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HERMITIAN-TOEPLITZ DETERMINANTS FOR A SUBCLASS OF ANALYTIC FUNCTIONS

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Abstract

In this study, we obtain sharp bounds for the second Hermitian- Toeplitz determinants of a subclass of analytics functions in the open unit disk.

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1 Introduction and definitions

Let H be the class of analytic functions in the unit disk $\mathbb{E} := \{\xi \in \mathbb{C} : |\xi| < 1\}$, and let A be the subclass normalized by $l(0) := l'(0) - 1 := 0$, that is, functions of the form

$$l(\xi) = \sum_{r=1}^{\infty} a_r \xi^r, \ a_1 := 1 \quad (\xi \in \mathbb{E}). \tag{1}$$

Let S be a subclass of A that consists of univalent (one-to-one) functions. A function $l \in A$ is said to be starlike (with respect to the origin) if $l(\mathbb{E})$ is starlike with respect to the origin, and convex if $l(\mathbb{E})$ is convex. Let $S^*(\alpha)$ and $C(\alpha)$ denote, respectively, the classes of starlike and convex functions of order α $(0 \leq \alpha < 1)$ in S. It is well known that a function $l \in A$ belongs to $S^*(\alpha)$ if, and only if,

$$\mathrm{Re}\left(\frac{\xi l'(\xi)}{l(\xi)}\right) > \alpha \quad (\xi \in \mathbb{E}),$$

and that $l \in C(\alpha)$ if, and only if,

$$\mathrm{Re}\left(1 + \frac{\xi l''(\xi)}{l'(\xi)}\right) > \alpha \quad (\xi \in \mathbb{E}).$$

Note that $S^*(0) =: S^*$and $C(0) =: C$.

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Definition 1. *Let $l \in A$ and be locally univalent for $\xi \in \mathbb{E}$, and $0 \leq \alpha < 1$. Then, $l \in M(\alpha)$ if and only if*

$$\operatorname{Re}\left(\left(1-\xi^2\right)\frac{l(\xi)}{\xi}\right) > \alpha \qquad (\xi \in \mathbb{E}). \tag{2}$$

Due to their geometrical characteristics, this class has a significant impact on the theory of geometric functions. This function $l \in M(\alpha)$ maps univalently $\mathbb{E}$ onto a domain $l(\mathbb{E})$ convex in the direction of the imaginary axis, i.e., for $w_1, w_2 \in l(\mathbb{E})$ such that $\operatorname{Re} w_1 = \operatorname{Re} w_2$ the line segment $[w_1, w_2]$ lies in $l(\mathbb{E})$, with the additional property that there exist two points w_1, w_2 on the boundary of $l(\mathbb{E})$ for which $\{w_1 + it : t > 0\} \subset \mathbb{C}\backslash l(\mathbb{E})$ and $\{w_2 - it : t > 0\} \subset \mathbb{C}\backslash l(\mathbb{E})$ (see, e.g., [11, p.199]).

Definition 2. *Let $l \in A$, and be given by (1). Then, for $q \geq 1$ and $n \geq 0$, define*

$$T_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ \overline{a}_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{a}_{n+q-1} & \overline{a}_{n+q-2} & \cdots & a_n \end{vmatrix},$$

where $\overline{a}_k := \overline{a_k}$. When a_n is a real number , $T_{q,n}(f)$ is qth Hermitian-Toeplitz determinant.

In particular

$$T_{3,1}(f) = 1 - 2|a_2|^2 + 2\operatorname{Re}\left(a_2^2\overline{a}_3\right) - |a_3|^2.$$

Finding sharp bounds for the Hankel determinants of functions in A has been the subject of a great many papers in the recent years. In particular, many results are known concerning the second Hankel determinant $H_2(2) = a_2a_4 - a_3^2$ when $l \in S$ and its subclasses, and asummary of some of the more important results can be found in [19]. On the other hand, investigations concerning Toeplitz determinants were introduced only recently in [2]. Similarly, problems concerning Hermitian-Toeplitz determinants were first considered in [7].

We next discuss the Zalcman functional, its relationship with the Zalcman conjecture, and a generalization due to Ma [18]. In the early 70s, Lawrence Zalcman posed the conjecture that if $l \in S$, and is given by (1) then

$$\left|a_n^2 - a_{2n-1}\right| \leq (n-1)^2 \quad \text{for} \quad n \geq 2, \tag{3}$$

with equality for the Koebe function $k(z) = z/(1-z)^2$ for $z \in \mathbb{E}$, or a rotation. This conjecture implies the celebrated Bieberbach conjecture $|a_n| \leq n$ for $l \in S$ [5]. The elementary area theorem shows that the conjecture is true when $n = 2$ [8]. Kruskal established the conjecture when $n = 3$ [14], and more recently for $n = 4, 5, 6$ [15]. However, the Zalcman conjecture for $n > 6$ remains an open problem.

The conjecture has been proved for several subclasses of S, e.g., starlike, typically real, and close-to-convex functions [5], [17] and it is known that the Zalcman conjecture is asymptotically true [10]. Recently, Abu Muhana et al. [1] proved the conjecture for the class $M(\alpha)$.

Relevant to this paper is Ma's generalization of the Zalcman functional $a_n^2 - a_{2n-1}$, defined as follows.

Definition 3. *Let $l \in A$, and be given by (1). For $m, n \in \mathbb{N}\setminus\{1\}$, let $j_{m,n}(f) := a_m a_n - a_{m+n-1}$, and in particular, $J_{2,3}(f) = a_2 a_3 - a_4$.*

In [18], Ma conjecture that if $l \in S$, then for $m, n \in \mathbb{N}\setminus\{1\}$

$$|J_{m,n}(f)| \leq (n-1)(m+1).$$

The following results will be used for functions $p \in P$, the class of functions with positive real part in $\mathbb{E}$ given by

$$p(\xi) = 1 + \sum_{r=1}^{\infty} d_r \xi^r, \tag{4}$$

and because the coefficients a_2, a_3, and a_4 will be our main focus, we also need Lemma 4, which can easily be deduced from (1), (2) and (4).

Lemma 1. *([8]) Let $p \in P$ be given by (4), then $|d_r| \leq 2$, when $r \geq 1$. Also*

$$\left|d_2 - \frac{\upsilon}{2} d_1^2\right| \leq \max\{2,\ 2|\upsilon - 1|\} = \begin{cases} 2, & 0 \leq \upsilon \leq 2, \\ 2|\upsilon - 1|, & \textit{elsewhere.} \end{cases} \tag{5}$$

Lemma 2. *([9]) If $p \in P$ is given by (4), then*

$$|d_r - \upsilon d_k d_{r-k}| \leq 2\max\{1,\ |2\upsilon - 1|\}$$

for $\upsilon \in \mathbb{C}$, and $1 \leq k \leq r-1$.

Lemma 3. *([16]) Assume that $p \in P$, with coefficients given by (4), and $d_1 \geq 0$. Then, for some complex valued ζ with $|\zeta| \leq 1$ and some complex-valued y with $|y| \leq 1$*

$$2d_2 = d_1^2 + y\left(4 - d_1^2\right),$$

$$4d_3 = d_1^3 + 2\left(4 - d_1^2\right) d_1 y - d_1\left(4 - d_1^2\right) y^2 + 2\left(4 - d_1^2\right)\left(1 - |y|^2\right)\zeta.$$

Lemma 4. *Assume that $l \in M(\alpha)$, and is given by (1). Then*

$$a_2 = (1-\alpha) d_1, \tag{6}$$
$$a_3 = (1-\alpha) d_2 + 1, \tag{7}$$
$$a_4 = (1-\alpha)(d_3 + d_1), \tag{8}$$
$$a_5 = (1-\alpha)(d_2 + d_4) + 1, \tag{9}$$

where d_1, d_2, and d_3, d_4 are given by (4).

Proof. By (2) there exists $p \in P$ of the form (4) such that

$$\left(1-\xi^2\right)\frac{l(\xi)}{\xi} = p\left(\xi\right)\left(1-\alpha\right)+\alpha \qquad \left(\xi \ \in \mathbb{E}\right). \tag{10}$$

Substituting the series (1) and (4) into (10) by equating the coefficients we obtain (6), (7), (8) and (9). □

2 Hermitian-Toeplitz determinants

In this section, we compute sharp lower and upper bounds for

$$T_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ \overline{a}_2 & 1 & a_2 \\ \overline{a}_3 & \overline{a}_2 & 1 \end{vmatrix} = 2\,\mathrm{Re}\left(a_2^2\overline{a}_3\right) - 2\left|a_2\right|^2 - \left|a_3\right|^2 + 1 \tag{11}$$

over the class $M(\alpha)$.

Theorem 1. *If* $l \in M(\alpha)$, $0 \le \alpha < \frac{1}{2}$, *then*

$$T_{3,1}(f) \le \begin{cases} 2, & \alpha \in [0, 0.0554] \\ 4\alpha(1-\alpha)(4\alpha-5)+2, & \alpha \in \left[0.0554, \frac{1}{2}\right) \end{cases} \tag{12}$$

and

$$T_{3,1}(f) \ge \frac{(2\alpha^2-3\alpha+2)^2}{2\alpha-1} + 4\alpha(1-\alpha)+2. \tag{13}$$

Both inequalities are sharp.

Proof. First note that both $M(\alpha)$ and $T_{3,1}(f)$ are rotationally invariant, and so, we can assume that $d_1 = 2x$ for $x \in [0,1]$ .Thus, using (11), Lemma 3 and Lemma 4, we obtain

$$\begin{aligned} T_{3,1}(f) \;&=\; 8(1-\alpha)^2\left[2(1-\alpha)x^4 + 2x^2(1-\alpha)(1-x^2)\,\mathrm{Re}\,y\right] \\ &\quad -4(1-\alpha)^2\left(x^4 + 2x^2(1-x^2)\,\mathrm{Re}\,y + \left|y\right|^2(1-x^2)^2\right) \\ &\quad -4(1-\alpha)(x^2 + (1-x^2)\,\mathrm{Re}\,y) \end{aligned} \tag{14}$$

for some complex y with $|y| \le 1$.

We consider two cases : **A,** when $y \neq 0$. Then, $y = \left|y\right| e^{i\varphi}$ with $|y| \in (0,1]$, and $\varphi \in [0, 2\pi)$.

Thus, setting $t = x^2$, from 14, we get

$$T_{3,1}(f) = G(t, \left|y\right|, \varphi), \tag{15}$$

where

$$\begin{aligned} G(t,u,\varphi) \;&=\; 16(1-\alpha)^3(t^2 + t(1-t)u\cos\varphi \\ &\quad -4(1-\alpha)^2\left(t^2 + 2t(1-t)u\cos\varphi + u^2(1-t)^2\right) \\ &\quad -4(1-\alpha)(t + (1-t)u\cos\varphi) \end{aligned} \tag{16}$$

for $t \in [0,1]$, $u \in [0,1]$, $\varphi \in [0,2\pi]$. Since

$$\begin{aligned} G(t,u,\varphi) &= -4(1-\alpha)^2(1-t)^2u^2 + \\ &+4(1-\alpha)\left(2(1-\alpha)\left(1-2\alpha\right)t-1\right)(1-t)u\cos\varphi \\ &+4(1-\alpha)^2(3-4\alpha)t^2 - 4(1-\alpha)t \end{aligned} \tag{17}$$

and $\alpha \in \left[0, \frac{1}{2}\right)$, we see that

$$L(t,u) \le G(t,u,\varphi) \le P(t,u), \quad t \in [0,1]\,,\; u \in [0,1]\,,\; \varphi \in [0,2\pi]\,, \tag{18}$$

where

$$L(t,u) = G(t,u,\pi), \qquad P(t,u) = G(t,u,0).$$

I. We first discuss the inequality (12). We have

$$\begin{aligned} P(t,u) &= -4(1-\alpha)^2(1-t)^2u^2 + \\ &+4(1-\alpha)\left(2(1-\alpha)\left(1-2\alpha\right)t-1\right)(1-t)u \\ &+4(1-\alpha)^2(3-4\alpha)t^2 - 4(1-\alpha)t. \end{aligned}$$

When $t = 1$, i.e., $x = 1$, then $d_1 = 2$, so

$$P(1,u) = 4(1-\alpha)(4\alpha^2 - 7\alpha + 2), \quad u \in [0,1]\,. \tag{19}$$

Assume next that $t \in [0,1)$, and let

$$u_w = \frac{2(1-\alpha)(1-2\alpha)t - 1}{2(1-\alpha)(1-t)}.$$

We consider two further cases.

Case1. Suppose that $u_w \ge 1$ i.e., that $\frac{3-2\alpha}{4(1-\alpha)^2} \le t \le 1$. Then

$$P(t,u) \le P(t,1) = 8(1-\alpha)^2(1-2\alpha)t - 4(1-\alpha)(2-\alpha) + 2.$$

Since

$$t_w = \frac{2(1-\alpha)(2-\alpha)-1}{4(1-\alpha)^2(1-2\alpha)} < \frac{(3-2\alpha)}{4(1-\alpha)^2},$$

$$P(t,1) \le P(1,1) = 4\alpha(1-\alpha)(4\alpha-5) + 2.$$

Case2. Suppose that $0 \le u_w < 1$, i.e., that $0 \le t < \frac{(3-2\alpha)}{4(1-\alpha)^2}$. Then

$$P(t,u) \le P(t,u_w) = 16(1-\alpha)^4t^2 - 8(1-\alpha)^2t + 1 \le P(0,u_w) = 1.$$

Noting now that

$$P(1,1) = 4\alpha(1-\alpha)(4\alpha-5) + 2 < 1 = P(0,u_w)$$

if, and only if, $\alpha \in (0,\, 0.0554)$, combining (19) with (18) and (15), inequality (12) follows in the case when $y \neq 0$.

II. We next discuss the inequality (13).

We have

$$\begin{aligned}L(t,u) &= -4(1-\alpha)^2(1-t)^2u^2\\ &\quad -4(1-\alpha)\left(2(1-\alpha)\left(1-2\alpha\right)t-1\right)(1-t)u\\ &\quad +4(1-\alpha)^2(3-4\alpha)t^2+4(1-\alpha)(2\alpha-3)t+2\end{aligned}$$

for $t\in[0,1]$, $u\in[0,1]$.

When $t=1$, i.e., $x=1$, so far $d_1=2$, we have

$$L(1,u)=4\alpha(1-\alpha)(4\alpha-5)+2,\quad u\in[0,1]\,. \tag{20}$$

Assume next that $t\in[0,1)$ and let

$$u_w^{'}=\frac{-2(1-\alpha)(1-2\alpha)t-1}{2(1-\alpha)(1-t)}.$$

Since $u_w^{'}\leq 0$, for $t\in[0,1)$, we have

$$\begin{aligned}L(t,u) \;\geq\; & L(t,1)=-4(1-\alpha)^2(1-t)^2-4(1-\alpha)\left(2(1-\alpha)\left(1-2\alpha\right)t-1\right)(1-t)\\ & +4(1-\alpha)^2(3-4\alpha)t^2+4(1-\alpha)(2\alpha-3)t+2.\end{aligned}$$

Let

$$t_w^{'}=\frac{(2\alpha^2-3\alpha+2)}{4(1-\alpha)(1-2\alpha)}.$$

It is easy to check, $t_w^{'}<1$, so

$$L(t,1)\geq L(t_w^{'},1)=\frac{-(2\alpha^2-3\alpha+2)^2}{(1-2\alpha)}+4\alpha(1-\alpha)+2,\quad t\in[0,1)\,. \tag{21}$$

Note now that the inequality $L(1,u)\geq L(t_w^{'},1)$, i.e., the inequality

$$4\alpha(1-\alpha)(4\alpha-5)+2\geq\frac{-(2\alpha^2-3\alpha+2)^2}{(1-2\alpha)}+4\alpha(1-\alpha)+2.$$

Therefore, (21), together with (20), (15) and (18), proves (13) in this case also.

This completes the proof of the theorem in case **A**.

B. Now, assume that $y=0$. Since

$$T_{3,1}(f)=4(1-\alpha)^2(3-4\alpha)t^2-4(1-\alpha)t=G(t,0,\varphi)$$

for $t\in[0,1]$ and $\varphi\in[0,2\pi]$, and noting that (18) is true for $u=0$, by Parts I and II above, both inequalities (12) and (13) are true. □

3 The functional $J_{2,3}(f)$

We give the sharp upper bound for $|J_{2,3}(f)|$ when $f \in M(\alpha)$.

Theorem 2. *If* $l \in M(\alpha)$, $0 \le \alpha < 1$, *then*

$$|J_{2,3}(f)| \le 2(1-\alpha)(1-2\alpha).$$

The inequality is sharp.

Proof. From Lemma 4, we have

$$a_2a_3 - a_4 = (1-\alpha)^2 d_1 d_2 - (1-\alpha)d_3. \tag{22}$$

Noting that both $M(\alpha)$ and $J_{2,3}(f)$ are rotationally invariant, we now use Lemma 3 to express the coefficients d_3 and d_2 in terms of d_1, and write $u = d_1$ to obtain with $0 \le u \le 2$

$$a_2a_3 - a_4 = \frac{(1-\alpha)}{4}\left[(1-2\alpha)u^3 + u(4-u^2)y^2 - 2(4-u^2)(1-|y|^2)\zeta\right].$$

$$|a_2a_3 - a_4| \le \frac{(1-\alpha)}{4}\left[(1-2\alpha)u^3 + u(4-u^2)t^2 - 2(4-u^2)(1-t^2)\right] = \psi(u,t),$$

where $t = |y| \in [0,1]$. When $u = 2$, $0 \le \alpha < 1$

$$|a_2a_3 - a_4| \le 2(1-\alpha)(1-2\alpha).$$

□

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