Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 5(67), No. 2 - 2025, 39-52
https://doi.org/10.31926/but.mif.2025.5.67.2.3

AN APPROACH TO REVERSE MINKOWSKI TYPE INEQUALITY WITH K-WEIGHTED FRACTIONAL INTEGRAL OPERATOR

Noureddine AZZOUZ[1] **and Bouharket BENAISSA**[*,2]

Abstract

In this research, we present an innovative approach to the reverse Minkowski type inequality using the k-weighted fractional integral operator ${}_{a^+}\mathbf{J}_{v}^{\psi}$. This operator has two positive summation parameters, $1 \leq p \leq q < \infty$, and our approach yields new results based on the selection of the function ψ.

2000 *Mathematics Subject Classification:* 26D10, 26E35, 26D15 .
Key words: Minkowski inequality, k-weighted fractional operator .

1 Introduction

In 2010, Dahmani presented a reverse Minkowski fractional integral inequality [5, Theorem 2.1]. For any f, g positive functions on $[0, +\infty]$, $\alpha > 0$, $p \geq 1$, if $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M$, for all $\tau \in [0, t]$, then

$$\left(J^{\alpha} f^{p}(t)\right)^{\frac{1}{p}} + \left(J^{\alpha} g^{p}(t)\right)^{\frac{1}{p}} \leq \frac{1 + M(m+2)}{(m+1)(M+1)} \left(J^{\alpha}(f+g)^{p}(t)\right)^{\frac{1}{p}}, \tag{1}$$

where J^{α} is the Riemann-Liouville fractional integral operator of order $\alpha > 0$. Several researchers have made significant contributions to the field by deriving extensions and generalizations of the above reverse Minkowski inequality for fractional integral operators [11, 13, 14].

In another way, in [8] for an integrable function f defined on the interval $[a, b]$ and for a differentiable function ψ such that $\psi'(t) > 0$ for all $t \in [a, b]$, the left

[1]Faculty of Sciences, University Center Nour Bachir El Bayadh, Algeria, e-mail: n.azzouz@cu-elbayadh.dz
[2*] *Corresponding author*, Faculty of Material Sciences, University of Tiaret, Algeria, e-mail: bouharket.benaissa@univ-tiaret.dz

weighted fractional integral of f with respect to the function ψ is defined as follows

$$ {}_{a^+}J_v^{\alpha} f(x) = \frac{1}{v(x)\Gamma(\alpha)} \int_a^x \psi'(s)(\psi(x)-\psi(s))^{\alpha-1} v(s) f(s) ds, \quad x > a, \qquad (2) $$

where $v(x) \neq 0$ is a weight function (positive measurable function).
This study aims to present a novel generalized form of the weighted fractional operator previously described in [8]. Additionally, the study seeks to establish new reverse Minkowski-type inequalities utilizing two positive parameters, with values restricted to $1 \leq p \leq q < \infty$.

2 k-weighted fractional operator

This section aims to provide a definition of the k-weighted fractional integral of a function f with respect to the function ψ. Additionally, we establish the space in which this integral is bounded.
Let $[a, b] \subseteq (0, +\infty)$, where $a < b$.

Definition 1. *Let $\alpha > 0$, $k > 0$ and ψ be an increasing differentiable function on $[a, b]$. The left sided k-weighted fractional integral of an integrable function f with respect to the function ψ on $[a, b]$ is defined as follows*

$$ {}_{a^+}\mathbf{J}_v^{\psi} f(x) = \frac{1}{v(x) k\,\Gamma_k(\alpha)} \int_a^x \psi'(s)(\psi(x)-\psi(s))^{\frac{\alpha}{k}-1} v(s) f(s) ds, \quad x > a. \qquad (3) $$

where $v(x) \neq 0$ is a weight function and the k-gamma function defined by

$$ \Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-\frac{t^k}{k}} dt. $$

For $f(s) = 1$, we denote

$$ {}_{a^+}\mathbf{J}_v^{\psi} \mathbf{1}(x) = \frac{1}{v(x) k\,\Gamma_k(\alpha)} \int_a^x \psi'(s)(\psi(x)-\psi(s))^{\frac{\alpha}{k}-1} v(s)\, ds, \quad x > a. $$

The space $L_p^w[a, b]$ of all real-valued Lebesgue measurable functions f on $[a, b]$ with norm condition :

$$ \| f \|_p^w = \left(\int_a^b | f(x) |^p \, w(x) dx \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < +\infty, $$

is known as weighted Lebesgue space, where w denotes a weight function.

1. Put $w \equiv 1$, the space $L_p^w[a, b]$ reduces to the classical Lebesgue space $L_p[a, b]$.
2. Choose $w(x) = v^p(x)\, \psi'(x)$, we get

$$ L_{X_v^p}[a, b] = \left\{ f : \| f \|_{X_v^p} = \left(\int_a^b | v(x) f(x) |^p \, \psi'(x) dx \right)^{\frac{1}{p}} < \infty \right\}. \qquad (4) $$

In the next theorem, we show that the k-weighted fractional integral is bounded.

Theorem 1. *For any functions $f \in L_{X_v^p}[a,b]$, the fractional integral operator (3) is defined and we have*

$$_{a^+}\mathbf{J}_v^{\psi} f(x) \in L_{X_v^p}[a,b]. \tag{5}$$

Moreover

$$\left\| _{a^+}\mathbf{J}_v^{\psi} f(x) \right\|_{X_v^p} \leq C \, \| f(x) \|_{X_v^p} \,, \tag{6}$$

where

$$C = \frac{(\psi(b)-\psi(a))^{\frac{p\alpha}{k}}}{\Gamma_k^p(\alpha+k)}. \tag{7}$$

Proof. Let $f \in L_{X_v^p}[a,b]$ and $\frac{1}{p}+\frac{1}{p'}=1$, we have

$$\left\| _{a^+}\mathbf{J}_v^{\psi} f(x) \right\|_{X_v^p}^p = \int_a^b \mid v(x) \ _{a^+}\mathbf{I}_v^{\psi} f(x) \mid^p \psi'(x)dx$$

$$= \frac{1}{k\,\Gamma_k(\alpha)} \int_a^b \left| \int_a^x v(s) f(s) \psi'(s)(\psi(x)-\psi(s))^{\frac{\alpha}{k}-1} ds \right|^p \psi'(x)\,dx$$

$$= \frac{1}{k^p\,\Gamma_k^p(\alpha)} \int_a^b \left| \int_a^x v(s) f(s) \left(\psi'(s)(\psi(x)-\psi(s))^{\frac{\alpha}{k}-1} \right)^{\frac{1}{p}+\frac{1}{p'}} ds \right|^p \psi'(x)\,dx.$$

Using Hölder inequality for $p \geq 1$, we get

$$\begin{aligned}
\left\| _{a^+}\mathbf{J}_v^{\psi} f(x) \right\|_{X_v^p}^p &\leq \frac{1}{k^p\,\Gamma_k^p(\alpha)} \int_a^b \left| \int_a^x v^p(s) f^p(s) \psi'(s)(\psi(x)-\psi(s))^{\frac{\alpha}{k}-1} ds \right| \\
&\times \left| \int_a^x \psi'(s)(\psi(x)-\psi(s))^{\frac{\alpha}{k}-1} ds \right|^{p-1} \psi'(x)\,dx \\
&\leq \frac{1}{k^p\,\Gamma_k^p(\alpha)} \int_a^b \left(\int_a^x |v(s)f(s)|^p \ \psi'(s)(\psi(x)-\psi(s))^{\frac{\alpha}{k}-1} ds \right) \\
&\times \left(\int_a^x \psi'(s)(\psi(x)-\psi(s))^{\frac{\alpha}{k}-1} ds \right)^{p-1} \psi'(x)\,dx \\
&= \frac{k^{p-1}}{\alpha^{p-1}} \frac{1}{k^p\,\Gamma_k^p(\alpha)} \int_a^b \int_a^x |v^p(s)\ f^p(s)|\, \psi'(s)(\psi(x)-\psi(s))^{\frac{\alpha}{k}-1} \\
&\times (\psi(x)-\psi(a))^{\frac{(p-1)\alpha}{k}} \psi'(x)\,ds\,dx.
\end{aligned}$$

Applying Fubini's Theorem and the monotonicity of ψ, we deduce

$$\begin{aligned}
\left\| {}_{a^+}\mathbf{J}_v^{\psi} f(x) \right\|_{X_v^p}^p &\leq \frac{\alpha \left(\psi(b)-\psi(a)\right)^{\frac{(p-1)\alpha}{k}}}{k\,\Gamma_k^p(\alpha+k)} \\
& \int_a^b |v^p(s) f^p(s)|\, \psi'(s) \left(\int_s^b (\psi(x)-\psi(s))^{\frac{\alpha}{k}-1} \psi'(x)\, dx \right) ds \\
&= \frac{(\psi(b)-\psi(a))^{\frac{(p-1)\alpha}{k}}}{\Gamma_k^p(\alpha+k)} \int_a^b |v^p(s)\ f^p(s)|\ \psi'(s)(\psi(b)-\psi(s))^{\frac{\alpha}{k}}\, ds \\
&\leq \frac{(\psi(b)-\psi(a))^{\frac{(p-1)\alpha}{k}} (\psi(b)-\psi(a))^{\frac{\alpha}{k}}}{\Gamma_k^p(\alpha+k)} \int_a^b |v^p(s)\, f^p(s)|\ \psi'(s)\, ds \\
&= \frac{(\psi(b)-\psi(a))^{\frac{p\alpha}{k}}}{\Gamma_k^p(\alpha+k)}\ \|f\|_{X_v^p}^p\ .
\end{aligned}$$

□

Putting $v(x)=1$, the operator ${}_{a+}\mathbf{J}_v^{\psi} f(x)$ is simplified to the k-Hilfer operator of order $\alpha>0$

$$ {}_{a+}\mathbf{J}_{\alpha}^{\psi} f(x) = \frac{1}{k\,\Gamma_k(\alpha)} \int_a^x (\psi(x)-\psi(s))^{\frac{\alpha}{k}-1} \psi'(s) f(s) ds,, \quad x>a, \tag{8}$$

which is bounded on

$$L_{X^p}[a,b] = \left\{ f : \|\ f\ \|_{X^p} = \left(\int_a^b |\ f(x)\ |^p\ \psi'(x) dx \right)^{\frac{1}{p}} < \infty \right\}.$$

Depending on the function ψ, we'll get different types of k-weighted fractional integral operators.

1. By choosing $\psi(\tau)=\tau$, the operator ${}_{a+}\mathbf{J}_v^{\psi} f(x)$ is reduced to the k-weighted Riemann-Liouville fractional integral operator of order $\alpha>0$

$$\mathcal{RL} f(x) = \frac{1}{v(x) k\,\Gamma_k(\alpha)} \int_a^x (x-s)^{\frac{\alpha}{k}-1} v(s) f(s) ds,, \quad x>a, \tag{9}$$

   which is bounded on

$$L_{X_v^p}[a,b] = \left\{ f : \|\ f\ \|_{X_v^p} = \left(\int_a^b |\ v(x) f(x)\ |^p\ \ dx \right)^{\frac{1}{p}} < \infty \right\}.$$

2. Choosing $\psi(\tau)=\ln\tau$, ${}_{a+}\mathbf{J}_v^{\psi} f(x)$ is reduced to the k-weighted Hadamard fractional integral operator of order $\alpha>0$

$$\mathcal{H} f(x) = \frac{1}{v(x)\, k\Gamma_k(\alpha)} \int_a^x \left(\ln\frac{x}{s}\right)^{\frac{\alpha}{k}-1} v(s) f(s) \frac{ds}{s}, \quad x>a>1, \tag{10}$$

which is bounded on

$$L_{X_v^p}[a,b] = \left\{ f : \| f \|_{X_v^p} = \left(\int_a^b | v(x)f(x) |^p \frac{dx}{x} \right)^{\frac{1}{p}} < \infty \right\}.$$

3. Choosing $\psi(\tau) = \frac{\tau^{\rho+1}}{\rho+1}$ where $\rho > 0$, ${}_{a+}\mathbf{J}_v^{\psi} f(x)$ is reduced to the k-weighted Katugompola fractional integral operator of order $\alpha > 0$

$$\mathcal{K}f(x) = \frac{(\rho+1)^{1-\frac{\alpha}{k}}}{v(x)k\,\Gamma_k(\alpha)} \int_a^x \left(x^{\rho+1} - s^{\rho+1}\right)^{\frac{\alpha}{k}-1} v(s)f(s)s^{\rho} ds, \ x > a \tag{11}$$

which is bounded on

$$L_{X_v^p}[a,b] = \left\{ f : \| f \|_{X_v^p} = \left(\int_a^b | v(x)f(x) |^p \ x^{\rho}\, dx \right)^{\frac{1}{p}} < \infty \right\}.$$

4. Choosing $\psi(\tau) = \frac{(\tau-a)^{\theta}}{\theta}$, ${}_{a+}\mathbf{J}_v^{\psi} f(x)$ is reduced to the k-weighted fractional conformable integral operator of order $\alpha > 0$

$$\mathcal{C}f(x) = \frac{\theta^{1-\frac{\alpha}{k}}}{v(x)k\,\Gamma_k(\alpha)} \int_a^x \left((x-a)^{\theta} - (s-a)^{\theta}\right)^{\frac{\alpha}{k}-1} v(s)\frac{f(s)}{(s-a)^{1-\theta}} ds, \ x > a, \tag{12}$$

which is bounded on

$$L_{X_v^p}[a,b] = \left\{ f : \| f \|_{X_v^p} = \left(\int_a^b | v(x)f(x) |^p \ \frac{dx}{(s-a)^{1-\theta}} \right)^{\frac{1}{p}} < \infty \right\}.$$

For example see [6].

We need the following Lemma to prove our results [2]-[3].

Lemma 1. *Let* $1 < p \le q < \infty$ *and* f, W *be non-negative measurable functions on* $[a, b]$. *we suppose that,* $0 < \int_a^b f^r(s)W(s)ds < \infty$, *for* $r > 1$, *then*

$$\int_a^b f^p(s)W(s)ds \le \left(\int_a^b W(s)ds \right)^{\frac{q-p}{q}} \left(\int_a^b f^q(s)W(s)ds \right)^{\frac{p}{q}}. \tag{13}$$

Proof. If $p = q$, then we get equality and for $p \ne q$, we use Hölder's integral inequality with $\dfrac{q}{p} > 1$. We have

$$\begin{aligned} \int_a^b f^p(s)W(s)ds &= \int_a^b \left(W^{\frac{q-p}{q}}(s)\right) \left(f^p(s)W^{\frac{p}{q}}(s)\right) ds \\ &\le \left(\int_a^b W(s)ds \right)^{\frac{q-p}{q}} \left(\int_a^b f^q(s)W(s)ds \right)^{\frac{p}{q}}. \end{aligned}$$

□

Corollary 1. *Let $1 < p \leq q < \infty$, f be non-negative measurable function on $[a,\, x]$ and $\mu \in \mathcal{C}^1[a, x]$ be a positive, increasing function and ${}_{a^+}\mathbf{J}_v^{\psi}$ is the left sided k-weighted fractional operator defined by (3), then*

$$\left({}_{a^+}\mathbf{J}_v^{\psi} f^p(x) \right)^{\frac{1}{p}} \leq \left[{}_{a^+}\mathbf{J}_v^{\psi} \mathbf{1}(x) \right]^{\frac{q-p}{pq}} \left({}_{a^+}\mathbf{J}_v^{\psi} f^q(x) \right)^{\frac{1}{q}}. \tag{14}$$

Proof. Using the inequality (13) by taking $W(s) = \frac{\psi'(s)(\psi(x)-\psi(s))v(s)}{v(x)\, k\, \Gamma_k(\alpha)}$, we obtain

$$\begin{aligned} \int_a^x \frac{\psi'(s)(\psi(x)-\psi(s))v(s)}{v(x)\, k\, \Gamma_k(\alpha)} f^p(s) ds \quad &\leq \left(\int_a^x \frac{\psi'(s)(\psi(x)-\psi(s))v(s)}{v(x)\, k\, \Gamma_k(\alpha)} ds \right)^{\frac{q-p}{q}} \\ &\times \left(\int_a^x \frac{\psi'(s)(\psi(x)-\psi(s))v(s)}{v(x)\, k\, \Gamma_k(\alpha)} f^q(s) ds \right)^{\frac{p}{q}}, \end{aligned}$$

this gives the desired results. □

3 Main results

In this section, we present new inverse Minkowski-type inequalities using the k-weighted fractional integral operator ${}_{a^+}\mathbf{J}_v^{\psi}$. Let $0 \leq a < b < +\infty$, $v(x) \neq 0$ be a weight function and $f, g \in L_{X_v^p}[a, b]$.

Theorem 2. *Let $f,\, g > 0,\ 1 \leq p \leq q < +\infty$, if*

$$0 < m \leq \frac{f(s)}{g(s)} \leq M, \quad for \quad all\ s \in [a, x], \tag{15}$$

then the following inequality yields

$$\left({}_{a^+}\mathbf{J}_v^{\psi} f^p(x) \right)^{\frac{1}{p}} + \left({}_{a^+}\mathbf{J}_v^{\psi} g^p(x) \right)^{\frac{1}{p}} \leq K_{m,M}^{p,q} \left({}_{a^+}\mathbf{J}_v^{\psi} (f(x)+g(x))^q \right)^{\frac{1}{q}}, \tag{16}$$

where

$$K_{m,M}^{p,q} = \frac{1+M(m+2)}{(m+1)(M+1)} \left[{}_{a^+}\mathbf{J}_v^{\psi} \mathbf{1}(x) \right]^{\frac{q-p}{pq}}. \tag{17}$$

Proof. From the supposition (15) we result

$$\frac{M+1}{M} \leq \frac{f(s)+g(s)}{f(s)},$$

hence

$$f(s) \leq \frac{M}{M+1} \left(f(s) + g(s) \right),$$

taking the p^{th} power of the above inequality and multiplying by the positive $\psi'(s)(\psi(x)-\psi(s))^{\frac{\beta}{k}-1} v(s)$, we obtain

$$\psi'(s)(\psi(x)-\psi(s))^{\frac{\beta}{k}-1} v(s) f^p(s) \leq$$

$$\left(\frac{M}{M+1}\right)^p \psi'(s)(\psi(x)-\psi(s))^{\frac{\beta}{k}-1} v(s)(f(s)+g(s))^p,$$

integrating with respect to s over $[a,x]$, we get

$$\left({}_{a^+}\mathbf{J}_v^{\psi} f^p(x)dx\right)^{\frac{1}{p}} \leq \frac{M}{M+1}\left({}_{a^+}\mathbf{J}_v^{\psi}(f(x)+g(x))^p dx\right)^{\frac{1}{p}}, \tag{18}$$

applying the inequality (14) on the right-hand side of (18), we get

$$\left({}_{a^+}\mathbf{J}_v^{\psi} f^p(x)dx\right)^{\frac{1}{p}} \leq \frac{M}{M+1}\left[{}_{a^+}\mathbf{J}_v^{\psi}\mathbf{1}(x)\right]^{\frac{q-p}{pq}} \left({}_{a^+}\mathbf{J}_v^{\psi}(f(x)+g(x))^q(x)\right)^{\frac{1}{q}}. \tag{19}$$

Using the assumption (15), we obtain

$$(1+m)g(s) \leq f(s)+g(s),$$

for $p \geq 1$ we deduce that

$$g^p(s) \leq \left(\frac{1}{1+m}\right)^p (f(s)+g(s))^p,$$

multiplying by $\psi'(s)(\psi(x)-\psi(s))^{\frac{\beta}{k}-1}v(s)$ and integrating with respect to s over $[a,x]$, thus

$$\left({}_{a^+}\mathbf{J}_v^{\psi} g^p(x)dx\right)^{\frac{1}{p}} \leq \frac{1}{1+m}\left({}_{a^+}\mathbf{J}_v^{\psi}(f(x)+g(x))^p dx\right)^{\frac{1}{p}}. \tag{20}$$

Now applying the inequality (14) on the right-hand side of (20), we get

$$\left({}_{a^+}\mathbf{J}_v^{\psi} g^p(x)dx\right)^{\frac{1}{p}} \leq \frac{1}{1+m}\left[{}_{a^+}\mathbf{J}_v^{\psi}\mathbf{1}(x)\right]^{\frac{q-p}{pq}} \left({}_{a^+}\mathbf{J}_v^{\psi}(f(x)+g(x))^q(x)\right)^{\frac{1}{q}}, \tag{21}$$

adding the inequalities (19) with (21), we get the required inequality (16). □

In the following corollaries, we present some special cases of two-parameter reverse Minkowski's inequalities using the k-weighted fractional integral operator (3):

1. Setting $\psi(\tau)=\tau$, $v(\tau)=1$ and $\alpha=k=1$, then we get ${}_{a^+}\mathbf{J}_v^{\psi}\mathbf{1}(x)=x-a$ and

$$ {}_{a^+}\mathbf{J}_v^{\psi} f(x) = \mathcal{R}f(x) = \int_a^x f(s)\,ds; \quad x>a, \tag{22}$$

Corollary 2. *(Reverse Minkowski's inequality via Riemann integral operator.)*
Let $f,\, g>0,\ 1\leq p\leq q<+\infty$ *, if*

$$0<m\leq \frac{f(s)}{g(s)} \leq M, \quad \textit{for} \ \ \textit{all}\ s\in[a,x], \tag{23}$$

then

$$\left(\int_a^x f(s)\,ds\right)^{\frac{1}{p}} + \left(\int_a^x g(s)\,ds\right)^{\frac{1}{p}} \leq K \left(\int_a^x (f(s)+g(s))\,ds\right)^{\frac{1}{q}}, \quad (24)$$

where

$$K = \left[\frac{1+M(m+2)}{(m+1)(M+1)}\right][x-a]^{\frac{q-p}{pq}}. \quad (25)$$

Inequality (24) is a new generalization on $[a,x]$ with two parameters $1 < p \leq q$, for $q = p$ we get [4, Theorem 1.2].

2. Setting Setting $v(\tau) = 1$ and $\psi(\tau) = \tau$, we get ${}_{a^+}\mathbf{J}_v^{\psi}\mathbf{1}(x) = \frac{1}{\Gamma_k(\alpha+k)}(x-a)^{\frac{\alpha}{k}}$ and

$$_{a+}\mathbf{J}_v^{\psi} f(x) = \mathcal{RL}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-s)^{\frac{\alpha}{k}-1} f(s)\,ds, \quad x > a. \quad (26)$$

Corollary 3. *(Reverse Minkowski's inequality via k-Riemann-Liouville operator.) Under the assumptions of Corollary 2, we have.*

$$(\mathcal{RL}f(x))^{\frac{1}{p}} + (\mathcal{RL}g(x))^{\frac{1}{p}} \leq K\ (\mathcal{RL}(f(x)+g(x)))^{\frac{1}{q}}, \quad (27)$$

where

$$K = \left[\frac{1+M(m+2)}{(m+1)(M+1)}\right]\left[\frac{1}{\alpha\Gamma_k(\alpha)}(x-a)^{\frac{\alpha}{k}}\right]^{\frac{q-p}{pq}}. \quad (28)$$

Inequality (27) is a generalization on $[a,x]$ with two parameters $1 < p \leq q$, taking $q = p$ we get [12, Theorem3.1].

3. Setting $v(\tau) = 1$ and $\psi(\tau) = \ln \tau$, we deduce ${}_{a^+}\mathbf{J}_v^{\psi}\mathbf{1}(x) = \frac{1}{\Gamma_k(\alpha+k)}\left(\ln \frac{x}{a}\right)^{\frac{\alpha}{k}}$ and

$$_{a+}\mathbf{J}_v^{\psi} f(x) = \mathcal{H}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x \left(\ln\frac{x}{s}\right)^{\frac{\alpha}{k}-1}\frac{f(s)}{s}\,ds, \quad x > a > 1. \quad (29)$$

Corollary 4. *(Reverse Minkowski type inequality via k-Hadamard operator.) Under the assumptions of Corollary 2, we have.*

$$(\mathcal{H}f(x))^{\frac{1}{p}} + (\mathcal{H}g(x))^{\frac{1}{p}} \leq K\ (\mathcal{H}(f(x)+g(x)))^{\frac{1}{q}}, \quad (30)$$

where

$$K = \left[\frac{1+M(m+2)}{(m+1)(M+1)}\right]\left[\frac{1}{\alpha\Gamma_k(\alpha)}\left(\ln\frac{x}{a}\right)^{\frac{\alpha}{k}}\right]^{\frac{q-p}{pq}}. \quad (31)$$

Inequality (30) is a generalization on $[a,x]$ with two parameters $1 < p \leq q$, taking $q = p$ we get [7, Theorem11].

4. Setting $v(\tau) = 1$ and $\psi(\tau) = \frac{\tau^{\rho+1}}{\rho+1}$ we get ${}_{a^+}\mathbf{J}_v^{\psi}\mathbf{1}(x) = \frac{1}{\Gamma_k(\alpha+k)}\left(\frac{x^{\rho+1}-a^{\rho+1}}{\rho+1}\right)^{\frac{\alpha}{k}}$ and
$$
{}_{a^+}\mathbf{J}_v^{\psi} f(x) = \mathcal{K}f(x) = \frac{(\rho+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x (x^{\rho+1}-s^{\rho+1})^{\frac{\alpha}{k}-1} s^{\rho} f(s)\, ds, \quad x > a. \tag{32}
$$

Corollary 5. *(Reverse Minkowski's inequality via k-Katugompola operator.) Under the assumptions of Corollary 2, we have for all* $\rho > -1$
$$
(\mathcal{K}f(x))^{\frac{1}{p}} + (\mathcal{K}g(x))^{\frac{1}{p}} \leq K\ (\mathcal{K}(f(x)+g(x)))^{\frac{1}{q}}, \tag{33}
$$
where
$$
K = \left[\frac{1+M(m+2)}{(m+1)(M+1)}\right]\left[\frac{1}{\Gamma_k(\alpha+k)}\left(\frac{x^{\rho+1}-a^{\rho+1}}{\rho+1}\right)^{\frac{\alpha}{k}}\right]^{\frac{q-p}{pq}}. \tag{34}
$$

Inequality (33) is a generalization on $[a, x]$ with two parameters $1 < p \leq q$, taking $k = 1$ and $q = p$ we get [9, Theorem 3].

5. Setting $v(\tau) = 1$ and $\psi(\tau) = \frac{(\tau-a)^{\theta}}{\theta}$, we have ${}_{a^+}\mathbf{J}_v^{\psi}\mathbf{1}(x) = \frac{1}{\Gamma_k(\alpha+k)}\left(\frac{(x-a)^{\theta}}{\theta}\right)^{\frac{\alpha}{k}}$ and for $x > a$
$$
{}_{a^+}\mathbf{J}_v^{\psi} f(x) = \mathcal{C}f(x) = \frac{\theta^{1-\frac{\alpha}{k}}}{k\,\Gamma_k(\alpha)}\int_a^x \Big((x-a)^{\theta}-(s-a)^{\theta}\Big)^{\frac{\alpha}{k}-1}\frac{f(t)}{(s-a)^{1-\theta}}\, ds.
$$

Corollary 6. *(Reverse Minkowski's inequality via fractional k-conformal integral operator.) Under the assumptions of Corollary 2, we get*
$$
(\mathcal{C}f(x))^{\frac{1}{p}} + (\mathcal{C}g(x))^{\frac{1}{p}} \leq K\ (\mathcal{C}(f(x)+g(x)))^{\frac{1}{q}}, \tag{35}
$$
where
$$
K = \left[\frac{1+M(m+2)}{(m+1)(M+1)}\right]\left[\frac{1}{\Gamma_k(\alpha+k)}\left(\frac{(x-a)^{\theta}}{\theta}\right)^{\frac{\alpha}{k}}\right]^{\frac{q-p}{pq}}. \tag{36}
$$

Now, we give the second result.

Theorem 3. *Let* $1 \leq p \leq q < +\infty$, $\eta \geq 1$ *and* $f, g > 0$, *such that for all* $x > a$, ${}_{a^+}\mathbf{J}_v^{\psi} f^{\eta}(x) < \infty$, ${}_{a^+}\mathbf{J}_v^{\psi} g^{\eta}(x) < \infty$. *If*
$$
0 < m \leq \frac{f(s)}{g(s)} \leq M, \quad \text{for all } s \in [a, x],
$$
then
$$
\left({}_{a^+}\mathbf{J}_v^{\psi} f^{q}(x)\right)^{\frac{2}{q}} + \left({}_{a^+}\mathbf{J}_v^{\psi} g^{q}(x)\right)^{\frac{2}{q}} \geq \left(\frac{(m+1)(M+1)}{M} - 2\right)\left[{}_{a^+}\mathbf{J}_v^{\psi}\mathbf{1}(x)\right]^{\frac{2p-2q}{pq}} \times \left[{}_{a^+}\mathbf{J}_v^{\psi} f^{p}(x)\right]^{\frac{1}{p}}\left[{}_{a^+}\mathbf{J}_v^{\psi} g^{p}(x)\right]^{\frac{1}{p}}. \tag{37}
$$

Proof. Multiplying the inequalities (19) with (21), we get

$$\left({}_{a^+}\mathbf{J}_v^{\psi}(f(x)+g(x))^q(x)\right)^{\frac{2}{q}} \geq \frac{(M+1)(m+1)}{M}\left[{}_{a^+}\mathbf{J}_v^{\psi}\mathbf{1}(x)\right]^{\frac{p-q}{pq}} \times \left({}_{a^+}\mathbf{J}_v^{\psi}f^p(x)dx\right)^{\frac{1}{p}}\left({}_{a^+}\mathbf{J}_v^{\psi}g^p(x)dx\right)^{\frac{1}{p}},$$

hence for $p=q$ we get

$$\left({}_{a^+}\mathbf{J}_v^{\psi}(f(x)+g(x))^p(x)\right)^{\frac{2}{p}} \geq \frac{(M+1)(m+1)}{M} \times \left({}_{a^+}\mathbf{J}_v^{\psi}f^p(x)dx\right)^{\frac{1}{p}}\left({}_{a^+}\mathbf{J}_v^{\psi}g^p(x)dx\right)^{\frac{1}{p}}. \tag{38}$$

By applying Minkowski inequality, we get

$$\left[\left({}_{a^+}\mathbf{J}_v^{\psi}(f(x)+g(x))^p(x)\right)^{\frac{1}{p}}\right]^2 \leq \left[\left({}_{a^+}\mathbf{J}_v^{\psi}f^p(x)dx\right)^{\frac{1}{p}}+\left({}_{a^+}\mathbf{J}_v^{\psi}g^p(x)dx\right)^{\frac{1}{p}}\right]^2, \tag{39}$$

consequently, by setting the inequalities (38) and (39), we obtain

$$\left({}_{a^+}\mathbf{J}_v^{\psi}f^p(x)\right)^{\frac{2}{p}}+\left({}_{a^+}\mathbf{J}_v^{\psi}g^p(x)\right)^{\frac{2}{p}} \geq B_1\left({}_{a^+}\mathbf{J}_v^{\psi}f^p(x)\right)^{\frac{1}{p}}\left({}_{a^+}\mathbf{J}_v^{\psi}g^p(x)\right)^{\frac{1}{p}}, \tag{40}$$

where

$$B_1=\left(\frac{(m+1)(M+1)}{M}-2\right). \tag{41}$$

On the other hand, from the inequality (14), we deduce that for $1<p\leq q<\infty$

$$\left({}_{a^+}\mathbf{J}_v^{\psi}f^q(x)\right)^{\frac{1}{q}} \geq \left[{}_{a^+}\mathbf{J}_v^{\psi}\mathbf{1}(x)\right]^{\frac{p-q}{pq}}\left({}_{a^+}\mathbf{J}_v^{\psi}f^p(x)\right)^{\frac{1}{p}},$$

and

$$\left({}_{a^+}\mathbf{J}_v^{\psi}g^q(x)\right)^{\frac{1}{q}} \geq \left[{}_{a^+}\mathbf{J}_v^{\psi}\mathbf{1}(x)\right]^{\frac{p-q}{pq}}\left({}_{a^+}\mathbf{J}_v^{\psi}g^p(x)\right)^{\frac{1}{p}},$$

this gives us

$$\left({}_{a^+}\mathbf{J}_v^{\psi}f^q(x)\right)^{\frac{2}{q}}+\left({}_{a^+}\mathbf{J}_v^{\psi}g^q(x)\right)^{\frac{2}{q}} \geq \left[{}_{a^+}\mathbf{J}_v^{\psi}\mathbf{1}(x)\right]^{2(\frac{p-q}{pq})}\left[\left({}_{a^+}\mathbf{J}_v^{\psi}f^p(x)\right)^{\frac{2}{p}}+\left({}_{a^+}\mathbf{J}_v^{\psi}g^p(x)\right)^{\frac{2}{p}}\right], \tag{42}$$

using the inequalities (40) and (42), we obtain the desired inequality (37). □

Corollary 7. *Under the assumptions of Theorem 3, we result the following cases.*

1. *The reverse Minkowski's inequality associated with Riemann integral*

$$\left(\int_a^x f^q(s)\,ds\right)^{\frac{2}{q}}+\left(\int_a^x g^q(s)\,ds\right)^{\frac{2}{q}}\geq B\,\left(\int_a^x f^p(s)\,ds\right)^{\frac{1}{p}}\left(\int_a^x f^p(s)\,ds\right)^{\frac{1}{p}}, \tag{43}$$

where

$$B=\left(\frac{(m+1)(M+1)}{M}-2\right)[x-a]^{\frac{2p-2q}{pq}}. \tag{44}$$

2. *The reverse Minkowski's inequality associated with k-Riemann-Liouville integral*

$$(\mathcal{RL}f^q(x))^{\frac{2}{q}}+(\mathcal{RL}g^q(x))^{\frac{2}{q}}\geq B\,(\mathcal{RL}f^p(x))^{\frac{1}{p}}\,(\mathcal{RL}g^p(x))^{\frac{1}{p}}, \tag{45}$$

where

$$B=\left(\frac{(m+1)(M+1)}{M}-2\right)\left[\frac{1}{\Gamma_k(\alpha+k)}(x-a)^{\frac{\alpha}{k}}\right]^{\frac{2p-2q}{pq}}. \tag{46}$$

3. *The reverse Minkowski's inequality associated with k-Hadamard integral*

$$(\mathcal{H}f^q(x))^{\frac{2}{q}}+(\mathcal{H}g^q(x))^{\frac{2}{q}}\geq B\,(\mathcal{H}f^p(x))^{\frac{1}{p}}\,(\mathcal{H}g^p(x))^{\frac{1}{p}}, \tag{47}$$

where

$$B=\left(\frac{(m+1)(M+1)}{M}-2\right)\left[\frac{1}{\Gamma_k(\alpha+k)}\left(\ln\frac{x}{a}\right)^{\frac{\alpha}{k}}\right]^{\frac{2p-2q}{pq}}. \tag{48}$$

4. *The reverse Minkowski's inequality associated with k-Katugompola integral*

$$(\mathcal{K}f^q(x))^{\frac{2}{q}}+(\mathcal{K}g^q(x))^{\frac{2}{q}}\geq B\,(\mathcal{K}f^p(x))^{\frac{1}{p}}\,(\mathcal{K}g^p(x))^{\frac{1}{p}}, \tag{49}$$

where

$$B=\left(\frac{(m+1)(M+1)}{M}-2\right)\left[\frac{1}{\Gamma_k(\alpha+k)}\left(\frac{x^{\rho+1}-a^{\rho+1}}{\rho+1}\right)^{\frac{\alpha}{k}}\right]^{\frac{2p-2q}{pq}}. \tag{50}$$

5. *The reverse Minkowski's inequality associated with k-fractional conformal integral*

$$(\mathcal{C}f^q(x))^{\frac{2}{q}}+(\mathcal{C}g^q(x))^{\frac{2}{q}}\geq B\,(\mathcal{C}f^p(x))^{\frac{1}{p}}\,(\mathcal{C}g^p(x))^{\frac{1}{p}}, \tag{51}$$

where

$$B=\left(\frac{(m+1)(M+1)}{M}-2\right)\left[\frac{1}{\Gamma_k(\alpha+k)}\left(\frac{(x-a)^\theta}{\theta}\right)^{\frac{\alpha}{k}}\right]^{\frac{p-q}{pq}}. \tag{52}$$

Remark 1. *The above inequalities are generalizations with two parameters respectively to the inequalities in [10, Theorem 1.], [12, Theorem3.2], [7, Theorem12] and [9, Theorem 3].*

Now we consider that M and N are functions of a variable $t \in [a,b]$.

Theorem 4. *Let* $f,\, g > 0,\ 1 \leq p \leq q < +\infty$ *, if*

$$0 < m(t) \leq \frac{f(s)}{g(s)} \leq M(t), \quad \textit{for} \ \ \textit{all}\ s,\, t \in [a,x], \tag{53}$$

then the following inequality yield

$$\left({}_{a^+}\mathbf{J}_v^{\psi} f^p(x)\right)^{\frac{1}{p}} + \left({}_{a^+}\mathbf{J}_v^{\psi} g^p(x)\right)^{\frac{1}{p}} \leq K_1 \left({}_{a^+}\mathbf{J}_v^{\psi} (f(x)+g(x))^q\right)^{\frac{1}{q}}, \tag{54}$$

where

$$K_1 =: K_1^{p,q}(x) = {}_{a^+}\mathbf{J}_v^{\psi} \left[\frac{1+M(x)(m(x)+2)}{(m(x)+1)(M(x)+1)}\right] \left[{}_{a^+}\mathbf{J}_v^{\psi} \boldsymbol{1}(x)\right]^{\frac{q-p}{pq}-1}. \tag{55}$$

Proof. From the inequality (16) and the hypothesis (53), we get

$$\left({}_{a^+}\mathbf{J}_v^{\psi} f^p(x)\right)^{\frac{1}{p}} + \left({}_{a^+}\mathbf{J}_v^{\psi} g^p(x)\right)^{\frac{1}{p}} \leq \left(\frac{1+M(t)(m(t)+2)}{(m(t)+1)(M(t)+1)}\right)$$

$$\times \left[{}_{a^+}\mathbf{J}_v^{\psi} \mathbf{1}(x)\right]^{\frac{q-p}{pq}} \left({}_{a^+}\mathbf{J}_v^{\psi} (f(x)+g(x))^q\right)^{\frac{1}{q}},$$

multiplying by $\frac{\psi'(t)(\psi(x)-\psi(t))v(t)}{v(x)\,k\,\Gamma_k(\alpha)}$ and by integrating with respect to t on $[a,x]$ and we apply the inequality (14), we give us

$${}_{a^+}\mathbf{J}_v^{\psi} \mathbf{1}(x) \left[\left({}_{a^+}\mathbf{J}_v^{\psi} f^p(x)\right)^{\frac{1}{p}} + \left({}_{a^+}\mathbf{J}_v^{\psi} g^p(x)\right)^{\frac{1}{p}}\right] \leq$$

$${}_{a^+}\mathbf{J}_v^{\psi} \left[\frac{1+M(x)(m(x)+2)}{(m(x)+1)(M(x)+1)}\right] \left[{}_{a^+}\mathbf{J}_v^{\psi} \mathbf{1}(x)\right]^{\frac{q-p}{pq}} \left({}_{a^+}\mathbf{J}_v^{\psi} (f(x)+g(x))^q\right)^{\frac{1}{q}},$$

therefore

$$\left({}_{a^+}\mathbf{J}_v^{\psi} f^p(x)\right)^{\frac{1}{p}} + \left({}_{a^+}\mathbf{J}_v^{\psi} g^p(x)\right)^{\frac{1}{p}} \leq$$

$${}_{a^+}\mathbf{J}_v^{\psi} \left[\frac{1+M(x)(m(x)+2)}{(m(x)+1)(M(x)+1)}\right] \left[{}_{a^+}\mathbf{J}_v^{\psi} \mathbf{1}(x)\right]^{\frac{q-p}{pq}-1} \left({}_{a^+}\mathbf{J}_v^{\psi} (f(x)+g(x))^q\right)^{\frac{1}{q}}.$$

□

So, we get the inequality (54).

Putting $q = p$, we get the following Corollary.

Corollary 8. *Let* $f, g > 0,\ 1 \leq p < +\infty$, *if*

$$0 < m(t) \leq \frac{f(s)}{g(s)} \leq M(t), \quad for \ \ all \ s, t \in [a, x],$$

then the following inequality yield

$$\left({}_{a^+}\mathbf{J}_v^{\psi} f^p(x) \right)^{\frac{1}{p}} + \left({}_{a^+}\mathbf{J}_v^{\psi} g^p(x) \right)^{\frac{1}{p}} \leq K_2 \left({}_{a^+}\mathbf{J}_v^{\psi} (f(x) + g(x))^p \right)^{\frac{1}{p}},$$

where

$$K_2 =: K_2^p(x) = {}_{a^+}\mathbf{J}_v^{\psi} \left[\frac{1 + M(x)(m(x) + 2)}{(m(x) + 1)(M(x) + 1)} \right] \left[{}_{a^+}\mathbf{J}_v^{\psi} \boldsymbol{1}(x) \right]^{-1}.$$

4 Conclusion

This paper presents a novel approach to reverse Minkowski-type inequalities using the k-weighted fractional integral operator, as well as several related inequalities. Our results generalize previously obtained inequalities with two positive parameters $1 \leq p \leq q < \infty$, in certain special cases, which depend on the choice of the ψ function.

References

[1] Aljaaidi, T.A. et al., *Generalized proportional fractional integral functional bounds in Minkowski's inequalities*, Adv. Differ. Equ. **202** (2021), Article number 419.

[2] Benaissa, B. and Benguessoum, A., *Reverses Hardy-Type inequalities via Jensen integral inequality*, Math. Montisnigri. **52** (2021), 43-51.

[3] Benaissa, B., Sarikaya, M.Z. and Senouci, A., *On some new Hardy-type inequalities*, Math. Meth. Appl .Sci. **43** (2020), 8488-8495.

[4] Bougoffa, L., *On Minkowski and Hardy integral inequality*, J. Inequal. Pure and Appl. Math. **7** (2006), no. 2, Article number 60

[5] Dahmani, Z., *On Minkowski and Hermite-Hadamard integral inequalities via fractional integral*, Ann. Funct. Anal. **1** (2010), 51-58.

[6] Feng, Q., Siddra, H., Shahid, M. and Muhammad, N.N., *Generalized k-fractional conformable integrals and related inequalities*, AIMS Mathematics. **4** (2019), no. 3, 343-358.

[7] Iqbal. S., Mubeen, S. and Tomar, M., *On Hadamard k-fractional integrals*, J. Fract. Calc. Appl. **9** (2018), no. 2, 255-267.

[8] Jarad, F., Abdeljawad, T. and Shah, K., *On the weighted fractional operators of a function with respect to another function*, Fractal Fract. **28** (2020), no. 8, Article ID 2040011, 12 pages.

[9] Naz, S. and Naeem, M.N., *New generalized reverse Minkowski inequality and related integral inequalities via generalized fractional Hilfer-Katugampola derivative*, Punjab Univ. J. Math. (Lahore). **53** (2021), no. 4, 247-260.

[10] Ozdemir, E. and Dragomir. M., *On the Hermite-Hadamard inequality and other integral inequalities involving two functions*, J. Inequal. Appl. **2010**, (2010), Article ID 148102.

[11] Usta, F., Budak, H., Ertugral, F., Sarikaya, M.Z., *The Minkowski's inequalities utilizing newly defined generalized fractional integral operators*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **68** (2019), no. 1, 686-701.

[12] Rahman, G. et al., *The Minkowski inequalities via generalized proportional fractional integral operators*, Adv. Differ. Equ. **2019** (2019), Article number 287.

[13] Rashid, S. et al., *New generalized reverse Minkowski and related integral inequalities involving generalized fractional conformable integrals*, J. Inequal. Appl. **2020** (2020), Article number 177.

[14] Vanterler, J.D., Sousa, C. and Capelas de Oliveira. E., *The Minkowski's inequality by means of a generalized fractional integral*, AIMS Ser. Appl. Math. **3** (2018), 131-147.