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NUMERICAL SOLUTION FOR STOCHASTIC MIXED NONLINEAR SCHRÖDINGER EQUATION

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Abstract

In this paper, a numerical study of a mixed nonlinear stochastic Schrödinger equation in the case of an additive white noise and with mixed concave-convex, sub-super nonlinearities is developed. The influence of the stochastic part on the deterministic solutions such as stationary states and blow-up solutions is investigated. Numerical examples are provided with error estimates.

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1 Introduction

In this work, we are interested to the study of a one-dimensional stochastic nonlinear Schrödinger (NLS) equation with both a sublinear and a superlinear power law nonlinearities, and an additive noise. The deterministic equation occurs as a basic model in many areas of physics, hydrodynamics, plasma physics, nonlinear optics, molecular biology, etc. It describes the propagation of waves

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in media with both nonlinear and dispersive responses. The deterministic case has been widely studied for existence, uniqueness, nodal solutions, phase plane, non-radially symmetric solutions, as well as the asymptotic problem on the power laws ([1, 2, 5, 6, 7, 8, 17, 32]).

However, although it is an idealized model, it does not take into account many aspects such as non-homogeneity, high order terms, thermal fluctuations, and external forces, which may be modeled as a random excitation (see [18, 19, 24, 25, 28]). Propagation in random media may also be considered. The resulting rescaled equation is a random perturbation of the dynamical system of the following form

$$\begin{cases} i\dfrac{\partial u}{\partial t}+\dfrac{\partial^2 u}{\partial x^2}+|u|^{p-1}u+\alpha|u|^{q-1}u=\varepsilon f(t,x) & , \quad L_1<x<L_2,\ t>t_0, \\ u(t_0,x)=u_0(x) & , \quad L_1\leq x\leq L_2, \\ \dfrac{\partial u}{\partial x}(t,L_1)=\dfrac{\partial u}{\partial x}(t,L_2)=0 & , \quad t\geq t_0, \end{cases} \tag{1}$$

where $u=u(t,x)$, $t\geq 0$, $x\in\mathbb{R}$ is a complex-valued function, $\alpha\in\mathbb{R}$, L_1 and L_2 are real parameters. The term $f(t,x)$ includes the stochastic contribution. For an additive noise, $f(t,x)=\dot{\chi}(t,x)$ is real-valued, Gaussian, white in time and either white or correlated in space. In this case, the noise does not depend on the solution. The size of the noise is controlled by the parameter $\varepsilon>0$. Finally, the nonlinearity powers p and q satisfy $0<q<1<p$.

Here, we are particularly interested in the influence of a noise acting as a potential on the behavior of the solution. Such noise has been considered in [26] where its paths are smooth functions and the nonlinearity is subcritical. The case of a white noise, considered here, has been introduced in the context of crystals (see [3, 4] and also [27, 33] for other models). It is expected that such noise has a strong influence on the solution blow-up. It may delay or even prevent the formation of a singularity. In [15], some numerical simulations tend to show that this is the case for a very irregular noise such as a space-time white noise. However, in the supercritical case and for a space-correlated and non-degenerate noise, it has been observed that, on the contrary, any solution seems to blow-up in a finite time. Recall that in the deterministic case, only a restricted class of solutions blows up. Our aim is to prove rigorously such a behavior. It is mathematically very difficult to consider space-time white noises due to the lack of smoothing effect in the Schrödinger equation. Thus, we restrict our attention to the study of correlated noise.

The case of an additive noise has been considered in [9, 10], where it has been proved that for any initial data, blow-up occurs in the sense that, for arbitrary $t>0$, the probability that the solution blows up before the time t is strictly positive. Thus, the noise strongly influences this blow-up phenomenon. This result is in perfect agreement with the numerical simulations. The argument is based on three ingredients: first, we generalize the deterministic argument to prove that blow-up occurs for some initial data: this is based on a stochastic version of the variance identity (see [31, 34]). Then, we use the fact that the

NLS equation is controllable by a forcing term. Thus, any initial data can be transformed into a state which yields a singular solution. Finally, since the noise is non-degenerate and the solution depends continuously on the path of the noise, we can argue that, with positive probability, the noise will be close to the control so that blow-up will happen afterward.

The paper is organized as follows: In section 2, we formulate the discrete problem associated to the continuous one revised in (1). Discrete derivatives as well as discrete functional space framework are introduced. Section 3 is concerned to the solvability of the discrete problem. Standard calculus and matrix spaces are applied to reach our aim. Section 4 is devoted to the proof of convergence of the discrete method. It is proved to be convergent of order 2 in both time and space. In section 5, consistency and stability are investigated by applying local truncation error method for the consistency and Lax-Richtmyer concept for stability. Section 6 is devoted to the implementation of some numerical simulations due to the theoretical results provided in the eventual discussions and interpretations. Finally, we conclude in section 7.

2 The discrete stochastic Schrödinger equation

In the present work, we will apply noises having their paths in $\mathcal{H}^1$ as used in [9] when studying the influence of a noise on the formation of singularities and in [11] to give a numerical discretization of a similar problem to (1).

Taking into account these facts, the system (1) will be rewritten on the form

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \dfrac{\partial^2 u}{dx^2} + |u|^{p-1}u + \alpha |u|^{q-1}u = \varepsilon \dot{\chi} & , \quad L_1 < x < L_2,\ t > t_0, \\ u(t_0, x) = u_0(x) & , \quad L_1 \leq x \leq L_2 \\ \dfrac{\partial u}{\partial x}(t, L_1) = \dfrac{\partial u}{\partial x}(t, L_2) = 0 & , \quad t \geq t_0. \end{cases} \tag{2}$$

Consider a time step $l = \delta t$ and denote

$$t_k = t_0 + kl\,,\ k \in \mathbb{N}.$$

Fix next an integer N and consider a space step

$$h = \delta x = \frac{L_2 - L_1}{N+1}.$$

We subdivide the interval $[L_1,\, L_2]$ into subintervals $[x_m,\, x_{m+1}]$ where

$$x_m = L_0 + mh\,,\ m = 0, ..., N+1.$$

Consider also positive parameters λ_i and μ_i, $i = 1, 2, 3$, such that

$$\lambda_1 + \lambda_2 + \lambda_3 = \mu_1 + \mu_2 + \mu_3 = 1.$$

These parameters play the role of calibrators for the discrete derivatives to be introduced later. Denote next u_m^k the approximation of $u(t_k, x_m)$ and U_m^k the numerical solution of (2). We introduce the following notations

$$\partial_m^k U = \frac{U_m^{k+1} - U_m^{k-1}}{2l} \quad \text{and} \quad \frac{\partial U_m^k}{\partial t} = \lambda_1 \partial_{m-1}^k U + \lambda_2 \partial_m^k U + \lambda_3 \partial_{m+1}^k U,$$

for the time derivatives, and

$$\frac{\partial U_m^k}{\partial x} = \frac{U_{m+1}^k - U_{m-1}^k}{2h} \quad \text{and} \quad \Delta_k^m U = \frac{U_{m+1}^k - 2U_m^k + U_{m-1}^k}{h^2},$$

and

$$\frac{\partial^2 U_m^k}{\partial x^2} = \mu_1 \Delta_{k+1}^m U + \mu_2 \Delta_k^m U + \mu_3 \Delta_{k-1}^m U,$$

for the space derivatives. Denote next

$$g_p(u) = |u|^{p-1} u\,, \quad g_q(u) = \alpha |u|^{q-1} u,$$

and $g(u) = g_p(u) + g_q(u)$. We discretize problem (1) as

$$i \frac{\partial U_m^k}{\partial t} + \frac{\partial^2 U_m^k}{\partial x^2} + g(U_m^k) = \varepsilon f_m^{k+\frac{1}{2}}, \tag{3}$$

where

$$f_m^{k+\frac{1}{2}} = \frac{\varepsilon}{\sqrt{hl}}\, \chi_m^{k+\frac{1}{2}}$$

is an additive noise, with $(\chi_m^{k+\frac{1}{2}})_{k\geq 0}$, $m = 0, \ldots, N+1$ being sequences of independent random variables with normal law $\mathcal{N}(0,1)$.

The numerical problem is considered under the initial data

$$\begin{cases} U_m^0 = u(t_0, x_m) = u_0(x_m), \\ U_m^1 = U_m^0 + il(u_0''(x_m) + g(u_0(x_m))), \\ 0 \leq\ m \leq\ N+1, \end{cases} \tag{4}$$

and the boundary conditions

$$U_{-1}^k = U_1^k \text{ and } U_{N+2}^k = U_N^k,\ \forall k. \tag{5}$$

The nonlinear part $g(u)$ will be approximated by

$$g(U_m^k) = \frac{1}{2}\Big[\widetilde{g}(U_m^k + U_m^{k-1}) + g_q(U_m^k) + g_q(U_m^{k-1})\Big],$$

where $\widetilde{g} = \max\limits_m |U_m^0|^{p-1}$. Next, denote

$$\begin{array}{llll} a_1 & = \mu_1\sigma + i\lambda_1, & b_1 & = b_3 = -\mu_2\sigma, \\ a_2 & = -2\mu_1\sigma + i\lambda_2, & b_2 & = 2\mu_2\sigma - \tilde{g}l, \\ a_3 & = \mu_1\sigma + i\lambda_3, & c_1 & = -\mu_3\sigma + i\lambda_1, \\ c_2 & = 2\mu_3\sigma - \tilde{g}l + i\lambda_2, & c_3 & = -\mu_3\sigma + i\lambda_3, \end{array}$$

where $\sigma = \dfrac{2l}{h^2}$. We obtain, for $1 \leq m \leq N$, and $k \geq 0$,

$$\left\{\begin{array}{l} a_1 U_{m-1}^{k+1} + a_2 U_m^{k+1} + a_3 U_{m+1}^{k+1} = b_1 U_{m-1}^k + b_2 U_m^k + b_3 U_{m+1}^k \\ \qquad\qquad\qquad\qquad + c_1 U_{m-1}^{k-1} + c_2 U_m^{k-1} + c_3 U_{m+1}^{k-1} + H_m^k + F_m^k, \\ a_2 U_0^{k+1} + (a_1 + a_3) U_1^{k+1} = b_2 U_0^k + (b_1 + b_3) U_1^k \\ \qquad\qquad\qquad\qquad + c_2 U_0^{k-1} + (c_1 + c_3) U_1^{k-1} + H_0^k + F_0^k, \\ (a_1 + a_3) U_N^{k+1} + a_2 U_{N+1}^{k+1} = (b_1 + b_3) U_N^k + b_2 U_{N+1}^k \\ \qquad\qquad\qquad\qquad + (c_1 + c_3) U_N^{k-1} + c_2 U_{N+1}^{k-1} + H_N^k + F_N^k, \end{array}\right. \tag{6}$$

where

$$H_m^k = -l(g_q(U_m^k) + g_q(U_m^{k-1})), \quad \text{and} \quad F_m^k = -2l f_m^{k+1/2}.$$

Recall now the net function $u_m^k = u(t^k, x_m)$, $0 \leq m \leq N+1$, and $k \geq 0$. By Taylor expansion, we find

$$\left\{\begin{array}{l} i\dfrac{\partial u_m^k}{\partial t} = \dfrac{\partial^2 u_m^k}{\partial x^2} + g(u_m^k) + R_m^k\,,\ 0 \leq m \leq N+1\,, \quad k \geq 0, \\ u_m^0 = u_0(x_m)\,,\ 0 \leq m \leq N+1, \\ u_m^1 = u_0(x_m) + k\Big(u_0''(x_m) + g(u_0(x_m))\Big) + S_m\,,\ 0 \leq m \leq N+1, \end{array}\right. \tag{7}$$

where R_m^k and S_m are the truncation errors, which satisfy

$$|R_m^k| \leq c_1(h^2 + k^2), \ \text{ and } \ |S_m| \leq c_1(h^2 + k^2)\,,\ 0 \leq m \leq N+1\,,\ k \geq 0, \tag{8}$$

for some constant $c_1 > 0$. Letting $e_m^k = u_m^k - U_m^k$, it follows from (6) and (7) that

$$\left\{\begin{array}{l} i\dfrac{\partial e_m^k}{\partial t} = \dfrac{\partial^2 e_m^k}{\partial x^2} + g(u_m^k) - g(U_m^k) + R_m^k\,,\ 0 \leq m \leq N+1\,,\ k \geq 0, \\ e_m^0 = 0\,, \quad e_m^1 = S_m\,,\ 0 \leq m \leq N+1. \end{array}\right. \tag{9}$$

To tackle our problem, we introduce the functional space settings. Define the set $\Omega_h = \{x_m\,;\, 0 \leq m \leq N+1\}$, and the space W_h of functions defined on Ω_h. For $v \in W_h$, we denote $v_m = v(x_m)$. The discrete inner product and the discrete L^2-norm on W_h will be, respectively,

$$(u,v)_h = h\sum_{m=0}^{N+1} u_m \overline{v_m}, \quad \text{and} \quad ||v||_2 = (v,v)_h^{1/2} = \left[h\sum_{m=0}^{N+1} |v_m|^2\right]^{1/2}.$$

3 Solvability of the discrete problem

From equation (6), we obtain the following matrix system

$$AU^{k+1} = BU^k + CU^{k-1} + H^k + F^k, \tag{10}$$

where F is the white noise vector, and A, B and C are the matrices defined by

$$A = \begin{pmatrix} a_2 & a_1+a_3 & 0 & \cdots & \cdots & 0 \\ a_1 & a_2 & a_3 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_1 & a_2 & a_3 \\ 0 & \cdots & \cdots & 0 & a_1+a_3 & a_3 \end{pmatrix},$$

$$B = \begin{pmatrix} b_2 & b_1 & b_3 & 0 & \cdots & \cdots & 0 \\ b_1 & b_2 & b_3 & \ddots & \ddots & \vdots & \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \\ \vdots & \ddots & \ddots & b_1 & b_2 & b_3 & \\ 0 & \cdots & \cdots & 0 & b_1+b_3 & b_2 & \end{pmatrix},$$

$$C = \begin{pmatrix} c_2 & c_1+c_3 & 0 & \cdots & \cdots & 0 \\ c_1 & c_2 & c_3 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & c_1 & c_2 & c_3 \\ 0 & \cdots & \cdots & 0 & c_1+c_3 & c_2 \end{pmatrix},$$

and where

$$H^k = \left(H_0^k, \ldots, H_m^k, \ldots, H_{N+1}^k\right)^T,$$

and

$$F^k = \left(F_0^k, \ldots, F_m^k, \ldots, F_{N+1}^k\right)^T,$$

where the upper-script ^{T} stands for the transpose.

In order to prove the solvability of the discrete problem (6), we have to calculate the determinant of the matrix A. This is based on techniques developed in [20, 21, 22, 23] and treating the invertibility of a general tri-diagonal matrix. We recall the basic result in what follows

Lemma 1. *[20, 21, 22, 23] For $n \in \mathbb{N}$, consider a real-valued $n \times n$-matrix*

$$A = \begin{pmatrix} d_1 & a_1 & 0 & \ldots & \ldots & 0 \\ b_2 & d_2 & a_2 & \ldots & \ldots & 0 \\ 0 & b_3 & d_3 & a_3 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & b_{n-1} & d_{n-1} & a_{n-1} \\ 0 & 0 & \ldots & 0 & b_n & d_n \end{pmatrix},$$

and define the vector $\alpha = (\alpha_0, \alpha_1, \ldots \alpha_n)$ *by*

$$\alpha_i = \begin{cases} 1 & \textit{if} \quad i = 0, \\ d_1 & \textit{if} \quad i = 1, \\ d_i\alpha_{i-1} - b_i a_{i-1}\alpha_{i-2} & \textit{if} \quad i = 2, 3, \ldots, n. \end{cases}$$

Then, $det(A) = \alpha_n$.

Now, we are able to state the main result of this section

Theorem 1. *The discrete problem (6) is uniquely solvable.*

Proof. Taking in Lemma 1, $n = N+2$, and $\alpha_n = Det_{N+2}(A)$ the determinant of the matrix A of our discrete problem (6), we get

$$Det_{N+2}(A) - a_2 Det_{N+1}(A) + a_3(a_1 + a_3)Det_N(A) = 0. \tag{11}$$

The characteristic equation associated to (11) is

$$\lambda^2 - a_2\lambda + a_3(a_1 + a_2) = 0.$$

Its determinant is

$$\Delta_{N+2} = -4\mu_1^2\sigma^2 - \lambda_2^2 + 4\lambda_1\lambda_3 + 4\lambda_3^2 - 4\mu_1\sigma^2 i(1 + 2\lambda_3).$$

We now split the proof into cases.

Case 1. $\mu_1 = 0$, $\lambda_2 \in]0, 2(\sqrt{2} - 1)[$, $\lambda_1 \in]0, 1[$, and $\lambda_3 = \dfrac{\lambda_2^2}{4(1 - \lambda_2)}$: In this case, we obtain $\Delta_{N+2} = 0$, and the characteristic equation (11) has a double solution $\lambda = i\lambda_2$. Simple computation yields that

$$Det_{N+2}(A) = (i\lambda_2)^{N+2} \neq 0.$$

Case 2. $\mu_1 = 0$, $\lambda_2 \in]0, 2(\sqrt{2} - 1)[$, $\lambda_1 \in]0, 1[$ and $\lambda_3 < \dfrac{\lambda_2^2}{4(1 - \lambda_2)}$. We get a determinant $\Delta_{N+2} = -\omega_0^2 < 0$, with $\omega_0 \in (0, \infty)$. In this case, we obtain two different complex roots for the characteristic equation (11),

$$r_1 = \frac{a_2 + i\omega_0}{2}, \quad \text{and} \quad r_2 = \frac{a_2 - i\omega_0}{2},$$

which yields that

$$Det_{N+2}(A) = -i\frac{r_1^{N+3} - r_2^{N+3}}{\omega_0} \neq 0.$$

Case 3. $\mu_1 = 0$, $\lambda_2 \in]0, 2(\sqrt{2} - 1)[$, $\lambda_1 \in]0, 1[$, and $\lambda_3 > \dfrac{\lambda_2^2}{4(1 - \lambda_2)}$. We get a determinant $\Delta_{N+2} = \omega_0^2 > 0$, with $\omega_0 \in (0, \infty)$. In this case, we obtain two different real number roots for the characteristic equation (11),

$$r_1 = \frac{a_2 + \omega_0}{2}, \quad \text{and} \quad r_2 = \frac{a_2 - \omega_0}{2},$$

which yields that

$$Det_{N+2}(A) = \frac{r_1^{N+3} - r_2^{N+3}}{\omega_0} \neq 0.$$

Case 4. $\mu_1 = 0$, $\lambda_2 \in [2(\sqrt{2}-1), 1[$, $\lambda_1, \lambda_3 \in]0,1[$: In this case,

$$\lambda_3 < \frac{\lambda_2^2}{4(1-\lambda_2)}.$$

Therefore, we obtain

$$\Delta = 4(1-\lambda_2)(\lambda_3 - \frac{\lambda_2^2}{4(1-\lambda_2)}) = -\omega_0^2 < 0.$$

with $\omega_0 \in (0, \infty)$. There are in this case two complex roots

$$r_1 = i\frac{\lambda_2 - \omega_0}{2}, \quad \text{and} \quad r_2 = i\frac{\lambda_2 + \omega_0}{2},$$

for the characteristic equation (11). Standard computations yield that

$$Det_{N+2}(A) = \left[\left(\frac{\lambda_2 - \omega_0}{2}\right)^{N+3} - \left(\frac{\lambda_2 - \omega_0}{2}\right)^{N+3}\right]\frac{i^{N+2}}{\omega_0} \neq 0.$$

Case 5. $\mu_1 \neq 0$: Let in this case $\omega_0 \in \mathbb{C}^\star$ be such that $\Delta_{N+2} = -\omega_0^2$. Let also

$$r_1 = \frac{a_2 + i\omega_0}{2}, \quad \text{and} \quad r_2 = \frac{a_2 - i\omega_0}{2}$$

be the complex roots of the characteristics equation (11). We easily obtain

$$Det_{N+2}(A) = \frac{r_2^{N+3} - r_1^{N+3}}{\omega_0} i \neq 0.$$

From all the cases above, we conclude that the matrix A is invertible. It follows that the problem (10) has a unique solution.

4 Convergence of the discrete problem

The main result of this section is to prove the convergence of the difference scheme. We will prove precisely that the method is unconditionally convergent with order 2 in both time and space.

Theorem 2. *Let u and U^k be the solutions of (2) and (6), respectively. Assume further that u is sufficiently regular. Then, for h, l small enough, we have*

$$\|U^k - u^k\|_2 \leq C(h^2 + l^2),$$

where $u^k = u(t^k, .)$, and $C > 0$ is a constant independent of h and l.

Proof. The proof reposes on the recurrence rude. We deduce from (8)-(9) that

$$\|e^0\|_2 \leq C(h^2+l^2), \quad \text{and} \quad \|e^1\|_2 \leq C(h^2+l^2).$$

So, suppose that this occurs for all $l \leq k$, that is

$$\|e_m^l\|_2 \leq C(h^2+l^2),\ \forall l \leq k. \tag{12}$$

Denote

$$\widetilde{e}_m^k = \frac{e_m^{k+1}+e_m^{k-1}}{2l}.$$

Using (9), we obtain

$$i(\frac{\partial e^k}{\partial t},\widetilde{e}^k)_h = (\frac{\partial^2 \widetilde{e}^k}{\partial X^2},\widetilde{e}^k)_h + (g(u^k)-g(U^k),\widetilde{e}^k)_h + (R^k,\widetilde{e}^k)_h. \tag{13}$$

We will examine now each term of the above equality. We have, for all k, m,

$$\frac{\partial e_m^k}{\partial t}\widetilde{e}_m^k = \frac{1}{4l}\left[|e_m^{k+1}|^2 + 2iIm(e_m^{k+1}\overline{e_m^{k-1}}) - |e_m^{k-1}|^2\right]. \tag{14}$$

Similarly, we have

$$\begin{aligned}\frac{\partial^2 e_m^k}{\partial x^2}\widetilde{e}_m^k &= \mu_1\frac{e_{m+1}^{k+1}-2e_m^{k+1}+e_{m-1}^{k+1}}{h^2}\widetilde{e}_m^k + \mu_2\frac{e_{m+1}^k-2e_m^k+e_{m-1}^k}{h^2}\widetilde{e}_m^k\\ &+\mu_3\frac{e_{m+1}^{k-1}-2e_m^{k-1}+e_{m-1}^{k-1}}{h^2}\widetilde{e}_m^k.\end{aligned}$$

Denote $I_{m,k}^1$, $I_{m,k}^2$ and $I_{m,k}^3$, respectively, the first, the second, and the third right-hand terms of the last equality. It results that

$$\begin{aligned}I_{m,k}^1 &= \frac{\mu_1}{2h^2}(e_{m+1}^{k+1}\overline{e_m^{k+1}} + e_{m+1}^{k+1}\overline{e_m^{k-1}} - 2|e_m^{k+1}|^2\\ &\quad -2e_m^{k+1}\overline{e_m^{k-1}} + e_{m-1}^{k+1}\overline{e_m^{k+1}} + e_{m-1}^{k+1}\overline{e_m^{k-1}}).\end{aligned} \tag{15}$$

Similarly,

$$\begin{aligned}I_{m,k}^2 &= \frac{\mu_2}{h^2}(e_{m+1}^{k}\overline{e_m^{k+1}} + e_{m+1}^{k}\overline{e_m^{k-1}} - 2e_m^{k}\overline{e_m^{k+1}}\\ &\quad -2e_m^{k}\overline{e_m^{k-1}} + e_{m-1}^{k}\overline{e_m^{k+1}} + e_{m-1}^{k}\overline{e_m^{k-1}}).\end{aligned} \tag{16}$$

Finally, for the third part, we get

$$\begin{aligned}I_{m,k}^3 &= \frac{\mu_3}{h^2}(e_{m+1}^{k-1}\overline{e_m^{k+1}} + e_{m+1}^{k-1}\overline{e_m^{k-1}} - 2e_m^{k-1}\overline{e_m^{k+1}}\\ &\quad -2|e_m^{k-1}|^2 + e_{m-1}^{k-1}\overline{e_m^{k+1}} + e_{m-1}^{k-1}\overline{e_m^{k-1}}).\end{aligned} \tag{17}$$

For the nonlinear part, denote

$$X_m^k = (g(u_m^k)-g(U_m^k))\tilde{e}_m^k,$$

and

$$Y_m^k = (g_q(u_m^k)-g_q(U_m^k))\tilde{e}_m^k.$$

Since $p>1$, the function g_p is locally Lypschitz continuous. Hence,

$$|g_p(u_m^k)-g_p(U_m^k)|\leq C|e_m^k|,$$

for some constant C independent of k,m. Consequently,

$$\begin{array}{rl}|X_m^k| & \leq \dfrac{1}{2}\big|(g_p(u_m^k)-g_p(U_m^k))\overline{e_m^{k+1}}\big|+\dfrac{1}{2}\big|(g_p(u_m^k)-g_p(U_m^k))\overline{e_m^{k-1}}\big| \\ & \leq C_1\big(|e_m^k e_m^{k+1}|+|e_m^k e_m^{k-1}|\big),\end{array}$$

for some constant C_1 independent of k,m. Similarly, using the fact that the function g_q is locally q-Hölder continuous, we get

$$|Y_m^k|\leq CA_k(|e_m^{k+1}|^2+|e_m^{k-1}|^2),$$

where $A_k=\|e^k\|_2^q+\|e^{k-1}\|_2^q$. As a result, we obtain

$$\begin{array}{rcl}|(g(u_m^k)-g(U_m^k))\widetilde{e}_m^k| & \leq & C\big(|e_m^k e_m^{k+1}|+|e_m^k e_m^{k-1}| \\ & & +A_k(q)(|e_m^{k+1}|^2+|e_m^{k-1}|^2)\big).\end{array} \tag{18}$$

Denote

$$\begin{array}{l}\varphi_{1,k}(l,h)=2lC_1+4lA_k+(3+5\mu_1)\dfrac{\sigma}{2}, \\ \varphi_{2,k}(l,h)=4(\mu_2\sigma+lC_1), \\ \varphi_{3,k}(l,h)=2lC_1+4lA_k+(2+2\mu_3)\sigma.\end{array}$$

Taking the real and imaginary parts in (13), and taking into account (14)-(18), we obtain

$$(1-\varphi_{1,k}(l,h))\|e^{k+1}\|_2^2\leq\varphi_{2,k}(l,h))\|e^k\|_2^2+(1+\varphi_{3,k}(l,h))\|e^{k+1}\|_2^2. \tag{19}$$

Using the recurrence hypothesis (12), we immediately decuce that

$$\|e^{k+1}\|_2\leq C, \tag{20}$$

where C is independent of the time index k.

5 Consistency and stability of the method

The present section is twofold. We propose to investigate firstly the consistency of the discrete method developed above using the so-called local truncation error. The second part will be devoted to the stability of the numerical scheme.

Lemma 2. *The following assertions hold.*

- *For $\lambda_1\neq\lambda_3$, $\mu_1\neq\mu_3$, and $l=o(h^{4+\eta})$ for $\eta>0$ small enough, the numerical scheme is consistent with order (h^2+l^2).*
- *For $\lambda_1\neq\lambda_3$, $\mu_1=\mu_3$, and $l=o(h^{2+\eta})$ for $\eta>0$ small enough, the numerical scheme is consistent with order (h^2+l^2).*

- *For $\lambda_1 = \lambda_3$, $\mu_1 = \mu_3$, and $l = o(h^{1+\eta})$ for $\eta > 0$ small enough, the numerical scheme is consistent with order $(h^2 + l^2)$.*

Proof. The principal part of the local truncation error of the method arising from the scheme (3) is given by

$$\begin{aligned}
&\mathcal{L}_u(t,x) = ih\frac{\partial^2 u}{\partial t\partial x}(\lambda_3 - \lambda_1) + l(\mu_1 - \mu_3)\left(\frac{2}{h^2}\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^2 \partial t}\right)\\
&+l^2\left(\frac{1}{h^2}(\mu_1+\mu_3)\frac{\partial^2 u}{\partial t^2} + \frac{1}{6}\frac{\partial^3 u}{\partial t^3}\right) + h^2\left(\frac{i}{2}(\lambda_1+\lambda_3)\frac{\partial^3 u}{\partial x^2\partial t} + \frac{1}{24}\frac{\partial^4 u}{\partial x^4}\right)\\
&+o(l^2+h^2).
\end{aligned} \tag{21}$$

It is clear that $\mathcal{L}_u$ tend toward 0 as l and h tend to 0, which ensures the consistency of the method. Furthermore, the method may be always consistent with an order 2 in time and space by setting $l = o(h^{4+\eta})$ for $\eta > 0$ small enough. Moreover, this shows the utility of the calibrating barycenter parameters λ_i and μ_i, for $i = 1, 2, 3)$ which permits effectively to minimize the numerical error and to regulate the order of consistency. A special case may be obtained by following [12, 13] and [14] where the authors considered

$$\begin{cases} \lambda_1 = \lambda_3 = \lambda & \in\ ]0,1[, \\ \lambda_2 = 1 - 2\lambda & \in\ ]0,1[\end{cases} \qquad \text{and} \qquad \begin{cases} \mu_1 = \mu_3 = \mu & \in\ ]0,1[, \\ \mu_2 = 1 - 2\mu & \in\ ]0,1[. \end{cases}$$

We obtain in this case,

$$\mathcal{L}_u(t,x) = l^2\left(\frac{2\mu}{h^2}\frac{\partial^2 u}{\partial^2 t} + \frac{1}{6}\frac{\partial^3 u}{\partial t^3}\right) + h^2\left(i\lambda\frac{\partial^3 u}{\partial x^2\partial t} + \frac{1}{24}\frac{\partial^4 u}{\partial x^4}\right) + o(l^2+h^2).$$

This particular case makes that the scheme is consistent just when taking the assumption $l = o(h^{1+\eta})$, and not necessarily $l = o(h^2)$ as in the general cases. This is due to the fact that the second term of the summation in the right hand of the formula (21), which was in the origin of the assumption ($l = o(h^2)$), is simplified as $\mu_1 = \mu_3$. A particular numerical example will be treated and presented in the next section.

Now, we move to the study of the stability of this method. We will apply the well known Lyapunov criterion of stability, which states that a system $\Phi(X_k, X_{k-1}, \ldots, X_0) = 0$ is stable if for any bounded initial solution X_0 the solution X_k remains bounded, for all $k \geq 0$. We have in our case the following result.

Lemma 3. $\mathcal{P}_k$: *The solution U^k is bounded independently of the time index k whenever the initial solution U^0 is bounded.*

Proof. The proof follows similar techniques as in the case of the convergence treated in section 4 (Theorem 2) by applying the induction rule on k for problem (10). Indeed, using equation (10), we obtain, for all k,

$$\|U^{k+1}\| \leq \|A^{-1}B\|.\|U^k\| + \|A^{-1}C\|.\|U^{k-1}\| + \|A^{-1}H^k\| + \|A^{-1}F^k\|. \tag{22}$$

Next, recall that, for $l = o(h^{4+\eta})$ small enough, we obtain the uniform limits

$$A \longrightarrow iI_\lambda, \quad B \longrightarrow 0, \quad \text{and} \quad C \longrightarrow iI_\lambda,$$

whenever $l, h \to 0$, where

$$I_\lambda = i \begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_3 & 0 & \cdots & \cdots & 0 \\ \lambda_1 & \lambda_2 & \lambda_3 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & \cdots & \cdots & 0 & \lambda_1 & \lambda_2 + \lambda_3 \end{pmatrix}.$$

As a consequence, for l, h small enough,

$$\|A^{-1}\|, \ \|A^{-1}C\|, \ \|A^{-1}B\| \leq C_\lambda, \tag{23}$$

for some constant $C_\lambda = C(\lambda_1, \lambda_2, \lambda_3) > 0$. As a result, (22) yields that

$$\|U^{k+1}\| \leq C_\lambda \Big(\|U^k\| + \|U^{k-1}\| + \|H^k\| + \|F^k\| \Big). \tag{24}$$

Now, using the recurrence method the result is immediate.

Remark 1. *The last result on stability may induce another proof of the convergence of the numerical scheme by using the well-known Lax-Richtmayer equivalence theorem, which states that for consistent numerical approximations, stability and convergence are equivalent. Recall here that we have already proved that the numerical scheme is consistent.*

6 Numerical implementations

We want to investigate the noise effects on stationary solutions in different cases. As mentioned in the introduction, stationary waves play an important role in physics, while the effect of white noise on propagation in not well-known. Noise effects on solitary waves have already been studied for the NLS equation, and for the Korteweg-de-Vries equation (see [15, 16, 29, 30]). One of the famous classes of NLS solution in the deterministic case is the so-called single-soliton-type solutions for which $u(t, x)$ is given by

$$u(t, x) = \sqrt{\frac{2a}{q_s}} \exp(i(\frac{1}{2}cx - \theta t + \varphi)) sech(\sqrt{a}(x - ct) + \phi),$$

where $a, q_s, \theta, c, \varphi$, and ϕ are some appropriate constants. For fixed t, this function decays exponentially as $|x| \to \infty$, and is considered as a soliton-type disturbance traveling with a speed c, and with a governed amplitude. In the cubic NLS equation, we have the restriction $\theta = \dfrac{c^2}{4} - a$.

In a first example, and in order to compare with existing studies, such as [12, 13, 14], we chose the following parameters

$$\lambda_1 = \lambda_3 = \lambda = \frac{1}{5}, \qquad \text{and} \qquad \mu_1 = \mu_3 = \mu = \frac{1}{3}.$$

In a second general example, we take different values for the parameters λ_i, and μ_i, $1 \leq i \leq 3$, such as

$$\lambda_1 = 0.29\,, \quad \lambda_2 = 0.38\,, \quad \lambda_3 = 0.33,$$

and

$$\mu_1 = 0.25\,, \quad \mu_2 = 0.45\,, \quad \mu_3 = 0.3.$$

In the both studied cases, the presented simulations of equation (2) are given considering an additive noise. In the numerical scheme (3), the computations are done for $-80 \leq x \leq 100$ with a space step $h = 1$, and for $0 \leq t \leq 10$, with a time step $l = 0.01$. We fix also the soliton parameters $a = 0.01$, $q_s = 1$, and $c = 0.1$, and the phase parameters $\phi = \varphi = 0$. Finally, we consider the parameters of the nonlinearity $q = 0.73$, and $p = 1.5$.

For small amplitudes of the noise, corresponding to small values of the parameter ε, we may see that the solitary wave is not strongly perturbed and the noise does not prevent its propagation. This is clearly expressed in the particular case where $\lambda_1 = \lambda_3 = \lambda = \frac{1}{5}$, and $\mu_1 = \mu_3 = \mu = \frac{1}{3}$ by Figure 1 (e,f,g). It is also confirmed, in the more general case, where $\lambda_1 \neq \lambda_3$, and $\mu_1 \neq \mu_3$, by Figure 3 (b,c,d). However, from Figure 1 (c,d,e), we notice that as the noise level becomes higher, the wave is progressively destroyed. We remark the same behavior in the general case, especially in Figure 2 (c,d,e), and Figure 3-a. For the deterministic case, corresponding to $\varepsilon = 0$, and physically interpreted by the absence of noise, the solution of the problem is given, in the first case by Figure 1-h, and in the second one by Figure 3-j. It represents the stationary wave.

Now, in both cases, taking the amplitude of the noise $\varepsilon \geq 0.3$, it is clearly seen that the wave explodes under the influence of the additive noise. The phenomenon appears respectively in Figure 1 (a,b and slightly c) for the particular case where $\lambda_1 = \lambda_3$, and $\mu_1 = \mu_3$, and in Figure 2 (a,b and slightly c) for the more general case where $\lambda_1 \neq \lambda_3$, and $\mu_1 \neq \mu_3$.

In the next, we will look at the general case. Being interested in the right side of Figures 2 and 3, we can see that the infinite norm of u presents many observable peaks. As we decrease the value of ε, the amplitude of the noise decreases consequently and its influence on the deterministic solution disappears progressively. That is why the blow-up phenomenon appears less and less, and so are the peaks. More precisely, we can remark that the soliton wave starts to appear with small perturbations of the deterministic equation, corresponding to weak values of the parameter ε. As examples, we can cite $\varepsilon = 0.04$, $\varepsilon = 0.02$ and $\varepsilon = 0.01$ respectively in Figures 3-b,c,d.

Finally, taking $\varepsilon = 0$ in Figure 3-e, we can say that we did proceed to a simulation of the solution in the deterministic case and the infinite norm's figure tends to a linear shape. We can say that we did start with a blow-up phenomenon to convert it into a soliton wave, also we did straightening the infinite norm of the solution. Looking at the effect of noise on one trajectory, we show, in Figure 4, the profiles of the solution with additive noise at a final computation time for several values of ε. We see that in any case the profile has the same shape as the solitary wave and has been diffused and damped. In some cases, we see in Figure 4 that the noise effect is really strong and the wave has been completely destroyed.

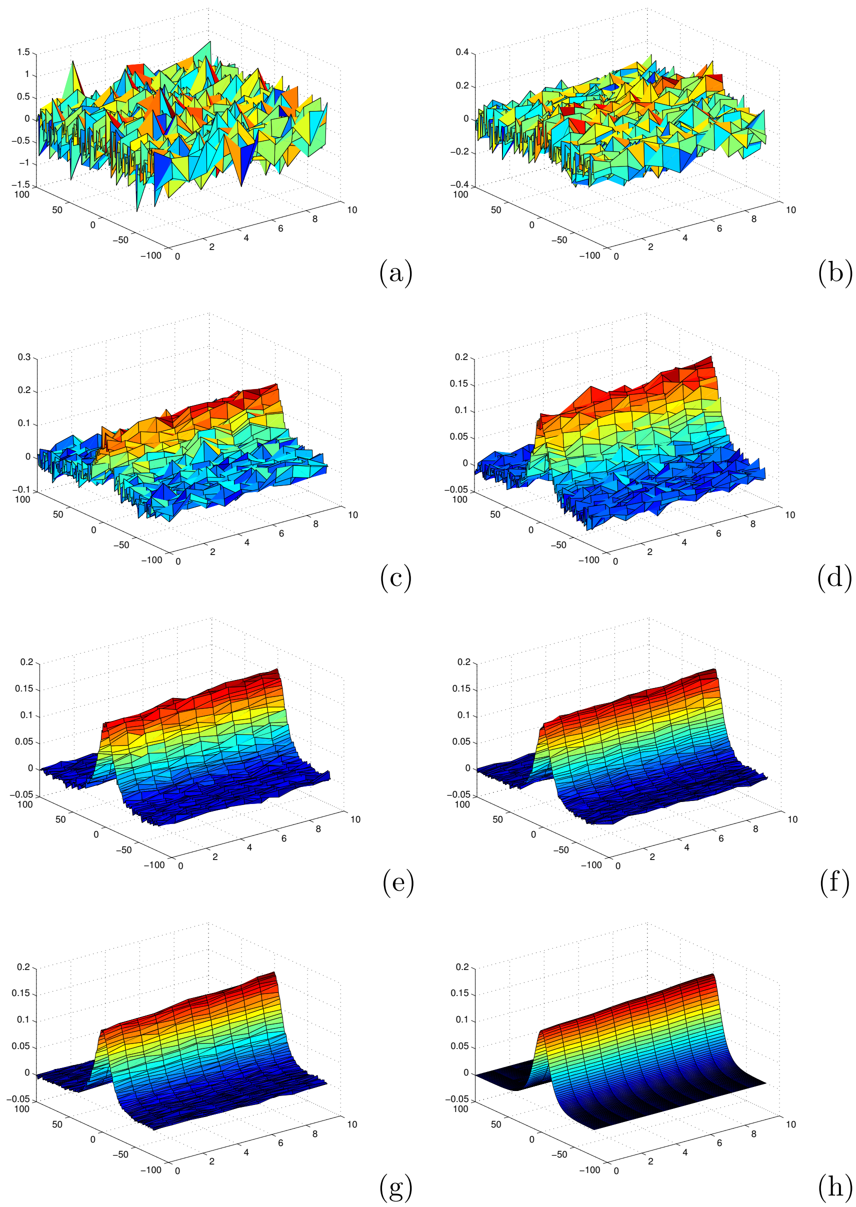

Figure 1: Plots of $(t, x, |u(t, x)|)$ for $\lambda_1 = \lambda_3 = \lambda = \dfrac{1}{5}$ and $\mu_1 = \mu_3 = \mu = \dfrac{1}{3}$, (a) $\varepsilon = 0.45$, (b) $\varepsilon = 0.35$, (c) $\varepsilon = 0.25$, (d) $\varepsilon = 0.15$, (e) $\varepsilon = 0.1$, (f) $\varepsilon = 0.05$, (g) $\varepsilon = 0.04$, (h) $\varepsilon = 0$.

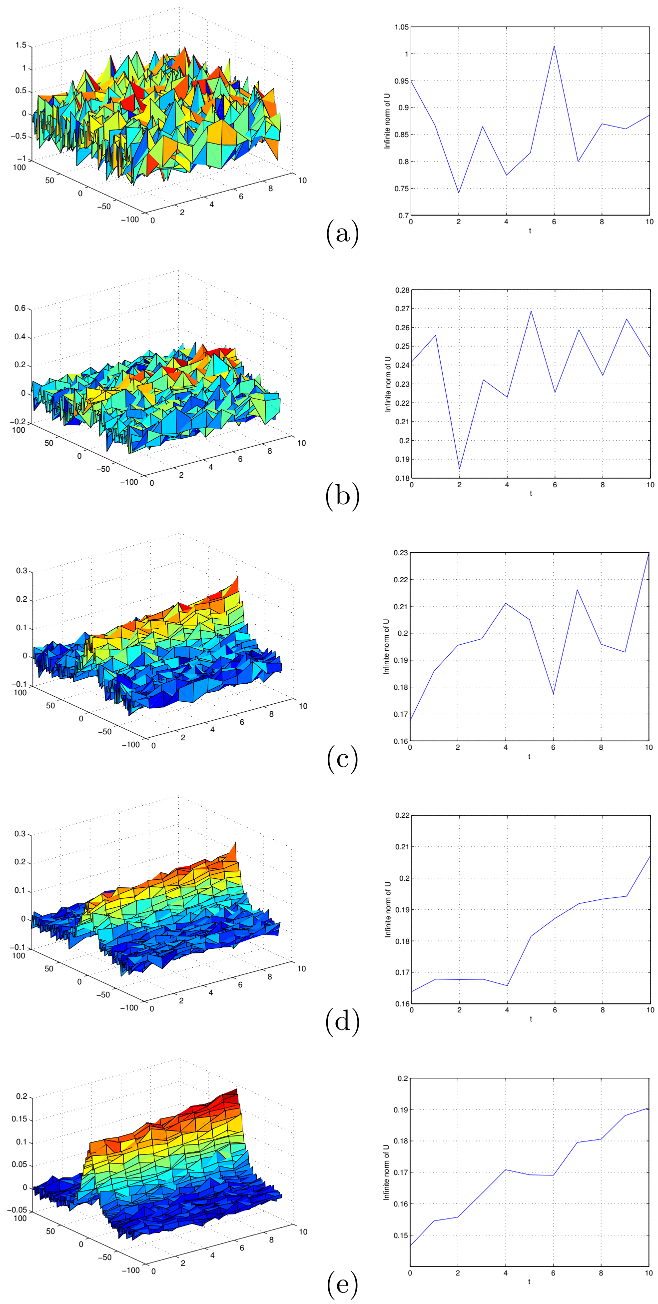

Figure 2: Left: $(t, x, |u(t, x)|)$, (a) $\varepsilon = 0.45$, (b) $\varepsilon = 0.35$, (c) $\varepsilon = 0.3$, (d) $\varepsilon = 0.25$, (e) $\varepsilon = 0.15$, Right: $\|u\|_\infty = f(\varepsilon)$.

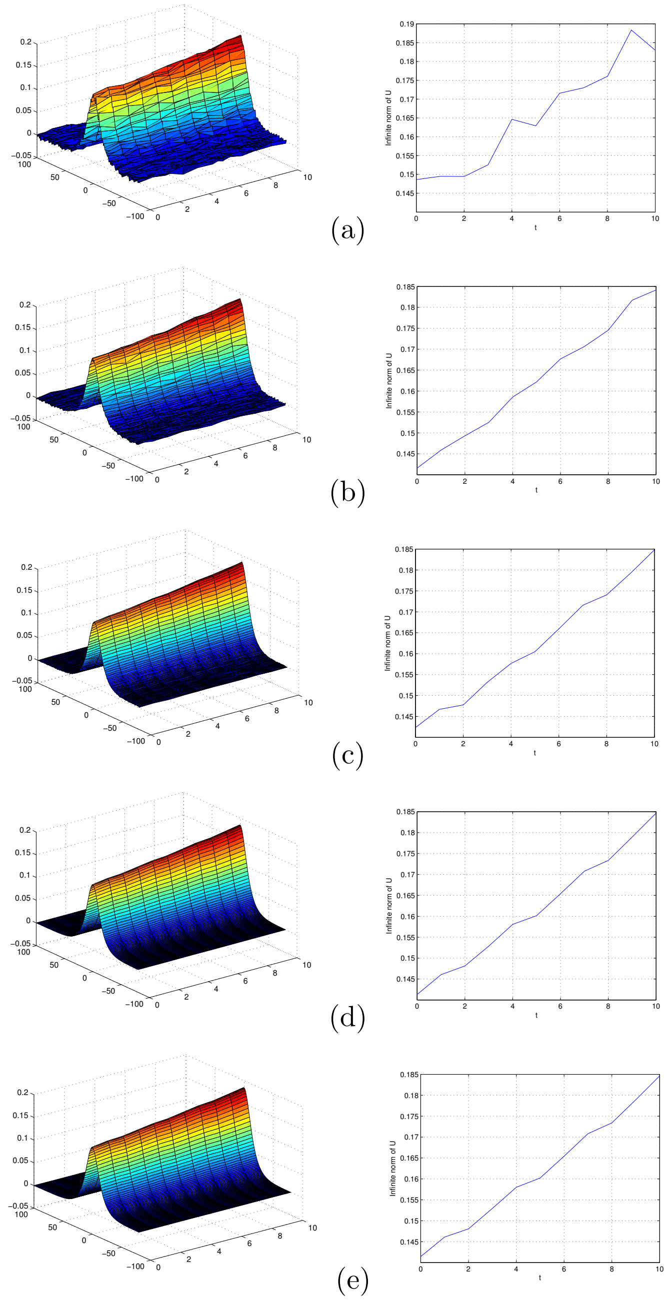

Figure 3: Left: $(t, x, |u(t,x)|)$, (a) $\varepsilon = 0.1$, (b) $\varepsilon = 0.04$, (c) $\varepsilon = 0.02$, (d) $\varepsilon = 0.01$, (e) $\varepsilon = 0$. Right: $\|u\|_\infty = f(\varepsilon)$.

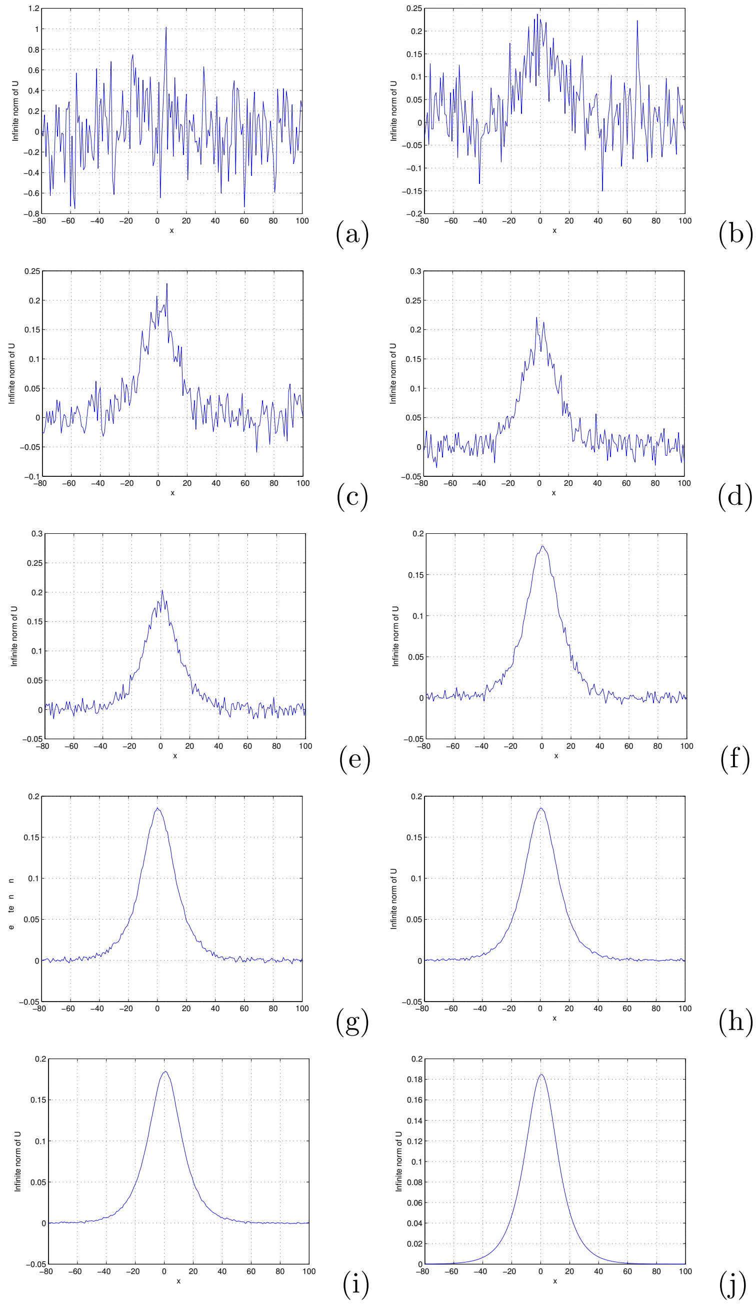

Figure 4: $(x, |u(t,x)|)$ for t fixed (one trajectory), (a) $\varepsilon = 0.45$, (b) $\varepsilon = 0.35$, (c) $\varepsilon = 0.3$, (d) $\varepsilon = 0.25$, (e) $\varepsilon = 0.15$, (f) $\varepsilon = 0.1$, (g) $\varepsilon = 0.04$, (h) $\varepsilon = 0.02$, (i) $\varepsilon = 0.01$, (j) $\varepsilon = 0$.

7 Conclusion

It is noted that the stochastic nonlinear equation (1) can be considered as a white noise random perturbation of the deterministic equation defined by ($\varepsilon = 0$). Such a perturbation occurs when the size of the noise, described by the real-value parameter ε, is positive. We proved, in this work, that as $\varepsilon \to 0$, the solution of the perturbed case converges to the unique trajectory of the deterministic equation. Then, we may conclude that the stochastic model would be more realistic, and we thus observe a similar evolution phenomena about the solution as that given by the deterministic case. We propose in future directions to investigate the case where the noise depends on the solution u, and also the case of other types of noise such as the multiplicative case, correlated noises, and so on.

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