach to construct a harmonic univalent function by $q$-calculus in place of ordinary calculus, and illustrate an example as follows:

**Example 1.** *Let $f = h + \overline{g}$ with $h(z) - g(z) = z - \frac{1}{[2]_q}z^2$ and $\omega_q(z) = \frac{D_qg(z)}{D_qh(z)} = z$. We compute $h$ and $g$ explicitly so that $f \in S^0_{\mathcal{H}_q}$. So*

$$D_qh(z) - D_qg(z) = 1 - z, \qquad D_qg(z) = zD_qh(z), \quad (21)$$

*On $q$-integration and normalization gives*

$$h(z) = z \qquad g(z) = \frac{1}{[2]_q}z^2$$

*Thus the mapping $f = h + \bar{g}$ is given by*

$$f = z + \frac{1}{[2]_q}\overline{z^2}$$

In Example 1, by using $q$-calculus with a suitable dilatation, we generate a harmonic function.

## References

[1] Ahuja, O.P., Çetinkaya, A. and Polatoğlu, Y., *Bieberbach-de Branges and Fekete-Szegö inequalities for certain families of q-convex and q-close-to-convex functions*, J. Comput. Anal. Appl. **26** (2019), no. 4, 639–649.

[2] Ahuja, O.P. and Çetinkaya, A., *Connecting quantum calculus and harmonic starlike functions*, Filomat, **34** (2020), no. 5, 1431–1441.

[3] Ahuja, O.P., Çetinkaya, A. and Polatoğlu, Y., *Harmonic univalent convex functions using a quantum calculus approach*, Acta Universitatis Apulensis, **58** (2019), 67–81

[4] Beig, S. and Ravichandran, V., *Directional convexity of combinations of harmonic half-plane and strip mappings*, Commun. Korean Math. Soc. **37** (2022), 125–136.

[5] Clunie, J. and Sheil-Small, T., *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I Math. **9** (1984), 3–25.

[6] Dorff, M., *Convolutions of planar harmonic convex mappings*, Comp. Var. Theory Appl. **45** (2001), no. 3, 263–271.

[7] Dorff, M., Nowak, M. and Wołoszkiewicz, M., *Convolutions of harmonic convex mappings*, Complex Var. Elliptic Equ. **57** (2012), no. 5, 489–503.

[8] Dorff, M. and Rolf, J.S., *Anamorphosis, mapping problems, and harmonic univalent functions*, In Book: Explorations in Complex Analysis, 197–269, Math. Assoc. of America, Inc., Washington, DC, 2012.

[9] Gasper, G. and Rahman, M., *Basic hypergeometric series*, Cambridge University Press, Cambridge, 2004.

[10] Govindaraj, M. and Sivasubramanian, S., *On a class of analytic functions related to conic domains involving q-calculus*, Anal. Mathematica, **43** (2017), 475–487.

[11] Hussain, S., Khan, S., Zaighum, M.A. and Darus, M. *Certain subclass of analytic functions related with conic domains and associated with Sălăgean q-differential operator*, AIMS Mathematics, **2** (2017), 622–634.

[12] Ismail, M.E.H., Merkes, E. and Styer, D., *A generalizations of starlike functions*, Complex Variables, **14** (1990), 77–84.

[13] Jackson, F.H., *On q-functions and a certain difference operator*, Trans. Royal Soc. Edinburgh, **46** (1909), 253–281.

[14] Jackson, F.H., *On q-definite integrals*, Quart. J. Pure Appl. Math. **41** (1910), 193–203.

[15] Kac, V. and Cheung, P., *Quantum calculus*, Springer-Verlag, New York, 2002.

[16] Kanas, S. and Răducanu, D., *Some class of analytic functions related to conic domains*, Math. Slovaca. **64** (2014), 1183–1196.

[17] Lewy, H., *On the non-vanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc. **42** (1936), no. 10, 689–692.

[18] Li, Y. and Liu, Z., *Convolutions of harmonic right half-plane mappings*, Open Math. **14** (2016), 789–800.

[19] Li, L. and Ponnusamy, S., *Convolutions of slanted half-plane harmonic mappings*, Analysis, **33** (2013), no. 2, 159–176.

[20] Li, L. and Ponnusamy, S., *Note on the convolution of harmonic mappings*, Bull. Aust. Math. Soc. **99** (2019), no. 3, 421–431.

[21] Liu, Z., and Ponnusamy, S., *Univalency of convolutions of univalent harmonic right half-plane mappings*, Comput. Methods Funct. Theory, **17** (2017), no. 2, 289–302.

[22] Mahmood, S. and Sokol, J., *New subclass of analytic functions in conical domain associated with Ruscheweyh q-differential operator*, Results Math. **71** (2017), 1345–1357.

[23] Muir, S., *Harmonic mappings convex in one or every direction*, Comput. Methods Funct. Theory, **12** (2012), 221–239.

[24] Mishra, O., and Sokol, J., *Generalized q-starlike harmonic functions*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. 115, 176 (2021).

[25] Pommerenke, C., *On starlike and close-to-convex functions*, Proc. London Math. Soc. **13** (1963), 290–304.

[26] Ruscheweyh, S. and Sheil-Small, T., *Hadamard products of Schlicht functions and the Pólya-Schoenberg conjecture*, Comment. Math. Helv. **48** (1973), 119–136.

[27] Sharma, P. and Mishra, O., *A class of harmonic functionc associated with a q-Sălăgean operator*, U.P.B. Sci. Bull. Series A, **82** (2020), 3–12.