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SOLUTIONS FOR A NONLOCAL ELLIPTIC EQUATION WITH CRITICAL SOBOLEV EXPONENT AND SINGULAR TERM

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Abstract

In this paper, we consider a class of nonhomogeneous fractional elliptic equations involving critical Hardy Sobolev exponents as follows

$$\begin{cases} (-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = |u|^{2_s^*-2}u + \lambda \frac{u}{|x|^{2s-\alpha}} + f(x), & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $0 < s < 1$, $\lambda > 0$ is a parameter. We prove the existence of multiple solutions using the variational methods and the Nehari manifold decomposition.

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1 Introduction

The paper deals with the following fractional Hardy-Sobolev equation with nonhomogeneous term

$$\begin{cases} (-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = |u|^{2_s^*-2}u + \lambda \frac{u}{|x|^{2s-\alpha}} + f(x), & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1}$$

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being $0 < s < 1$, where Ω is a bounded domain in $\mathbb{R}^N$, $(N > 2s)$ containing the origin 0 in its interior, $0 \leq \mu < \overline{\mu_s} := 2^{2s}\frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}$, λ is a positive parameter, $0 < \alpha < 2s$, $2^*_s = \frac{2N}{N-2s}$ is the fractional critical Hardy-Sobolev exponent. The fractional Laplacian $(-\Delta)^s$ is defined by

$$-2(-\Delta)^s u(x) = C_{N,s}\int\limits_{\mathbb{R}^N}\frac{u(x+y)+u(x-y)-2u(x)}{|x-y|^{N+2s}}dy$$

where

$$C_{N,s} = \frac{4^s\Gamma(N\setminus 2+s)}{\pi^{N\setminus 2}|\Gamma(-s)|}.$$

Γ is the Gamma function, f is a given bounded measurable function.

It has been seen that the fractional differential equations have better effects in many realistic applications than the classical ones. Qualitative theory and its applications in physics, engineering, economics, biology, and ecology are extensively discussed and demonstrated in [5, 6, 8, 11, 12, 13] and the references therein.

There have been by now a large number of papers concerning the existence, nonexistence as well as qualitative properties of nontrivial solutions to critical elliptic problems of Hardy potential and fractional Laplace operator. For instance, Bennour and all in [1] handled the following singular equation

$$\begin{cases}\Delta^2 u - \mu\left(\frac{u}{|x|^4}\right) = \left(\frac{|u|^{2^*(s)-2}u}{|x|^s}\right) + \lambda\left(\frac{u}{|x|^{4-\alpha}}\right) + f(x), & x \in \Omega,\\ u = \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega,\end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and $N \geq 5$,Under sufficient conditions on the data, the existence and multiplicity of solutions was proven, via Ekland's variation principle and the Mountain Pass Lemma principle.

In the local setting case $(s = 1)$ the problem (1) is reduced to the semilinear problem with Sobolev-Hardy exponents

$$\begin{cases}-\Delta u - \mu\frac{u}{|x|^2} = |u|^{2^*_s-2}u + \lambda\frac{u}{|x|^{2-\alpha}} + f(x), & x \in \Omega,\\ u = 0 & x \in \partial\Omega.\end{cases}$$

This problem was further studied by Chen and Rocha [4], who based on variational methods obtained the existence of four non-trivial solutions.

Recently, the existence of nontrivial solutions for nonlinear fractional elliptic equations with Hardy's potential type

$$(-\Delta)^s u - \mu\frac{u}{|x|^{2s}} = g(u) \tag{2}$$

have been studied by several authors. Wang and all [15] studied (2) with $g(u) = |u|^{2^*-2}u + au$, $a > 0$ and discussed the infinitely many solutions. Daoues and all [7] studied (2) with $g(u) = \lambda|u|^{q-2}u + \frac{|u|^{2^*(t)-2}u}{|x|^t}$ and obtained the existence and

nonexistence of nonnegative distributional solutions.
In what follows, we state the main result for which we consider the following hypothesis

$$\inf\left\{\gamma_{N,s}(T(u))^{\frac{N+2s}{4s}}-\int\limits_{\Omega} fudx:\ u\in X,\int\limits_{\Omega}|u|^{2^*_s}dx=1\right\}>0.\quad (\mathcal{F})$$

Where X is a Hilbert space defined as

$$X=\{u\in H^{2s}(\mathbb{R}^N):u=0\text{ in }\mathbb{R}^N\setminus\Omega\},$$

where $H^{2s}(\mathbb{R}^N)$ the usual fractional Sobolev space,

$$\gamma_{N,s}=\frac{4s}{N-2s}(\frac{N-2s}{N+2s})^{\frac{N+2s}{4s}}$$

and

$$T(u)=C_{N,s}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy-\mu\int\limits_{\Omega}\frac{u^2}{|x|^{2s}}dx-\lambda\int\limits_{\Omega}\frac{u^2}{|x|^{2s-\alpha}}dx.$$

Thus, we write our results in the following theorem:

Theorem 1. *Let* $0<\mu<\overline{\mu_s}$, $0<\lambda<\lambda_1$ *and* f *is a bounded measurable function satisfying the condition* $(\mathcal{F})$, *then* (1) *has at least two nontrivial solutions, if* $0<\alpha<2\beta^+(\mu)+2s-N$.

This paper is organized as follows: in the forthcoming section, we give some preliminaries and technical lemmas used in our work. Section 3 is concerned by the proofs of our main results. In the following discussions, we shall denote various positive constants as c. $\mathcal{O}(\varepsilon^t)$ means that $|\mathcal{O}(\varepsilon^t)\varepsilon^{-t}|\leq c$, as $\varepsilon\to 0$, and $o(1)$ is an infinitesimal value, $\to$ (respectively, $\rightharpoonup$) will denote strongly (respectively, weakly) convergence. We denote the norme of X^- (the dual of X) by $\|.\|_-$.

2 Notations and preliminary results

2.1 A functional framework for the nonlocal problems

The embedding $X\hookrightarrow L^r(\Omega)$ is continuous for any $r\in[1;2^*]$ and compact for any $r\in[1;2^*)$. The space X is endowed with the norm defined as

$$\|u\|^2:=C_{N,s}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy-\mu\int\limits_{\Omega}\frac{u^2}{|x|^{2s}}dx,$$

by using fractional Hardy inequality [10],

$$\int_{\Omega} \frac{u^2}{|x|^{2s}} dx \leq \frac{1}{\overline{\mu_s}} C_{N,s} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy, \quad \forall u \in X,$$

we find that the norm $||.||$ is equivalent to the usual norm

$$\left(C_{N,s} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{\frac{1}{2}}.$$

From the fractional Sobolev-Hardy inequality, for $\mu \in [0; \overline{\mu_s}[$, we can define the best constant of fractional Sobolev-Hardy

$$A_{\mu,s}(\Omega) := \inf_{u \in X\setminus\{0\}} \frac{C_{N,s} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy - \mu \int_{\Omega} \frac{u^2}{|x|^{2s}} dx}{\left(\int_{\Omega} |u|^{2_s^*} dx \right)^{\frac{2}{2_s^*}}}.$$

Ghoussoub and Shakerian [9] proved that there exists radial solutions $U_\mu(x) \in H^s(\mathbb{R}^N)$ positive, symmetric, decreasing and solves

$$\begin{cases} (-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = |u|^{2_s^*-2} u & \text{in } \mathbb{R}^N, \\ u \geq 0 & \text{in } \mathbb{R}^N, \end{cases}$$

satisfying $\lim_{|x|\to\infty} U_\mu(x) = 0$ and $U_\mu \in C^1(\mathbb{R}^N \setminus \{0\})$. Furthermore, U_μ has the following properties

$$\lim_{|x|\to 0} |x|^{\beta^-(\mu)} U_\mu(x) = \delta_0,$$
$$\lim_{|x|\to\infty} |x|^{\beta^+(\mu)} U_\mu(x) = \delta_\infty,$$

where δ_0 and δ_∞ are positive constants and $\beta^-(\mu)$, $\beta^+(\mu)$ are zeros of the function

$$\Psi_{N,s}(\beta) = 4^s \frac{\Gamma(\frac{N-\beta}{2})\Gamma(\frac{2s+\beta}{2})}{\Gamma(\frac{N-2s-\beta}{2})\Gamma(\frac{\beta}{2})} - \mu, \;\; \beta > 0, \;\; 0 \leq \mu < \overline{\mu_s},$$

and satisfying

$$0 \leq \beta^-(\mu) < \frac{N-2s}{2} < \beta^+(\mu) \leq N-2s.$$

The author in [9] proved that $A_{\mu,s}$ is attained in $\mathbb{R}^N$ by the function

$$y_\varepsilon(x) = \varepsilon^{\frac{2s-N}{2}} U_\mu\left(\frac{x}{\varepsilon}\right), \quad \forall \varepsilon > 0,$$

and achieved

$$C_{N,s}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|y_\varepsilon(x) - y_\varepsilon(y)|^2}{|x-y|^{N+2s}} dxdy - \mu\int\limits_\Omega \frac{y_\varepsilon^2}{|x|^{2s}} dx = \int\limits_\Omega |y_\varepsilon|^{2_s^*} dx = A_{\mu,s}^{\frac{N}{2s}}.$$

Due to the fractional Hardy inequality, the operator $\mathcal{L} = (-\Delta)^s - \mu\frac{1}{|x|^{2s}}$ is defined on X. Moreover, the following eigenvalue problem with Hardy potentials and singular coefficient

$$\begin{cases} (-\Delta)^s u - \mu\frac{u}{|x|^{2s}} = \lambda\frac{u}{|x|^{2s-\alpha}} & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases}$$

where $0 < \alpha < 2s$, $\lambda \in \mathbb{R}$, has the first eigenvalue λ_1 given by:

$$\lambda_1 = \inf_{u \in X\setminus\{0\}} \frac{C_{N,s}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy - \mu\int\limits_\Omega \frac{u^2}{|x|^{2s}} dx}{\int\limits_\Omega \frac{u^2}{|x|^{2s-\alpha}} dx}.$$

Definition 1. *A functional $I \in C^1(X, \mathbb{R})$ satisfies the Palais-Smaile condition at level c, ($(PS)_c$ for short), if any sequence $(u_n) \subset X$ such that*

$$I(u_n) \to c \text{ and } I'(u_n) \to 0 \text{ in } X^{-1} \text{ (dual of } X),$$

contains a strongly convergent subsequence.

Definition 2. *We say that $u \in X$ is a weak solution of the problem (1) if for all $\varphi \in X$, one has*

$$C_{N,s}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} dxdy - \mu\int\limits_\Omega \frac{u\varphi}{|x|^{2s}} dx - \lambda\int\limits_\Omega \frac{u\varphi}{|x|^{2s-\alpha}} dx$$
$$- \int\limits_\Omega |u|^{2_s^*-2} u\varphi dx - \int\limits_\Omega f\varphi dx = 0.$$

2.2 Nehari manifold

The energy functional associated to (1) is given by the following expression:

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\int\limits_\Omega \frac{u^2}{|x|^{2s-\alpha}} dx - \frac{1}{2_s^*}\int\limits_\Omega |u|^{2_s^*} dx - \int\limits_\Omega fu dx, \qquad \forall u \in X.$$

We see that I is well defined in X and belongs to $C^1(X,\mathbb{R})$ and is not bounded. Moreover, all the critical points of I are precisely the solutions of (1). We define the Nehari manifold $\mathcal{N}$ associated with the functional by

$$\mathcal{N} := \{u \in X, \langle I'(u), u\rangle = 0\}.$$

It is usually effective to consider the existence of critical points in this smaller subset of the Sobolev space. We can split $\mathcal{N}$ for:

$$\mathcal{N}^+ := \{u \in \mathcal{N}, \langle I''(u), u\rangle > 0\},$$
$$\mathcal{N}^0 := \{u \in \mathcal{N}, \langle I''(u), u\rangle = 0\},$$
$$\mathcal{N}^- := \{u \in \mathcal{N}, \langle I''(u), u\rangle < 0\}.$$

Define

$$\begin{aligned} c\ &= \inf\{I(u)\ ,\ u \in \mathcal{N}\ \},\\ c^+ &= \inf\{I(u)\ ,\ u \in \mathcal{N}^+\},\\ c^- &= \inf\{I(u)\ ,\ u \in \mathcal{N}^-\}. \end{aligned}$$

2.3 Some technical lemmas

Lemma 1. *If $\mu \in]0;\overline{\mu_s}[$, $\alpha > 0$ and $0 < \lambda < \lambda_1$, then*

$$\inf\{(T(u))^{\frac{1}{2}} : \int_\Omega |u|^{2_s^*} dx = 1\} = M > 0.$$

In particular

$$T(u) \geq \eta\|u\|^2 \tag{3}$$

where $\eta = 1 - \frac{\lambda}{\lambda_1}$.

Proof. . The proof is similar to [2]. □

Lemma 2. *Let $f \neq 0$ satisfying the condition $(\mathcal{F})$, then $\mathcal{N}^0 = \emptyset$ and $c = c^+$.*

Proof. . The lemma is proved in the same way as in [14]. □

Let the cut-off function $\varphi(x) = \varphi(|x|) \in C_0^\infty(\Omega)$ such that $0 \leq \varphi(x) \leq 1$ in $B(0,R)$ and $\varphi(x) = 1$ in $B(0,\frac{R}{2})$. Set $u_\varepsilon = \varphi(x)y_\varepsilon(x)$, the following asymptotic properties hold.

Proposition 1. *Suppose that $N > 2s$, $\mu \in [0;\overline{\mu_s}[$. Then*

(1) $\|u_\varepsilon\|^2 = A_{\mu,s}^{\frac{N}{2s}} + \mathcal{O}(\varepsilon^{2\beta^+(\mu)+2s-N})$.

(2) $\int_\Omega |u_\varepsilon|^{2_s^*} dx = A_{\mu,s}^{\frac{N}{2s}} + \mathcal{O}(\varepsilon^{2_s^*\beta^+(\mu)-N})$.

(3) $\int_\Omega \frac{|u_\varepsilon|^2}{|x|^{2s-\alpha}} dx = \mathcal{O}(\varepsilon^\alpha)$, *where* $0 < \alpha < 2\beta^+(\mu) + 2s - N$.

(4) $\int\limits_{\Omega} u_\varepsilon dx = \mathcal{O}(\varepsilon^{\frac{2\beta^+(\mu)+2s-N}{2}})$.

Proof. . For the estimates (1), (2) and (4) one can see in [16], we only verify (3)

$$\begin{aligned}\int\limits_{\Omega} |x|^{\alpha-2s} u_\varepsilon^2 dx &= \int\limits_{\Omega\setminus B(0,\frac{R}{2})} |x|^{\alpha-2s} u_\varepsilon^2 dx + \int\limits_{B(0,\frac{R}{2})} |x|^{\alpha-2s} u_\varepsilon^2 dx \\ &= \mathcal{O}(\varepsilon^{2s-N+2\beta^+(\mu)}) + \omega_N \int_0^{\frac{R}{2}} \rho^{\alpha-2s} y_\varepsilon^2(\rho)\rho^{N-1} d\rho \\ &= \mathcal{O}(\varepsilon^{2s-N+2\beta^+(\mu)}) + \omega_N \varepsilon^{2s-N} \int_0^{\frac{R}{2}} \rho^{\alpha-2s-N-1} U_\mu^2(\frac{\rho}{\varepsilon})\rho^{N-1} d\rho \\ &= \mathcal{O}(\varepsilon^\alpha),\end{aligned}$$

ω_N is the area of the sphere S^{N-1}. □

Lemma 3. *Let $f \neq 0$ satisfies $(\mathcal{F})$. For every $u \in X$, $u \neq 0$ there exists a unique $t^+ = t^+(u) > 0$ such that $t^+u \in \mathcal{N}^-$. In particular:*

$$t^+ > \left[\frac{T(u)}{(2_s^*-1)\int\limits_{\Omega} |u|^{2_s^*} dx}\right]^{\frac{N-2s}{4s}} = t_{\max}(u)$$

and $I(t^+u) = \max\limits_{t\geq t_{\max}} I(tu)$. Moreover, if $\int\limits_{\Omega} f u dx > 0$, then there exists a unique $t^- = t^-(u) > 0$ such that $t^-u \in \mathcal{N}^+$, $t^- < t_{\max}(u)$ and $I(t^-u) = \min\limits_{0\leq t\leq t_{\max}} I(tu)$.

Proof. The lemma is proved in the same way as in [7]. □

Lemma 4. *Let $f \neq 0$ satisfies $(\mathcal{F})$. For each $u \in \mathcal{N}\setminus\{0\}$, there exist $\varepsilon > 0$ and a differentiable function $t = t(w) > 0$, $w \in X\setminus\{0\}$, $\|w\| < \varepsilon$, satisfying the following:*

$$t(0) = 1,\ t(w)(u-w) \in \mathcal{N},\ \forall\ \|w\| < \varepsilon,$$

$$\begin{aligned}\langle t'(0), v\rangle = \Bigg(&2C_{N,s}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dxdy - \\ &- \int\limits_{\Omega} (2(\frac{\mu}{|x|^4} + \frac{\lambda}{|x|^{4-\alpha}})uv + 2_s^*|u|^{2_s^*-2}uv + fv)dx\Bigg) \Big/ \Bigg(T(u) - (2_s^*-1)\int\limits_{\Omega} |u|^{2_s^*} dx\Bigg).\end{aligned}$$

Proof. . Define the map $F : \mathbb{R}\times X \to \mathbb{R}$,

$$F(t,w) = tT(u-w) - t^{2_s^*-1}\int\limits_{\Omega} |u-w|^{2_s^*} dx - \int\limits_{\Omega} (u-w) f dx.$$

Since

$$F(1,0)=0,\ \ \frac{\partial F}{\partial t}(1,0)=T(u)-(2_s^*-1)\int\limits_{\Omega}|u|^{2_s^*}dx\neq 0,$$

applying the implicit function theorem at the point $(1,0)$ we can get the result of this lemma. □

3 Proof of Theorem 1

The current section contains two subsections. We consider $0<\lambda<\lambda_1$ and $0<\mu<\overline{\mu_s}$.

3.1 Existence of solution in $\mathcal{N}^+$

Using Ekeland's variational principle, we prove the existence of a solution in $\mathcal{N}^+$.

Proposition 2. *Let f be a function satisfying $(\mathcal{F})$. Then $c=\inf\limits_{u\in\mathcal{N}}I(u)$ is achieved at a point $u_0\in\mathcal{N}^+$ which is a critical point and even a local minimum for I.*

Proof. . We start by showing that I is bounded from below in $\mathcal{N}$. Indeed for $u\in\mathcal{N}$ we have

$$T(u)-\int\limits_{\Omega}|u|^{2_s^*}dx-\int\limits_{\Omega}fudx=0.$$

Thus

$$\begin{aligned}I(u)=\frac{1}{2}T(u)-\frac{1}{2_s^*}\int\limits_{\Omega}|u|^{2_s^*}dx-\int\limits_{\Omega}fudx&=\frac{s}{N}T(u)-\frac{N+2s}{2N}\int\limits_{\Omega}fudx\\&\geq-\frac{(N+2s)^2}{16Ns\eta}\|f\|_-^2.\end{aligned}$$

In particular,

$$c\geq-\frac{(N+2s)^2}{16Ns\eta}\|f\|_-^2.$$

From Lemma 3, we can get $t_0=t^-(v)$ such that $t_0v\in\mathcal{N}^+$. Moreover,

$$\begin{aligned}I(t_0v)&=\frac{1}{2}t_0^2T(v)-\frac{t_0^{2_s^*}}{2_s^*}\int\limits_{\Omega}|v|^{2_s^*}dx-t_0\int\limits_{\Omega}fvdx\\&=-\frac{1}{2}t_0^2T(v)+\left(1-\frac{1}{2_s^*}\right)t_0^{2_s^*}\int\limits_{\Omega}|v|^{2_s^*}dx\\&<-\frac{s}{N}t_0^2T(v)<0.\end{aligned}$$

Hence,

$$c\leq I(t_0v)<0. \tag{4}$$

Applying the Ekeland's variational principle to the minimization problem (1), we can get a minimizing sequence $(u_n)_n \subset \mathcal{N}^+$ satisfying :

(i) $I(u_n) < c + \frac{1}{n}$,

(ii) $I(u_n) \leq I(w) + \frac{1}{n}\|w - u_n\|, \ \forall\, w \in \mathcal{N}$.

By taking n large enough, we get from (4) :

$$I(u_n) = \frac{s}{N}T(u_n) - \frac{N+2s}{2N}\int_\Omega f u_n dx < c + \frac{1}{n} \leq -\frac{s}{N}t_0^2 T(u_n).$$

This implies that

$$\int_\Omega f u_n dx \geq \frac{2st_0^2}{N+2s}T(u_n) \tag{5}$$

consequently, $u_n \neq 0$ and we have:

$$\frac{2s}{N+2s}\frac{t_0^2}{\|f\|_-}T(u_n) \leq \|u_n\| \leq \frac{N+2s}{2s\eta}\|f\|_-. \tag{6}$$

Next, we shall prove that $\|I'(u_n)\| \to 0$ as $n \to +\infty$.
Suppose that $\|I'(u_n)\| > 0$ for n be large enough. By Applying Lemma 4 with $u = u_n$ and $w = \sigma(\frac{I'(u_n)}{\|I'(u_n)\|})$, $\sigma > 0$, we can find some $t_n(\sigma) = t\sigma(\frac{I'(u_n)}{\|I'(u_n)\|})$ such that

$$w_\sigma = t_n(\sigma)\ \left[u_n - \sigma\frac{I'(u_n)}{\|I'(u_n)\|}\right]\ \in \mathcal{N}.$$

By condition (ii), we obtain:

$$\begin{aligned}\frac{1}{n}\|w - u_n\| &\geq I(u_n) - I(w_\sigma)\\ &= (1 - t_n(\sigma))\langle I'(w_\sigma), u_n\rangle + \sigma t_n(\sigma)\langle I'(w_\sigma), \frac{I'(u_n)}{\|I'(u_n)\|}\rangle + o(\sigma).\end{aligned}$$

Dividing by σ and passing to the limit as σ goes to zero we derive that:

$$\begin{aligned}\frac{1}{n}(1 + |t_n'(0)|\ \|u_n\|) &\geq -t_n'(0)\langle I'(u_n), u_n\rangle + \|I'(u_n)\|\\ &= \|I'(u_n)\|,\end{aligned}$$

where $t_n'(0) = \langle t'(0), \frac{I'(u_n)}{\|I'(u_n)\|}\rangle$. So, we conclude that:

$$\|I'(u_n)\| \leq \frac{C}{n}(1 + |t_n'(0)|),\ C > 0.$$

The proof will be completed once we have shown that $|t_n'(0)|$ uniformly bounded with respect to n. From Lemma 4 and the estimate (6), we get:

$$|t_n'(0)| \leq \frac{C_1}{|T(u_n) - (2_s^* - 1)\int_\Omega |u_n|^{2_s^*}dx}.$$

C_1 is a suitable constant.
Hence we must prove that $|T(u_n)-(2_s^*-1)\int\limits_\Omega |u_n|^{2_s^*}dx|$ is bounded away from zero. Arguing by contradiction, assume that for a subsequence still called (u_n), we have

$$|T(u_n)-(2_s^*-1)\int\limits_\Omega |u_n|^{2_s^*}dx| = o(1). \tag{7}$$

According to (6) and (7), there exists a constant $C_2>0$ such that $\int\limits_\Omega |u_n|^{2_s^*}dx \geq C_2$. In addition, from (7) and by the fact that $u_n \in \mathcal{N}$, we get

$$\int\limits_\Omega f u_n dx = (2_s^*-2)\int\limits_\Omega |u_n|^{2_s^*}dx + o(1).$$

This together with $(\mathcal{F})$ imply that

$$0 < (2_s^*-2)\left[\left(\frac{T(u_n)}{(2_s^*-1)\int\limits_\Omega |u_n|^{2_s^*}dx}\right)^{\frac{2_s^*-1}{2_s^*-2}} - 1\right] = o(1)$$

which is clearly impossible. In conclusion,

$$I'(u_n) \to 0 \text{ as } n \to +\infty. \tag{8}$$

Let $u_0 \in X$ be the weak limit in X of $(u_n)_n$.
From (5) we derive that $\int\limits_\Omega f u_0 > 0$, and from (8) that

$$\langle I'(u_0), w\rangle = 0, \ \ \forall w \in X,$$

i.e u_0 is a weak solution for (1). In fact, $u_0 \in \mathcal{N}$ and

$$c \leq I(u_0) \leq \lim_{n\to+\infty} I(u_n) = c.$$

So, we deduce that $u_n \to u_0$ strongly in X and $I(u_0) = c = \inf\limits_{u\in\mathcal{N}} I(u)$. Moreover, $u_0 \in \mathcal{N}^+$. So u_0 is a local minimum for I. □

3.2 Existence of solution in $\mathcal{N}^-$

In this subsection, to prove the existense of a solution in $\mathcal{N}^-$, we shall find the range of c^- where I verifies the $(PS)_{c^-}$ condition.

Lemma 5. *Let $(u_n)_n$ be any sequence of X satisfying the following conditions:*

(a) $I(u_n) \to c$ *with* $0 < c < \frac{s}{N} A_{\mu,s}^{\frac{N}{2s}}$,

(b) $I'(u_n) \to 0$ *as* $n \to +\infty$.

Then $(u_n)_n$ has a strongly convergent subsequence.

Proof. . From (a) and (b), we have

$$I(u_n) = c + o(1),$$

and

$$\langle I'(u_n), u_n\rangle = T(u_n) - \int_\Omega |u_n|^{2_s^*} dx - \int_\Omega f u_n dx + o(1), \tag{9}$$

then

$$\frac{s}{N}\int_\Omega |u_n|^{2_s^*} dx + o(1) = c + \frac{1}{2}\int_\Omega f u_n dx - \frac{1}{2}\langle I'(u_n), u_n\rangle.$$

By using Hölder inequality, we get

$$\frac{s}{N}\int_\Omega |u_n|^{2_s^*} dx \le c + \frac{1}{2}\|f\|_-\|u_n\| + \frac{1}{2}\|I'(u_n)\|_-\|u_n\|. \tag{10}$$

From (3), (9) and (10) then, we have for all ε positive

$$\begin{aligned}\eta\|u_n\|^2 \le T(u_n) &\le \int_\Omega |u_n|^{2_s^*} dx + \int_\Omega f u_n dx + \langle I'(u_n), u_n\rangle \\ &\le \frac{N}{s}c + \frac{N+2s}{2s}(\|f\|_- + \|I'(u_n)\|_-)\|u_n\| + \varepsilon\|u_n\|.\end{aligned}$$

So $T(u_n)$ is uniformly bounded. For a subsequence of $(u_n)_n$, we can get a $u \in X$ such that

$$u_n \rightharpoonup u.$$

So, from (b), we obtain that

$$\langle I'(u), w\rangle = 0, \quad \forall w \in X.$$

Then u is a weak solution for (1). In particular $u \neq 0$, $u \in \mathcal{N}$ and $I(u) \ge c$. We have

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } X \text{ and } L^{2_s^*}(\Omega), \\ u_n &\rightharpoonup u \quad \text{in } L^2(\Omega, |x|^{-2s}), \\ u_n &\to u \quad \text{in } L^2(\Omega, |x|^{\alpha-2s}), \\ u_n &\to u \quad \text{in } L^q(\Omega), \text{ for } 1 \le q < 2_s^*. \end{aligned}$$

Let $u_n = u + v_n$. So, $v_n \rightharpoonup 0$ in X. As in Brezis-Lieb Lemma (see [3]), we conclude that

$$o(1) = \frac{1}{2}T(v_n) - \frac{1}{2^*}\int_\Omega |v_n|^{2^*} dx, \tag{11}$$

and

$$o(1) = T(v_n) - \int_\Omega |v_n|^{2^*} dx.$$

Without loss of generality, as $n \to +\infty$ we may assume that

$$T(v_n) \to l \ , \ \int_\Omega |v_n|^{2_s^*} dx \to l.$$

By (11), we deduce that $l = 0$ and $u_n \to u$ strongly in X as $n \to +\infty$. Assume that $u = 0$ in Ω, from $\langle I'(u_n), u_n\rangle = o(1)$, we have

$$\|u_n\|^2 - \int_\Omega |u_n|^{2_s^*} dx = o(1), \tag{12}$$

by the definition of $A_{\mu,s}$

$$\|u_n\|^2 \geq A_{\mu,s} \left(\int_\Omega |u_n|^{2_s^*} dx \right)^{\frac{2}{2_s^*}}, \tag{13}$$

by (12) and (13)

$$o(1) \geq \|u_n\|^2 \left((A_{\mu,s})^{\frac{2_s^*}{2}} - \|u_n\|^{2_s^*-2} \right), \tag{14}$$

if $\|u_n\| \to 0$, this contradicts $c > 0$. Then by (14)

$$\|u_n\|^2 \geq (A_{\mu,s})^{\frac{N}{2s}} \tag{15}$$

It follows from (12) and (15) that

$$\begin{aligned} I(u_n) &= \frac{1}{2}\|u_n\|^2 - \frac{1}{2_s^*}\int_\Omega |u_n|^{2_s^*} dx + o(1) \\ &= \frac{s}{N}\|u_n\|^2 + o(1) \\ &\geq \frac{s}{N}(A_{\mu,s})^{\frac{N}{2s}}. \end{aligned}$$

this contradicts $c < \frac{s}{N}(A_{\mu,s})^{\frac{N}{2s}}$. Therefore $u \neq 0$ and u is a nontrival solution of problem (1). □

Lemma 6. *Let $f \neq 0$ be a function satisfying $(\mathcal{F})$ then for all $0 < \lambda < \lambda_1$, there exists $v \in X$ such that*

$$\sup_{t \geq 0} I(tv) < \frac{s}{N} A_{\mu,s}^{\frac{N}{2s}}. \tag{16}$$

Proof. . For $t \leq 0$, we consider the functions

$$\begin{aligned} g(t) &= I(tu_\varepsilon) \\ &= \frac{t^2}{2}\|u_\varepsilon\|^2 - \frac{\lambda t^2}{2}\int\limits_{\Omega}\frac{|u_\varepsilon|^2}{|x|^{2s-\alpha}}dx - \frac{t^{2^*}}{2^*}\int\limits_{\Omega}|u_\varepsilon|^{2^*}dx - \int\limits_{\Omega}fu_\varepsilon dx \end{aligned}$$

and
$\overline{g}(t) = \frac{t^2}{2}\|u_\varepsilon\|^2 - \frac{t^{2^*}}{2^*}\int\limits_{\Omega}|u_\varepsilon|^{2^*}dx.$
the function $\overline{g}$ attains it's maximum. By Proposition 1, we can get that

$$\sup_{t\geq 0}\overline{g}(t) \leq \frac{s}{N}A_{\mu,s}^{\frac{N}{2s}}. \tag{17}$$

On the other hand, using the definitions of g and u_ε, we get $g(t) = I(tu_\varepsilon) \leq \frac{t^2}{2}\|u_\varepsilon\|^2$, for all $t \geq 0$ and $0 < \lambda < \lambda_1$.
Combining this with Proposition 1, let $\varepsilon \in]0;1[$, then there exists $t_0 \in]0;1[$ independent of $\varepsilon > 0$ such that

$$\sup_{t_0\geq t\geq 0}\overline{g}(t) \leq \frac{s}{N}A_{\mu,s}^{\frac{N}{2s}} + \mathcal{O}(\varepsilon^{2\beta^+(\mu)+2s-N}).$$

Hence, for all $0 < \lambda < \lambda_1$ and, by (17), we have

$$\begin{aligned} \sup_{t\geq t_0}g(t) &= \sup_{t\geq t_0}\left(\overline{g}(t) - \frac{\lambda t^2}{2}\int\limits_{\Omega}\frac{|u_\varepsilon|^2}{|x|^{2s-\alpha}}dx - \int\limits_{\Omega}fu_\varepsilon dx\right) \\ &\leq \frac{s}{N}A_{\mu,s}^{\frac{N}{2s}} + \mathcal{O}(\varepsilon^{2\beta^+(\mu)+2s-N}) - \mathcal{O}(\varepsilon^{\alpha}) - \mathcal{O}(\varepsilon^{\frac{2\beta^+(\mu)+2s-N}{2}}) \\ &\leq \frac{s}{N}A_{\mu,s}^{\frac{N}{2s}} + \mathcal{O}(\varepsilon^{2\beta^+(\mu)+2s-N}) - \mathcal{O}(\varepsilon^{\alpha}). \end{aligned} \tag{18}$$

As $0 < \alpha < 2\beta^+(\mu) + 2s - N$.
Combining this with (17) and (18), for any $0 < \lambda < \lambda_1$, we can choose ε small enough such that

$$\sup_{t\geq 0}I(tu_\varepsilon) < \frac{s}{N}A_{\mu,s}^{\frac{N}{2s}}.$$

By taking $v = u_\varepsilon$. From Lemma 3, the definition of c^- and (16), for any $0 < \lambda < \lambda_1$, we see that there exists $t^- > 0$ such that $t^-v \in \mathcal{N}^-$ and

$$c^- \leq I(t^-v) \leq \sup_{t\geq 0}I(tv) < \frac{s}{N}A_{\mu,s}^{\frac{N}{2s}}.$$

□

Proposition 3. *Suppose that f verifies the conditions of Lemma 6. Then I has a minimizer $u \in \mathcal{N}^-$ such that $c^- = I(u)$. Moreover, u is a solution of problem (1).*

Proof. If $0 < \lambda < \lambda_1$, then, by Lemma 5 and Lemma 6, there exists a $(PS)_{c^-}$-sequence $(u_n) \subset \mathcal{N}^- \in X$ for I with $c^- \in (0; \frac{s}{N} A_{\mu,s}^{\frac{N}{2s}})$. Since I is bounded on $\mathcal{N}^-$, we see that (u_n) is bounded in X. From Lemma 5, there exist a subsequence still denoted by (u_n) and a nonzero solution $u \in X$ of (1). such that $u_n \to u$ strongly in X.
Now, we first prove that $u \in \mathcal{N}^-$. Arguing by contradiction, we assume $u \in \mathcal{N}^+$. Then, by Lemma 3, there exists a unique t^- such that $t^- u \in \mathcal{N}^-$. It follows that

$$c^- \leq I(t^- u) \leq \underset{n\to+\infty}{I(t^- u_n)} \leq \underset{n\to+\infty}{I(u_n)} = c^-.$$

This is a contradiction. Consequently, $u \in \mathcal{N}^-$. □

Proof of Theorem 1. By Proposition 2, 3, we obtain that the problem $(\mathcal{P})$ has two positive solutions u_0 and u such that $u_0 \in \mathcal{N}^+$, $u \in \mathcal{N}^-$. Since $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$, this implies that u_0 and u are distinct. This completes the proof. □

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