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EXISTENCE AND STABILITY OF SOLUTIONS FOR FUZZY FRACTIONAL MULTI-PANTOGRAPH DIFFERENTIAL EQUATIONS WITH ψ-CAPUTO DERIVATIVE

Elhoussain ARHRRABI[*,1] M'hamed ELOMARI[2] and Said MELLIANI[3]

Abstract

In the current work, we examine a novel type of fuzzy fractional multi-pantograph differential equations involving ψ-Caputo derivative. Firstly, we establish the existence result by using Schaefer fixed point theorem and then the uniqueness is proved by using Banach fixed point theorem. Secondly, with aid of generalized Grönwall inequality, we investigative the Ulam–Hyers–Mittag-Leffler stability of solution for the problem under consideration. Lately, two examples are provided to illustrate the theoretical results.

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1 Introduction

In the realm of mathematical modeling and differential equations, the study of fuzzy fractional multi-pantograph differential equations has emerged as a powerful and innovative approach. This specialized field brings together concepts from fuzzy logic, fractional calculus, and multi-pantograph differential equations to address the complexities of real-world problems where uncertainty, multiple factors, and non-integer order dynamics coexist.
Fuzzy logic [32], with its capacity to handle imprecise and uncertain information, provides a foundation for understanding problems in which conventional mathematics falls short. Fractional calculus, on the other hand, extends the traditional

[1*] *Corresponding author*, Laboratory of Systems, Control and Decision (LSCD), School of New Engineering Sciences (ENSI), Tangier 90060, Morocco. e-mail: arhrrabi.elhoussain@gmail.com
[2]LMACS, Laboratoire de mathématiques appliqués et Calcul Scientifique, Sultan Moulay Sliman University. PO Box 523, 23000 Beni mellal, Morocco., e-mail: m.elomari@usms.ma
[3]LMACS, Laboratoire de mathématiques appliqués et Calcul Scientifique, Sultan Moulay Sliman University. PO Box 523, 23000 Beni mellal, Morocco., e-mail: s.said@usms.ma

notion of derivatives and integrals to non-integer orders, enabling the modeling of phenomena with memory, hereditary properties, and fractal behavior. In this context, the ψ-Caputo derivative [5] further enhances the flexibility of fractional calculus by introducing a modifying function ψ that adapts the derivative to the specific characteristics of the problem under consideration.

Multi-pantograph differential equations, distinguished by their inclusion of multiple delayed terms with distinct time scales, are particularly suited for describing complex problems where various factors interact with differing intensities. These equations offer a rich framework for exploring how various components influence the rate of change of a variable. Recently, the theory of fractional pantograph differential equations have been the subject of important studies, then many scientists extended these equations into new forms and presented the solvability aspect of those problems both numerically and theoretically, see [11, 29, 6, 30, 18, 19, 3, 21, 27, 24, 12, 1]. Agilan et al. [2] studies the existence of solutions of the non-linear fuzzy fractional panto-graph equations using fixed point technique. Bica et al. [16] have used a numerical method based on an iterative algorithm to obtain the approximate solution of pantograph type fuzzy Volterra integral equation. In [26, 20] the authors used the homotopy analysis method to solve the fuzzy pantograph equation in approximate analytic form. Furthermore, several scientists [13, 3, 14, 4, 15] used various fractional derivatives to investigate the existence and Ulam-Hyers stability. What's more, Derbazi et al. [17] studied the existence and uniqueness of solution for initial value problem of fractional differential equations. Also, Wang et al. [31] studied the existence and stability of solutions of Caputo type FFDEs with time-delays. They established existence results by Schauder's fixed point theorem and a hypothetical condition. Also they showed the uniqueness of the solution by using Banach contraction principle. In addition, with aid of generalized Grönwall inequality the Ulam-Hyers stability are discussed. Arhrrabi et al. [7, 8, 9, 10, **?**, **?**, **?**, **?**, **?**, 23] studied different types of fuzzy stochastic and fuzzy fractional differential equations. To the best of our knowledge, no results have been published on the existence of solutions for fuzzy fractional multi-pantograph differential equations with ψ-Caputo derivatives. As a consequence, we want to fill the gap in the literature and advance this field of study.

Here, we are concerned with a novel class of fuzzy fractional multi-pantograph differential equations with ψ-Caputo derivative that are motivated by the aforementioned studies:

$$\begin{cases} \mathcal{D}_{0^+}^{q;\psi}\mathbf{z}(u) = \mathfrak{f}\big(u, \mathbf{z}(u), \mathbf{z}(\lambda_1 u), \mathbf{z}(\lambda_2 u)\big), \qquad u \in \mathbf{I} := [0, M], \\ \\ \mathbf{z}(0) = \mathbf{z}_0, \end{cases} \tag{1}$$

where $\lambda_1, \lambda_2 \in (0,1)$, $\mathcal{D}_{0^+}^{q;\psi}$ is the ψ-Caputo fractional derivative of order $q \in (0,1)$ and $\mathfrak{f} : \mathbf{I} \times \mathbf{E^1} \times \mathbf{E^1} \times \mathbf{E^1} \longrightarrow \mathbf{E^1}$ is continuous fuzzy function.

The format of this paper is as follows. The background information and preliminary materials needed for our investigation are provided in Section 2. The existence and uniqueness results of problem under consideration are given in Section 3. Afterwards, in Section 4 Ulam–Hyers–Mittag-Leffler stability result of (1) is established via generalized Grönwall inequality. Section 5 includes two examples to demonstrate the usefulness of our findings. The last section is where you come to a conclusion.

2 Preliminaries

The definitions and propositions that are utilized throughout this paper will be introduced in this part.

Definition 1. *[31] The set of fuzzy subsets of* $\mathbb{R}$ *is denoted by* $\mathbf{E^1} := \{\Upsilon : \mathbb{R} \longrightarrow [0,1]\}$ *which satisfies:*

(i) Υ *is upper semi-continuous on* $\mathbb{R}$*,*

(ii) Υ *is fuzzy convex, i.e, for* $0 \leq \lambda \leq 1$

$$\Upsilon\big(\lambda z_1 + (1-\lambda)z_2\big) \geq min\left\{\Upsilon(z_1), \Upsilon(z_2)\right\}, \ \forall z_1, z_2 \in \mathbb{R},$$

(iii) $[\Upsilon]^0 = \overline{\{z \in \mathbb{R} : \Upsilon(z) > 0\}}$ *is compact,*

(iv) Υ *is normal, i.e,* $\exists z_0 \in \mathbb{R}$ *such that* $\Upsilon(z_0) = 1$*.*

Remark 1. $\mathbf{E^1}$ *is called the space of fuzzy number.*

Definition 2. *[31] The p-level set of* $\Upsilon \in \mathbf{E^1}$ *is defined by:*
For $p \in (0,1]$*, we have* $[\Upsilon]^p = \{z \in \mathbb{R} | \Upsilon(z) \geq p\}$ *and for* $p = 0$ *we have* $[\Upsilon]^0 = \overline{\{z \in \mathbb{R} | \Upsilon(z) > 0\}}$*.*

Remark 2. *From Definition 1, it follows that the p-level set* $[\Upsilon]^p$ *of* Υ*, is a nonempty compact interval and* $[\Upsilon]^p = [\underline{\Upsilon}(p), \overline{\Upsilon}(p)]$*. Moreover,* $len([\Upsilon]^p) = diam([\Upsilon]^p) = \overline{\Upsilon}(p) - \underline{\Upsilon}(p)$*, where the parametric form* $[\Upsilon]^p = [\underline{\Upsilon}(p), \overline{\Upsilon}(p)]$ *is the p-level set of* Υ*.* $\underline{\Upsilon}, \overline{\Upsilon}$ *are called the left and right end points of* $[\Upsilon]^p$*, respectively.*

Definition 3. *[31] For addition and scalar multiplication in fuzzy set space* $\mathbf{E^1}$*, we have*

$$[\Upsilon_1 + \Upsilon_2]^p = [\Upsilon_1]^p + [\Upsilon_2]^p = \{z_1 + z_2 \mid z_1 \in [\Upsilon_1]^p, z_2 \in [\Upsilon_2]^p\},$$

and

$$[\alpha\Upsilon]^p = \alpha[\Upsilon]^p = \{\alpha z \mid z \in [\Upsilon]^p\},$$

for all $p \in [0,1]$*.*

Definition 4. *[31] The Hausdorff distance is given by*

$$\begin{aligned}\mathbf{D}_\infty\big(\Upsilon_1, \Upsilon_2\big) &= \sup_{0 \leq p \leq 1} \left\{|\underline{\Upsilon}_1(p) - \underline{\Upsilon}_2(p)|, |\overline{\Upsilon}_1(p) - \overline{\Upsilon}_2(p)|\right\}, \\ &= \sup_{0 \leq p \leq 1} \mathcal{D}_H\big([\Upsilon_1]^p, [\Upsilon_2]^p\big).\end{aligned}$$

Remark 3. *Note that $\mathbf{E^1}$ is complete metric space with the above definition (see [31, 32]) and we have the following properties of $\mathbf{D}_\infty$:*

$$\mathbf{D}_\infty(\Upsilon_1+\Upsilon_3,\Upsilon_2+\Upsilon_3)=\mathbf{D}_\infty(\Upsilon_1,\Upsilon_2),$$

$$\mathbf{D}_\infty(\lambda\Upsilon_1,\lambda\Upsilon_2)=|\lambda|\mathbf{D}_\infty(\Upsilon_1,\Upsilon_2),$$

$$\mathbf{D}_\infty(\Upsilon_1,\Upsilon_2)\leq\mathbf{D}_\infty(\Upsilon_1,\Upsilon_3)+\mathbf{D}_\infty(\Upsilon_3,\Upsilon_2),$$

for all $\Upsilon_1,\Upsilon_2,\Upsilon_3\in\mathbf{E^1}$ and $\lambda\in\mathbb{R}$.

Definition 5. *[31] Let Υ_1, $\Upsilon_2\in\mathbf{E^1}$, if there exists $\Upsilon_3\in\mathbf{E^1}$ such that $\Upsilon_1=\Upsilon_2+\Upsilon_3$, then Υ_3 is called the Hukuhara difference of Υ_1 and Υ_2 noted by $\Upsilon_1\ominus\Upsilon_2$.*

Definition 6. *[28] The generalized Hukuhara difference (gH-difference) of $\Upsilon_1,\Upsilon_2\in\mathbf{E^1}$ is defined as follows:*

$$\Upsilon_1\ominus_{gH}\Upsilon_2=\Upsilon_3\Leftrightarrow\begin{cases}(i)\quad \Upsilon_1=\Upsilon_2+\Upsilon_3,\ \ if\quad len([\Upsilon_1]^p)\geq len([\Upsilon_2]^p).\\ \\ (ii)\quad \Upsilon_2=\Upsilon_1+(-1)\Upsilon_3,\ \ if\quad len([\Upsilon_2]^p)\geq len([\Upsilon_1]^p).\end{cases}$$

Definition 7. *[31] Let a fuzzy function $\Upsilon:[a,b]\longrightarrow\mathbf{E^1}$. If for every $p\in[0,1]$, the function $u\longmapsto len[\Upsilon(u)]^p$ is increasing (decreasing) on $[a,b]$, then Υ is called d-increasing (d-decreasing) on $[a,b]$.*

Remark 4. *If Υ is d-increasing or d-decreasing, then we say that Υ is d-monotone on $[a,b]$.*

Remark 5. *The last definition is equivalent to $[\Upsilon]^p\subseteq[\Psi]^p$ ($[\Upsilon]^p\supseteq[\Psi]^p$), for all $p\in[0,1]$.*

Notation:
• $C\big([c,d],\mathbf{E^1}\big)$ denote the set of all continuous fuzzy functions.
• $AC\big([c,d],\mathbf{E^1}\big)$ denote the set of all absolutely continuous fuzzy functions on $[c,d]$ with value in $\mathbf{E^1}$.
• $C_{q;\psi}\big([c,d],\mathbf{E^1}\big)$ denote the weighted space of the fuzzy function $\mathbf{z}$ on $[c,d]$ defined by

$$C_{q;\psi}\big([c,d],\mathbf{E^1}\big)=\Big\{\mathbf{z}:[c,d]\longrightarrow\mathbf{E^1},\big((\psi(u)-\psi(c))\big)^q\mathbf{z}(u)\in C\big([c,d],\mathbf{E^1}\big)\Big\}.$$

Definition 8. *[28] The ψ-Riemann-Liouville fractional integral of order $q>0$ of function $\mathbf{z}\in\mathbf{E^1}$ on $[c,d]$ with respect to the non-decreasing differentiable function $\psi:[c,d]\longrightarrow\mathbb{R}^+$ with $\psi^{'}(u)\neq0$ is defined by*

$$\mathfrak{I}^{q;\psi}_{c+}\mathbf{z}(u)=\frac{1}{\Gamma(q)}\int_c^u\psi^{'}(v)\big(\psi(u)-\psi(v)\big)^{q-1}\mathbf{z}(v)dv,$$

Definition 9. *[28] Let $\mathbf{z} \in C^n\big([c,d], \mathbf{E^1}\big)$ and $\psi \in C^n\big([c,d], \mathbb{R}^+\big)$ be two functions such that ψ is non-decreasing with $\psi^{'}(u) \neq 0$ for all $u \in [c,d]$. The ψ-Caputo fractional derivative of order $q > 0$ of a continuous function $\mathbf{z}$ is given by*

$$\mathcal{D}^{q;\psi}_{c+}\mathbf{z}(u) := \mathcal{I}^{n-q;\psi}_{c^+}\left(\frac{1}{\psi'(v)}\frac{d}{du}\right)^n \mathbf{z}(u),$$

where $n = [q] + 1$ for $q \notin \mathbb{N}$ and $q = n$ for $q \in \mathbb{N}$.

Definition 10. *A d-monotone fuzzy function $\mathbf{z}(\cdot) \in C\big(\mathbf{I}, \mathbf{E^1}\big)$ is a solution of the problem (1) if and only if $\mathbf{z}(\cdot)$ satisfies*

$$\mathbf{z}(u) \ominus_{gH} \mathbf{z}_0 = \frac{1}{\Gamma(q)} \int_0^u \psi'(v)\big(\psi(u) - \psi(v)\big)^{q-1} \mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(\lambda_1 v), \mathbf{z}(\lambda_2 v)\big) dv, \quad (2)$$

and $u \mapsto \mathcal{I}^{q;\psi}_{0^+}\mathfrak{f}\big(u, \mathbf{z}(u), \mathbf{z}(\lambda_1 u), \mathbf{z}(\lambda_2 u)\big)$ is d-increasing on $\mathbf{I}$.

Remark 6. • *If $\mathbf{z}(\cdot) \in C\big(\mathbf{I}, \mathbf{E^1}\big)$ is d-increasing, then (2) becomes*

$$\mathbf{z}(u) = \mathbf{z}_0 + \frac{1}{\Gamma(q)} \int_0^u \psi'(v)\big(\psi(u) - \psi(v)\big)^{q-1} \mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(\lambda_1 v), \mathbf{z}(\lambda_2 v)\big) dv. \quad (3)$$

• *If $\mathbf{z}(\cdot) \in C\big(\mathbf{I}, \mathbf{E^1}\big)$ is d-decreasing, then (2) becomes*

$$\mathbf{z}(u) = \mathbf{z}_0 \ominus \frac{(-1)}{\Gamma(q)} \int_0^u \psi'(v)\big(\psi(u) - \psi(v)\big)^{q-1} \mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(\lambda_1 v), \mathbf{z}(\lambda_2 v)\big) dv. \quad (4)$$

Remark 7. • *Let $\psi(u) = u$, then the equation (2) becomes the following Riemann–Liouville fuzzy pantograph fractional integral equations*

$$\mathbf{z}(u) \ominus_{gH} \mathbf{z}_0 = \frac{1}{\Gamma(q)} \int_0^u \big(u - v\big)^{q-1} \mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(\lambda_1 v), \mathbf{z}(\lambda_2 v)\big) dv.$$

• *Let $\psi(u) = u^\rho$, then the equation (2) becomes the following Katugampola fuzzy pantograph fractional integral equations*

$$\mathbf{z}(u) \ominus_{gH} \mathbf{z}_0 = \frac{\rho^{1-q}}{\Gamma(q)} \int_0^u v^{\rho-1}\big(u^\rho - v^\rho\big)^{q-1} \mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(\lambda_1 v), \mathbf{z}(\lambda_2 v)\big) dv.$$

• *Let $\psi(u) = \ln(u)$, then the equation (2) becomes the following Hadamard fuzzy pantograph fractional integral equations*

$$\mathbf{z}(u) \ominus_{gH} \frac{\big(\ln(u) - \ln(v)\big)^{q-1}\mathbf{z}_0}{\Gamma(q)} = \frac{1}{\Gamma(q)} \int_0^u \big(\ln(u) - \ln(v)\big)^{q-1} \mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(\lambda_1 v), \mathbf{z}(\lambda_2 v)\big) \frac{dv}{v}.$$

3 Existence and uniqueness

We make the following hypotheses concerning the coefficients of the problem under consideration:

$(\mathbf{H1})$ For all $a_1, a_2, a_3,\ b_1, b_2, b_3 \in \mathbf{E^1}$ and $u \in \mathbf{I}$, there exist $\varpi > 0$ such that

$$\mathbf{D}_\infty\big[\mathfrak{f}(u, a_1, a_2, a_3), \mathfrak{f}(u, b_1, b_2, b_3)\big] \leq \varpi\Big(\mathbf{D}_\infty[a_1, b_1] + \mathbf{D}_\infty[a_2, b_2] + \mathbf{D}_\infty[a_3, b_3]\Big),$$

$(\mathbf{H2})$ For $u \in \mathbf{I}$, there exists $\varrho, \varsigma > 0$ such that

$$(i) \quad \mathbf{D}_\infty\big[\mathfrak{f}(u, a_1, a_2, a_3), \hat{0}\big] \leq \varrho,$$

$$(ii) \quad \mathbf{D}_\infty\big[\mathfrak{f}(u, \hat{0}, \hat{0}, \hat{0}), \hat{0}\big] \leq \varsigma.$$

We will now use Schaefer's fixed point theorem to demonstrate the existence result.

Theorem 1. *Suppose that the hypotheses* $(\mathbf{H1})$ *and* $(\mathbf{H2})$ *are hold, then the problem* (1) *has at least one solution in* $C\big(\mathbf{I}, \mathbf{E^1}\big)$.

Proof. Let define a mapping $\mathbf{L} : C\big(\mathbf{I}, \mathbf{E^1}\big) \longrightarrow C\big(\mathbf{I}, \mathbf{E^1}\big)$ as follow

$$\mathbf{L}\mathbf{z}(u) = \mathbf{z}_0 \circledcirc \frac{1}{\Gamma(q)} \int_0^u \psi'(v)\big(\psi(u) - \psi(v)\big)^{q-1} \mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(\lambda_1 v), \mathbf{z}(\lambda_2 v)\big) dv, \quad (5)$$

where $\circledcirc := \{+, \ominus(-1)\}$. We divide the subsequent proof into three steps.
Step 1: $\mathbf{L}$ is continuous. Indeed, let $(\mathbf{z}_n(u))_n \subset C\big(\mathbf{I}, \mathbf{E^1}\big)$ such that $\mathbf{z}_n$ converges to $\mathbf{z}$ in $C\big(\mathbf{I}, \mathbf{E^1}\big)$, then by using hypothesis $(\mathbf{H1})$, we have for $u \in \mathbf{I}$

$$\begin{aligned}
&\mathbf{D}_\infty\left[\mathbf{L}\mathbf{z}_n(u), \mathbf{L}\mathbf{z}(u)\right] \\
&\leq \frac{1}{\Gamma(q)} \int_0^u \frac{\psi'(v)}{\big(\psi(u) - \psi(v)\big)^{1-q}} \mathbf{D}_\infty\Big[\mathfrak{f}\big(v, \mathbf{z}_n(v), \mathbf{z}_n(\lambda_1 v), \mathbf{z}_n(\lambda_2 v)\big), \mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(\lambda_1 v), \mathbf{z}(\lambda_2 v)\big)\Big] dv, \\
&\leq \frac{\varpi}{\Gamma(q)} \int_0^u \frac{\psi'(v)}{\big(\psi(u) - \psi(v)\big)^{1-q}} \Big(\mathbf{D}_\infty[\mathbf{z}_n(v), \mathbf{z}(v)] + \mathbf{D}_\infty[\mathbf{z}_n(\lambda_1 v), \mathbf{z}(\lambda_1 v)] + \mathbf{D}_\infty[\mathbf{z}_n(\lambda_2 v), \mathbf{z}(\lambda_2 v)]\Big) dv, \\
&\leq \frac{3\varpi}{\Gamma(q)} \int_0^u \frac{\psi'(v)}{\big(\psi(u) - \psi(v)\big)^{1-q}} \mathbf{D}_\infty[\mathbf{z}_n(v), \mathbf{z}(v)] dv, \\
&\leq \frac{3\varpi}{\Gamma(q+1)} \big(\psi(M) - \psi(0)\big)^q \sup_{u \in \mathbf{I}} \mathbf{D}_\infty[\mathbf{z}_n(u), \mathbf{z}(u)].
\end{aligned}$$

We can conclude that $\mathbf{D}_\infty\big[\mathbf{L}\mathbf{z}_n(u), \mathbf{L}\mathbf{z}(u)\big] \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore, $\mathbf{L}$ is continuous on $C\big(\mathbf{I}, \mathbf{E^1}\big)$.

Step 2:

ⓐ- Let us prove that $\mathbf{L}$ is bounded. For this, let prove that there exists a positive constant ξ_1 and for all $\varsigma_1 > 0$ satisfying for all $\mathbf{z}(u) \in \mathbf{B}_{\varsigma_1} := \Big\{\mathbf{z}(u) \in$

$C\big(\mathbf{I},\mathbf{E^1}\big)|\mathbf{D}_\infty[\mathbf{z}(u),\hat{0}] \le \varsigma_1\Big\}$ one has $\mathbf{D}_\infty\big[\mathbf{L}\mathbf{z}(u),\hat{0}\big] \le \xi_1$. In fact, for all $u \in \mathbf{I}$ and $\mathbf{z}(u) \in \mathbf{B}_{\varsigma_1}$, by using hypothesis (**H2**) we get

$$\begin{aligned}
&\mathbf{D}_\infty\big[\mathbf{z}_n(u),\hat{0}\big] \\
&\quad \le \frac{1}{\Gamma(q)}\int_0^u \frac{\psi'(v)}{\big(\psi(u)-\psi(v)\big)^{1-q}}\mathbf{D}_\infty\big[\mathfrak{f}\big(v,\mathbf{z}_n(v),\mathbf{z}_n(\lambda_1 v),\mathbf{z}_n(\lambda_2 v)\big),\hat{0}\big]dv, \\
&\quad \le \frac{\big(\psi(M)-\psi(0)\big)^q\varrho}{\Gamma(q+1)} := \xi_1,
\end{aligned}$$

this implies that $\mathbf{L}(\mathbf{B}_{\varsigma_1}) \subseteq \mathbf{B}_{\xi_1}$. Thus, $\mathbf{L}(\mathbf{B}_{\varsigma_1})$ is bounded.
ⓑ- Let prove that $\mathbf{L}(\mathbf{B}_{\varsigma_1})$ equicontinuous. For each $\mathbf{z}(u) \in \mathbf{B}_{\varsigma_1}$ and $u_1, u_2 \in \mathbf{I}$ such that $u_1 < u_2$, we have

$$\begin{aligned}
&\mathbf{D}_\infty\big[\mathbf{z}(u_1),\mathbf{z}(u_2)\big] \\
&\quad \le \frac{1}{\Gamma(q)}\mathbf{D}_\infty\Big[\int_0^{u_1} \frac{\psi'(v)}{\big(\psi(u_1)-\psi(v)\big)^{1-q}}\mathfrak{f}\big(v,\mathbf{z}(v),\mathbf{z}(\lambda_1 v),\mathbf{z}(\lambda_2 v)\big)dv, \\
&\qquad\qquad \int_0^{u_2} \frac{\psi'(v)}{\big(\psi(u_2)-\psi(v)\big)^{1-q}}\mathfrak{f}\big(v,\mathbf{z}(v),\mathbf{z}(\lambda_1 v),\mathbf{z}(\lambda_2 v)\big)dv\Big],
\end{aligned}$$

since

$$\begin{aligned}
&\int_0^{u_2} \frac{\psi'(v)}{\big(\psi(u_2)-\psi(v)\big)^{1-q}}\mathfrak{f}\big(v,\mathbf{z}(v),\mathbf{z}(\lambda_1 v),\mathbf{z}(\lambda_2 v)\big)dv \\
&\quad = \int_0^{u_1} \frac{\psi'(v)}{\big(\psi(u_2)-\psi(v)\big)^{1-q}}\mathfrak{f}\big(v,\mathbf{z}(v),\mathbf{z}(\lambda_1 v),\mathbf{z}(\lambda_2 v)\big)dv \\
&\quad + \int_{u_1}^{u_2} \frac{\psi'(v)}{\big(\psi(u_2)-\psi(v)\big)^{1-q}}\mathfrak{f}\big(v,\mathbf{z}(v),\mathbf{z}(\lambda_1 v),\mathbf{z}(\lambda_2 v)\big)dv,
\end{aligned}$$

we get

$$\begin{aligned}
&\mathbf{D}_\infty\big[\mathbf{z}(u_1),\mathbf{z}(u_2)\big] \\
&\quad \le \frac{1}{\Gamma(q)}\int_0^{u_1} \psi'(v)\big|\big(\psi(u_1)-\psi(v)\big)^{q-1}-\big(\psi(u_2)-\psi(v)\big)^{q-1}\big| \\
&\qquad\qquad \mathbf{D}_\infty\big[\mathfrak{f}\big(v,\mathbf{z}(v),\mathbf{z}(\lambda_1 v),\mathbf{z}(\lambda_2 v)\big),\hat{0}\big]dv, \\
&\quad + \frac{1}{\Gamma(q)}\int_{u_1}^{u_2} \frac{\psi'(v)}{\big(\psi(u_2)-\psi(v)\big)^{1-q}}\mathbf{D}_\infty\big[\mathfrak{f}\big(v,\mathbf{z}(v),\mathbf{z}(\lambda_1 v),\mathbf{z}(\lambda_2 v)\big),\hat{0}\big]dv.
\end{aligned}$$

Thus, by using (**H2**), we obtain

$$\begin{aligned}
&\mathbf{D}_\infty\big[\mathbf{z}(u_1),\mathbf{z}(u_2)\big] \\
&\quad \le \frac{\varrho}{\Gamma(q+1)}\Big(\big(\psi(u_1)-\psi(0)\big)^q + 2\big(\psi(u_2)-\psi(u_1)\big)^q - \big(\psi(u_2)-\psi(0)\big)^q\Big), \\
&\quad \le \frac{2\varrho}{\Gamma(q+1)}\big(\psi(u_2)-\psi(u_1)\big)^q := \Psi.
\end{aligned}$$

We have Ψ is independent of $\mathbf{z}(u)$ and $\Psi \longrightarrow 0$ as $u_2 \longrightarrow u_1$. Then, we obtain

$$\mathbf{D}_\infty\big[\mathbf{Lz}(u_1), \mathbf{Lz}(u_2)\big] \longrightarrow 0.$$

It means that $\mathbf{L}(\mathbf{B}_{\varsigma_1})$ is equicontinuous. Then, according to Arzela-Ascoli theorem, $\mathbf{L}$ is relatively compact. As a result of the previous steps, $\mathbf{L}$ is completely continuous.

Step 3: we will prove that $\mathbf{B}_\delta = \Big\{\mathbf{z}(u) \in C\big(\mathbf{I}, \mathbf{E^1}\big) | \mathbf{z} = \delta(\mathbf{Lz}), \delta \in (0,1)\Big\}$ is bounded. Let $\mathbf{z} \in \mathbf{B}_\delta$, then $\mathbf{z} = \delta(\mathbf{Lz})$ for $\delta \in (0,1)$. So for each $u \in \mathbf{I}$, we have

$$\mathbf{z}(u) \ominus_{gH} \delta \mathbf{z}_0 = \frac{\delta}{\Gamma(q)} \int_0^u \psi'(v)\big(\psi(u) - \psi(v)\big)^{q-1} \mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(\lambda_1 v), \mathbf{z}(\lambda_2 v)\big) dv. \quad (6)$$

Therefore, using hypothesis (**H2**), we have

$$\begin{aligned}
&\mathbf{D}_\infty\big[\mathbf{z}(u), \hat{0}\big] \\
&\quad \leq \delta \mathbf{D}_\infty\big[\mathbf{z}_0, \hat{0}\big] + \frac{\delta}{\Gamma(q)} \int_0^u \frac{\psi'(v)}{\big(\psi(u) - \psi(v)\big)^{1-q}} \mathbf{D}_\infty\big[\mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(\lambda_1 v), \mathbf{z}(\lambda_2 v)\big), \hat{0}\big] dv, \\
&\quad \leq \delta \mathbf{D}_\infty\big[\mathbf{z}_0, \hat{0}\big] + \frac{\delta\big(\psi(M) - \psi(0)\big)^q \varrho}{\Gamma(q+1)} < \infty,
\end{aligned}$$

which implies that $\mathbf{B}_\delta$ is bounded. As a consequence of Schaefer's fixed point theorem, $\mathbf{L}$ has a fixed point which is a solution of problem (1). $\square$

For the uniqueness result, we have the following theorem:

Theorem 2. *Assume that the hypothesis* (**H1**) *hold. If*

$$\frac{3\varpi}{\Gamma(q+1)} \big(\psi(M) - \psi(0)\big)^q < 1,$$

then the solution of problem (1) *is unique.*

Proof. We know that $\mathbf{z}(u)$ is a solution of problem (1) if

$$\mathbf{z}(u) \ominus_{gH} \mathbf{z}_0 = \frac{1}{\Gamma(q)} \int_0^u \psi'(v)\big(\psi(u) - \psi(v)\big)^{q-1} \mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(\lambda_1 v), \mathbf{z}(\lambda_2 v)\big) dv. \quad (7)$$

hold. If $\mathbf{z}(u) \in C\big(\mathbf{I}, \mathbf{E^1}\big)$ is a fixed point of $\mathbf{L}$ which define as in Theorem 1, then $\mathbf{z}(u)$ is a solution of problem (1). Let $\mathbf{z}_1(u), \mathbf{z}_2(u) \in C\big(\mathbf{I}, \mathbf{E^1}\big)$. For all $u \in \mathbf{I}$, we have

$$\begin{aligned}
&\mathbf{D}_\infty\big[\mathbf{Lz}_1(u), \mathbf{Lz}_2(u)\big] \\
&\leq \frac{1}{\Gamma(q)} \int_0^u \frac{\psi'(v)}{\big(\psi(u) - \psi(v)\big)^{1-q}} \mathbf{D}_\infty\Big[\mathfrak{f}\big(v, \mathbf{z}_1(v), \mathbf{z}_1(\lambda_1 v), \mathbf{z}_1(\lambda_2 v)\big), \mathfrak{f}\big(v, \mathbf{z}_2(v), \mathbf{z}_2(\lambda_1 v), \mathbf{z}_2(\lambda_2 v)\big)\Big] dv, \\
&\leq \frac{\varpi}{\Gamma(q)} \int_0^u \frac{\psi'(v)}{\big(\psi(u) - \psi(v)\big)^{1-q}} \Big(\mathbf{D}_\infty[\mathbf{z}_1(v), \mathbf{z}_2(v)] + \mathbf{D}_\infty[\mathbf{z}_1(\lambda_1 v), \mathbf{z}_2(\lambda_1 v)] \\
&\qquad\qquad + \mathbf{D}_\infty[\mathbf{z}_1(\lambda_2 v), \mathbf{z}_2(\lambda_2 v)]\Big) dv, \\
&\leq \frac{3\varpi}{\Gamma(q)} \int_0^u \frac{\psi'(v)}{\big(\psi(u) - \psi(v)\big)^{1-q}} \mathbf{D}_\infty[\mathbf{z}_1(v), \mathbf{z}_2(v)] dv, \\
&\leq \frac{3\varpi}{\Gamma(q+1)} \big(\psi(M) - \psi(0)\big)^q \mathbf{D}_\infty[\mathbf{z}_1(u), \mathbf{z}_2(u)].
\end{aligned}$$

Based on the Banach fixed point theorem, we can deduce that $\mathbf{L}$ has a unique fixed point, which is the solution of problem (1). $\square$

4 Estimate on the solution

Here, we demonstrate an estimate of the solution to problem (1) using the generalized Grönwall inequality.

Theorem 3. *Assume that the function* $\mathfrak{f} : \mathbf{I} \times \mathbf{E^1} \times \mathbf{E^1} \times \mathbf{E^1} \longrightarrow \mathbf{E^1}$ *satisfies the hypothesis* $(\mathbf{H1})$ *and* (ii) *in* $(\mathbf{H2})$*. If* $\mathbf{z}(\cdot) \in C\big(\mathbf{I}, \mathbf{E^1}\big)$ *is any solution of the problem* (1) *then following holds:*

$$\mathbf{D}_\infty\big[\mathbf{z}(u), \hat{0}\big] \leq \left[\mathbf{D}_\infty\big[\mathbf{z}_0, \hat{0}\big] + \frac{\varsigma}{\Gamma(q+1)}\big(\psi(M) - \psi(0)\big)^q\right]\mathbf{E}_q\big(3\varpi(\psi(u) - \psi(0))^q\big). \tag{8}$$

Proof. From Definition 10, we have

$$\mathbf{z}(u) \ominus_{gH} \mathbf{z}_0 = \frac{1}{\Gamma(q)}\int_0^u \psi'(v)\big(\psi(u) - \psi(v)\big)^{q-1}\mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(\lambda_1 v), \mathbf{z}(\lambda_2 v)\big)dv.$$

Thus, by using hypothesis $(\mathbf{H1})$ and (ii) in $(\mathbf{H2})$, we get

$$\begin{aligned}
\mathbf{D}_\infty\big[\mathbf{z}(u), \hat{0}\big] &\leq \mathbf{D}_\infty\big[\mathbf{z}_0, \hat{0}\big] \\
&\quad + \frac{1}{\Gamma(q)}\int_0^u \frac{\psi'(v)}{\big(\psi(u) - \psi(v)\big)^{1-q}}\mathbf{D}_\infty\big[\mathfrak{f}\big(v, \mathbf{z}(v), \mathbf{z}(\lambda_1 v), \mathbf{z}(\lambda_2 v)\big), \mathfrak{f}\big(v, \hat{0}, \hat{0}, \hat{0}\big)\big]dv \\
&\quad + \frac{1}{\Gamma(q)}\int_0^u \frac{\psi'(v)}{\big(\psi(u) - \psi(v)\big)^{1-q}}\mathbf{D}_\infty\big[\mathfrak{f}\big(v, \hat{0}, \hat{0}, \hat{0}\big), \hat{0}\big]dv, \\
&\quad \leq \mathbf{D}_\infty\big[\mathbf{z}_0, \hat{0}\big] + \frac{3\varpi}{\Gamma(q)}\int_0^u \frac{\psi'(v)}{\big(\psi(u) - \psi(v)\big)^{1-q}}\mathbf{D}_\infty\big[\mathbf{z}(v), \hat{0}\big]dv \\
&\qquad + \frac{\varsigma}{\Gamma(q+1)}\big(\psi(M) - \psi(0)\big)^q,
\end{aligned}$$

then, using the generalized Grönwall inequality, we obtain

$$\mathbf{D}_\infty\big[\mathbf{z}(u), \hat{0}\big] \leq \left[\mathbf{D}_\infty\big[\mathbf{z}_0, \hat{0}\big] + \frac{\varsigma}{\Gamma(q+1)}\big(\psi(M) - \psi(0)\big)^q\right]\mathbf{E}_q\big(3\varpi(\psi(u) - \psi(0))^q\big).$$

Hence we obtain (8).

$\square$

5 Ulam–Hyers–Mittag-Leffler stability result

The Mittag-Leffler function can be defined in terms of a power series as

$$\mathbf{E}_q(z) := \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(qi+1)}, \quad q > 0.$$

Definition 11. *[22] The solution of problem* (1) *is said to be Ulam–Hyers–Mittag-Leffler stable with respect to* $\mathbf{E}_q\big((\psi(u)-\psi(0))^q\big)$ *if there exist a constant* $\Delta_{\mathbf{E}_q}>0$ *such that for all* $\varepsilon>0$ *and for each solution* $\mathbf{w}(u)\in C\big(\mathbf{I},\mathbf{E^1}\big)$ *of the following inequality*

$$\mathbf{D}_\infty\Big[\mathcal{D}^{q;\psi}_{0^+}\mathbf{w}(u),\mathfrak{f}\big(u,\mathbf{w}(u),\mathbf{z}(\lambda_1 u),\mathbf{w}(\lambda_2 u)\big)\Big]\le\varepsilon, \tag{9}$$

there exist a solution $\mathbf{z}(u)\in C\big(\mathbf{I},\mathbf{E^1}\big)$ *of problem* (1)*, such that*

$$\mathbf{D}_\infty\Big[\mathbf{w}(u),\mathbf{z}(u)\Big]\le\Delta_{\mathbf{E}_q}\mathbf{E}_q\big((\psi(u)-\psi(0))^q\big)\varepsilon,\qquad u\in\mathbf{I}.$$

Remark 8. *An d-monotone fuzzy function* $\mathbf{w}(u)\in C\big(\mathbf{I},\mathbf{E^1}\big)$ *is a solution of* (9) *if and only if* $\exists\phi\in C\big(\mathbf{I},\mathbf{E^1}\big)$ *such that*

(*i*)- $\mathbf{D}_\infty\big[\phi(u),\hat{0}\big]\le\mathbf{E}_q\big((\psi(u)-\psi(0))^q\big)\varepsilon,$

(*ii*)- *For* $u\in\mathbf{I}$,

$$\begin{cases}\mathcal{D}^{q;\psi}_{0^+}\mathbf{w}(u)=\mathfrak{f}\big(u,\mathbf{w}(u),\mathbf{w}(\lambda_1 u),\mathbf{w}(\lambda_2 u)\big)+\phi(u), & u\in\mathbf{I},\\ \mathbf{w}(0)=\mathbf{z}(0)=\mathbf{w}_0=\mathbf{z}_0\end{cases}\tag{10}$$

Theorem 4. *Assume that the hypotheses* (**H1**) *and* (**H2**) *hold. Then, the problem* (1) *is Ulam–Hyers–Mittag-Leffler stable.*

Proof. Let $\mathbf{w}(u)$ be the solution of the problem (9) and $\mathbf{z}(u)$ be the solution of the proposed problem (1). then by Remark 8, there exist $\phi\in C\big(\mathbf{I},\mathbf{E^1}\big)$ such that

$$\mathbf{D}_\infty\big[\phi(u),\hat{0}\big]\le\mathbf{E}_q\big((\psi(u)-\psi(0))^q\big)\varepsilon,$$

and

$$\mathcal{D}^{q;\psi}_{0^+}\mathbf{w}(u)=\mathfrak{f}\big(u,\mathbf{w}(u),\mathbf{w}(\lambda_1 u),\mathbf{w}(\lambda_2 u)\big)+\phi(u),\qquad u\in\mathbf{I}.$$

Thus, from Definition 10, we have

$$\mathbf{z}(u)\ominus_{gH}\mathbf{z}_0=\frac{1}{\Gamma(q)}\int_0^u\psi'(v)\frac{\mathfrak{f}\big(v,\mathbf{w}(v),\mathbf{w}(\lambda_1 v),\mathbf{w}(\lambda_2 v)\big)}{\big(\psi(u)-\psi(v)\big)^{1-q}}dv+\frac{1}{\Gamma(q)}\int_0^u\frac{\psi'(v)\phi(v)}{\big(\psi(u)-\psi(v)\big)^{1-q}}dv.$$

For $u\in\mathbf{I}$, we use (**H1**) and (**H2**), we get

$$\begin{aligned}
&\mathbf{D}_\infty\big[\mathbf{w}(u),\mathbf{z}(u)\big]\\
&\le\frac{1}{\Gamma(q)}\int_0^u\frac{\psi'(v)}{\big(\psi(u)-\psi(v)\big)^{1-q}}\mathbf{D}_\infty\Big[\mathfrak{f}\big(v,\mathbf{w}(v),\mathbf{w}(\lambda_1 v),\mathbf{w}(\lambda_2 v)\big),\mathfrak{f}\big(v,\mathbf{z}(v),\mathbf{z}(\lambda_1 v),\mathbf{z}(\lambda_2 v)\big)\Big]dv,\\
&+\frac{1}{\Gamma(q)}\int_0^u\frac{\psi'(v)}{\big(\psi(u)-\psi(v)\big)^{1-q}}\mathbf{D}_\infty\big[\phi(v),\hat{0}\big]dv\\
&\le\frac{\varpi}{\Gamma(q)}\int_0^u\frac{\psi'(v)}{\big(\psi(u)-\psi(v)\big)^{1-q}}\Big(\mathbf{D}_\infty[\mathbf{w}(v),\mathbf{z}(v)]+\mathbf{D}_\infty[\mathbf{w}(\lambda_1 v),\mathbf{z}(\lambda_1 v)]+\mathbf{D}_\infty[\mathbf{w}(\lambda_2 v),\mathbf{z}(\lambda_2 v)]\Big)dv\\
&+\frac{\varepsilon}{\Gamma(q)}\int_0^u\psi'(v)\big(\psi(u)-\psi(v)\big)^{q-1}\mathbf{E}_q\big((\psi(v)-\psi(0))^q\big)dv\\
&\le\frac{\varepsilon}{\Gamma(q+1)}\mathbf{E}_q\big((\psi(M)-\psi(0))^q\big)+\frac{3\varpi}{\Gamma(q)}\int_0^u\frac{\psi'(v)}{\big(\psi(u)-\psi(v)\big)^{1-q}}\mathbf{D}_\infty[\mathbf{w}(v),\mathbf{z}(v)]dv.
\end{aligned}$$

Thus, using the generalized Grönwall inequality, we obtain

$$\mathbf{D}_\infty\big[\mathbf{w}(u),\mathbf{z}(u)\big] \leq \frac{\varepsilon}{\Gamma(q+1)}\mathbf{E}_q\big((\psi(M)-\psi(0))^q\big)\mathbf{E}_q\big(3\varpi(\psi(u)-\psi(0))^q\big),$$

It follows that

$$\mathbf{D}_\infty\big[\mathbf{w}(u),\mathbf{z}(u)\big] \leq \Delta_{\mathbf{E}_q}\mathbf{E}_q\big((\psi(u)-\psi(0))^q\big)\varepsilon,$$

where $\Delta_{\mathbf{E}_q} = \frac{\mathbf{E}_q\big((\psi(M)-\psi(0))^q\big)}{\Gamma(q+1)}$.
Therefore, from Definition 11, the problem (1) is Ulam–Hyers–Mittag-Leffler stable. □

6 Examples

In this section, we provide an illustration of the results from the previous part. Note that the fuzzy number considered as initial value in the initial condition for Example 1 and Example 2 is triangular fuzzy number.

6.1 Example 1

Let

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{1}{2};u^2}\mathbf{z}(u) = -u^2\mathbf{z}(u) - 2\mathbf{z}(0.3u) + \frac{1}{2}\mathbf{z}(0.7u), & u \in (0,\frac{1}{10}], \\ \mathbf{z}(0) = (1,0,0), \end{cases} \tag{11}$$

where $\mathfrak{f}(u,\mathbf{z}(u),\mathbf{z}(\lambda_1 u),\mathbf{z}(\lambda_2 u)) = -u^2\mathbf{z}(u) - 2\mathbf{z}(0.3u) + \frac{1}{2}\mathbf{z}(0.7u)$.
Here $q=\frac{1}{2}$, $\psi(u)=u^2$, $\lambda_1 = 0.3$ and $\lambda_2 = 0.7$. The hypothesis (**H1**) is satisfied by choosing $\varpi = 5$. Moreover, we have

$$\frac{3\varpi}{\Gamma(q+1)}\big(\psi(M)-\psi(0)\big)^q = \frac{3\cdot 5}{\Gamma(0.5+1)}\big(\psi(0.1)-\psi(0)\big)^{\frac{1}{2}} \simeq 0.14 < 1.$$

The verification demonstrates that all assumptions in Theorem 1 are met in full. Then, the problem (11) has a unique solution. Also, we can verify that problem (11) satisfies all assumptions in Theorem 4. Then, problem (11) is Ulam–Hyers–Mittag-Leffler stable.

6.2 Example 2

Consider the following problem

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{4}{10};u^4}\mathbf{z}(u) = \mathbf{z}^2(u) + \mathbf{z}^3(0.2u) - \sin(u)\mathbf{z}(0.8u), & u \in (0,\frac{2}{10}], \\ \mathbf{z}(0) = (0.1,0,1), \end{cases} \tag{12}$$

where $\mathfrak{f}(u,\mathbf{z}(u),\mathbf{z}(\lambda_1 u),\mathbf{z}(\lambda_2 u)) = \mathbf{z}^2(u) + \mathbf{z}^3(0.2u) - \sin(u)\mathbf{z}(0.8u)$.
Here $q = \frac{4}{10}$, $\psi(u) = u^4$, $\lambda_1 = 0.2$ and $\lambda_2 = 0.8$. The hypothesis (**H1**) is satisfied by choosing $\varpi = \frac{6}{10}$. Moreover, we have

$$\frac{3\varpi}{\Gamma(q+1)}\big(\psi(M)-\psi(0)\big)^q = \frac{3\cdot 0.6}{\Gamma(0.4+1)}\big(\psi(0.2)-\psi(0)\big)^{\frac{4}{10}} \simeq 0.933 < 1.$$

The verification demonstrates that all assumptions in Theorem 1 are met in full. Then, the problem (11) has a unique solution. Also, we can verify that problem (11) satisfies all assumptions in Theorem 4. Then, problem (11) is Ulam–Hyers–Mittag-Leffler stable.

7 Conclusion

In this study, we have examined a class of ψ-Caputo fuzzy fractional multi-pantograph differential equations. The method of Schaefer and Banach fixed point theorem are employed under Lipschitz conditions to demonstrate the existence and uniqueness of solutions. Finally, by using the generalized Grönwall inequality, Ulam–Hyers-Mittag-Leffler stability result for the main problem is provided.

References

[1] Abuasbeh, K. and Shafqat, R., *Fractional Brownian motion for a system of fuzzy fractional stochastic differential equation*, Journal of Mathematics, **2022** (2022), Article ID 3559035, 14 pg.

[2] Agilan, K. and Parthiban, V., *Existence of solutions of fuzzy fractional pantograph equations*, Int. J. Math. Comput. Sci. **15** (2020), no. 4, 1117-1122.

[3] Ahmed, I., Kumam, P., Shah, K., Borisut, P., Sitthithakerngkiet, K. and Ahmed, D. *Stability results for implicit fractional pantograph differential equations via ϕ-Hilfer fractional derivative with a nonlocal Riemann-Liouville fractional integral condition*, Mathematics **8** (2020), no. 1, Article number 94.

[4] Ahmed, I., Kumam, P., Jarad, F., Borisut, P., Sitthithakerngkiet, K. and Ibrahim, A., *Stability analysis for boundary value problems with generalized nonlocal condition via Hilfer-Katugampola fractional derivative*, Adv. Differ. Equ. **1** (2020), Article number 225 .

[5] Almeida R., *A Caputo fractional derivative of a function with respect to another function*, Commun. Nonlinear Sci. Numer. Simul. **44** (2017), 460-481.

[6] Anguraj, A., Vinodkumar, A. and Malar, K., *Existence and stability results for random impulsive fractional pantograph equations* Filomat **30** (2016), no. 14, 3839–3854.

[7] Arhrrabi E., Elomari M., Melliani S. and Chadli L. S., *Existence and stability of solutions of fuzzy fractional stochastic differential equations with fractional Brownian motions*, Adv Fuzzy Syst **2021** (2021), Article ID 3948493.

[8] Arhrrabi E., Elomari M. and Melliani S., *Averaging principle for fuzzy stochastic differential equations*, Contrib. Math. **5** (2022), 25-28.

[9] Arhrrabi E., Taqbibt A., Elomari M. H. Melliani S. and saadia Chadli L., *Fuzzy fractional boundary value problem*, In 2021 7th International Conference on Optimization and Applications (ICOA), IEEE, 2021 1-6.

[10] Arhrrabi E., Elomari M., Melliani S. and Chadli L. S., *Existence and stability of solutions for a coupled problem of fuzzy fractional pantograph stochastic differential equations*, Asia Pac. J. Math. **9** (2022), Article number 20. doi:10.28924/APJM/9-20.

[11] Balachandran, K., Kiruthika, S. and Trujillo, J.J., *Existence of solutions of nonlinear fractional pantograph equations*, Acta Math. Sci. **33** (2013), no. 3, 712–720.

[12] Boulares, H., Benchaabane, A., Pakkaranang, N., Shafqat, R. and Panyanak, B., *Qualitative properties of positive solutions of a kind for fractional pantograph problems using technique fixed point theory*, Fractal and Fractional, **6** (2022), no. 10, Article number 593.

[13] Borisut, P., Kumam, P. and Ahmed, I. and Sitthithakerngkiet, K., *Nonlinear Caputo fractional derivative with nonlocal Riemann-Liouville fractional integral condition via fixed point theorems*, Symmetry **11** (2019), no. 6, Article number 829.

[14] Borisut, P., Kumam, P., Ahmed, I. and Jirakitpuwapat, W., *Existence and uniqueness for ψ-Hilfer fractional differential equation with nonlocal multi-point condition*, Math. Methods Appl. Sci. **44** (2020), 2506-2520.

[15] Borisut, P., Kumam, P., Ahmed, I. and Sitthithakerngkiet, K., *Positive solution for nonlinear fractional differential equation with nonlocal multi-point condition* Fixed Point Theory **21** (2020), no. 2, 427–440.

[16] Bica, A. M. and Satmari, Z., *Iterative numerical method for pantograph type fuzzy Volterra integral equations*, Fuzzy Sets and Systems **443** (2022), 262-285.

[17] Derbazi C., Baitiche Z., Benchohra M. and Cabada A., *Initial value problem for nonlinear fractional differential equations with ψ-Caputo derivative via monotone iterative technique*, Axioms **9** (2020), no. 2, Article number 57.

[18] Elsayed, E., Harikrishnan, S. and Kanagarajan, K., *Analysis of nonlinear neutral pantograph differential equations with Hilfer fractional derivative*, MathLAB **1** (2018), 231–240.

[19] Harikrishnan, S., Ibrahim, R.and Kanagarajan, K., *Establishing the existence of Hilfer fractional pantograph equations with impulses*, Fundam. J. Math. Appl. **1** (2018), no. 1, 36–42.

[20] Jameel, A. F., Saaban, A., Ahadkulov, H. and Alipiah, F. M., *Approximate solution fuzzy pantograph equation by using homotopy perturbation method*, Journal of Physics: Conference Series, IOP Publishing, **890** (2017), no. 1, Article ID 012023.

[21] Kanwal, A., Boulaaras, S., Shafqat, R., Taufeeq, B. and ur Rahman, M., *Explicit scheme for solving variable-order time-fractional initial boundary value problems*, Scientific Reports **14** (2024), no. 1, Article number 5396.

[22] Liu, K., Wang, J., and O'Regan, D., *Ulam–Hyers–Mittag-Leffler stability for ψ-Hilfer fractional-order delay differential equations*, Adv. Difference Equ. **2019** (2019), Article number 50, 12 p.

[23] Melliani, S., Arhrrabi, E., Elomari, M. and Chadli, L. S., *Existence and uniqueness solutions of fuzzy fractional integration-differential problem Under Caputo gH-differentiability*, In Applied Mathematics and Modelling in Finance, Marketing and Economics, Cham: Springer Nature Switzerland (2024), 99-117.

[24] Moumen, A., Shafqat, R., Hammouch, Z., Niazi, A. U. K. and Jeelani, M. B., *Stability results for fractional integral pantograph differential equations involving two Caputo operators*, AIMS Mathematics, **8** (2023), no. 3, 6009-6025.

[25] Rodriguez-Lopez R., *Monotone method for fuzzy differential equations*, Fuzzy Sets and Systems $\underline{1}$59 (2008), 2047–2076

[26] Shather, A. H., Jameel, A. F., Anakira, N. R., Alomari, A. K. and Saaban, A., *Homotopy analysis method approximate solution for fuzzy pantograph equation*, International Journal of Computing Science and Mathematics **14** (2021), no. 3, 286-300.

[27] Saleem, N., Shafqat, R., George, R., Hussain, A. and Yaseen, M., *A theoretical analysis on the fractional fuzzy controlled evolution equation*, Fractals 31 (2023), no. 10, Article ID 2340090.

[28] Vivek, R., Vivek, D. V., Kangarajan K. and Elsayed, E., *Some results on the study of-Hilfer type fuzzy fractional differential equations with time delay*, Proceedings of International Mathematical Sciences **4** (2022), no. 2, 65-76.

[29] Vivek, D., Kanagarajan, K. and Sivasundaram, S., *Theory and analysis of nonlinear neutral pantograph equations via Hilfer fractional derivative.* Nonlinear Stud. **24** (2017), no. 3, 699–712.

[30] Vivek, D., Kanagarajan, K. and Harikrishnan, S., *Dynamics and stability of Hilfer-Hadamard type fractional pantograph equations with boundary conditions*. J. Nonlinear Anal. Appl. **1** (2018), 1–13.

[31] Wang X., Luo D. and Zhu Q., *Ulam-Hyers stability of caputo type fuzzy fractional differential equations with time-delays*, Chaos, Solitons and Fractals **156** (2022), Ardicol ID 111822.

[32] Zadeh L., *Fuzzy sets*, Information and Control **6** (1965), no. 3, 338-353.