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THE SYMBOLIC SOLUTION FOR THE KINEMATIC TASK OF A BIPED ROBOT

Cs.Z. MATE[1] E. FALUVEGI[1] L. CRISTEA[1]

Abstract: *This paper deals with the kinematics solution of biped robots. It debates the direct and inverse kinematics problems of Leg's 5-dof biped robot, with which a leg can accumulate a position by multiple angle combinations. The symbolic solution for kinematics equations of biped robots is of great importance for the efficient controllability of these robots. The symbolic form of the kinematics equations describes explicitly in trigonometric form the biped robots' sole's position and orientation according to the joint coordinates.*

Key words: *symbolic solution, direct kinematics, inverse kinematics, model mathematic, biped robot.*

1. Introduction

For a biped robot the sole position and orientation is known, defined within the domain of exterior coordinates [6], [2], [3] if a $\vec{q}$ vector is given with join coordinates. In the case of a robot with n freedom degree, the vector of joint variables is the following:

$$\vec{q} = [q_1, q_2, ..., q_{n-1}, q_n]^T , \tag{1}$$

and the vector of unknown exterior coordinates is the following:

$$x_q = [x_{q1}, x_{q2}, ..., x_{qk-1}, x_{qn}]^T . \tag{2}$$

The equation below is the only solution for the so called direct kinematics problem:

$$x_q = f(\vec{q}) . \tag{3}$$

If we know sum of the joint's setup and from this we define the coordinate system's position, according to the sole's centre point, as well as its orientation, thus we solved the direct kinematics problem.

Inverse kinematics problem means that if the sole's expected position and orientation (within the exterior coordinates) is known, and then with which joint setups can we obtain this. In other words we can say that we are looking for an optimal solution:

$$\vec{q} = f^{-1}(x_q) . \tag{4}$$

This task is more complex, and for the direct kinematics problem, since it is not linear, we have to solve equations containing trigonometric functions.

2. The Symbolic Solution for the Direct Kinematics Task

The symbolic solution for kinematics equations of biped robots is of great importance for the efficient controllability of these robots.

[1] Dept. of Precision Mechanics and Mechatronics, *Transilvania* University of Braşov.

In the world of low-cost computers, the real-time motion control is an increasingly important [6], [3] requirement. In order to achieve this feature, the lowest computational demand requesting method has to be used. The use of symbolic solution, opposing to numerical methods, is important because accelerates the manipulator trajectory control signals needed to be determined according to the track.

The symbolic form of the kinematics equations describes explicitly in trigonometric form the biped robots sole's position and orientation according to the joint coordinates. In this case, the equation in the range of real numbers can be solved with the minimal possible operations.

3. Forward and Backward Progressing Symbolic links in the Symbolic Solution of Direct Kinematics Task

This method makes possible the homogenous transformation matrices forwards and backwards progressing recursive symbolic computation. The direction of the [6], [2], [3] computations is determined by, the sole's centre point coordinate systems position. Orientation is investigated backwards to the ankles, knees, hips. In basis coordinates, the study is focused on the torso, hip, knee, and finally on the sole's centre point coordinates.

The $T_{i\text{-}1,i}$ homogenous transformation matrix creates a bond with neighbouring coordinate systems so that the L_i-s coordinates are drafted in the $L_{i\text{-}1}$-s local coordinates. During the task's symbolic solution, these transformation matrices are used for making the $T_{0,n}$ homogenous transformation matrix. This matrix defines the sole centre point L_n coordinate system according to the basis. If we succeed to determine $T_{0,n}$ elements in the case of $\vec{q}$ joint coordinates, then we easily obtain the exterior coordinates. If Cartesian coordinates are used, then the fourth column of the matrix immediately provides the right position. In the case of cylindrical and spherical coordinates we get results with the help up well-known relationships from the literature. O_n this basis we assume that for the direct kinematics task's solution it is sufficient to determine $T_{0,n}$ homogenous transformation matrix in relation with $\vec{q}$ joint coordinates:

$$T_{0,n} = T_{0,1} \cdot T_{1,2} \cdot ...T_{n-1,n} \,. \tag{5}$$

The multiplication of matrices in the equation (5) can be done by starting on the right side. In this case, we respectively multiply with the $T_{i\text{-}1,i}$ $(i = n\text{-}1,...1)$ matrices, from the left side. By starting on the left side we multiply with the $T_{i\text{-}1,i}$ $(i = 2,..., n)$ matrices from the right.

For $T_{0,n}$ calculus, we multiplication from right to left. In this case, we are talking about backwards progressing symbolic solution (Figure 1):

$$T_{0,n} = T_{0,1}(T_{1,2}(...(T_{n-2,n-1} \cdot T_{n-1,n}))) \,. \tag{6}$$

Applying the formula from one foot to the biped robot (Figure 3) we get next symbolic solution:

$$T_{0,6} = T_{0,1}(T_{1,2}(T_{2,3}(T_{3,4}(T_{4,5} \cdot T_{5,6})))) \,. \tag{7}$$

The following backwards progressing relations provide the partial results:

$$T_{i-1,n} = T_{i-1,i} \cdot T_{i,n} ,\; i = n-1,...,1 \cdot \tag{8}$$

$T_{i\text{-}1,n}$-s upper left 3x3 part-matrix, describes the orientation of the L_n local system in the $L_{i\text{-}1}$ local system (Figure 1), while the fourth column the tool centre point (L_n origin's) is the position vector compared to $L_{i\text{-}1}$ local origin. $T_{i\text{-}1,n}$ matrices elements define the velocity data, and can be used in the future to solve the so-called inverse kinematics problem.

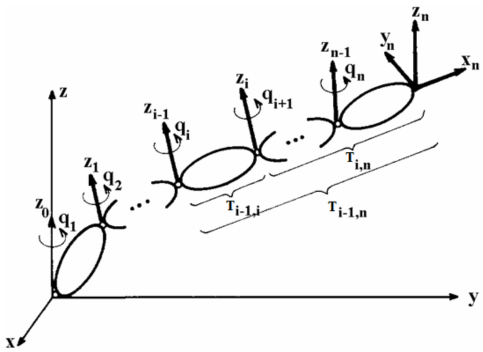

Fig. 1. *Backward recursive relations on the basis of Vukobratoviĉ* ($T_{i-1,n\,(i=0,\ldots,n-1)}$) *homogenous transformation matrix,* $Z_{0\ldots i}$ *coordinate axis*

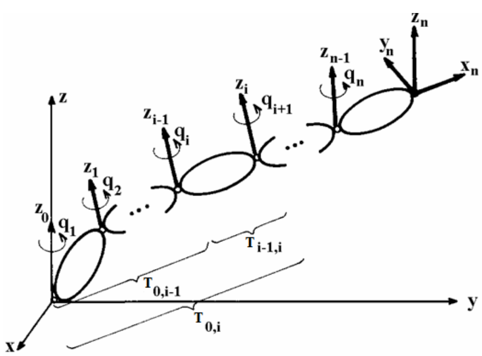

Fig. 2. *Forward recursive relations on the basis of Vukobratoviĉ* ($T_{i-1,\,n(i=0,\ldots,n-1)}$) *homogenous transformation matrix,* $Z_{0\ldots i}$ *coordinate axis*

If we start the multiplication from the other side, the $T_{0,n}$ matrix can be determined as follows (Figure 2):

$$T_{0,n} = (((T_{0,1} \cdot T_{1,2})\ldots)T_{n-2,n-1})T_{n-1,n}\,. \quad (9)$$

Applying the formula from one foot to the biped robot (Figure 3) we get next symbolic solution:

$$T_{0,6} = (((((T_{0,1} \cdot T_{1,2})T_{2,3})T_{3,4})T_{4,5})T_{5,6})\,. \quad (10)$$

The forward recursive symbolic links are provided by the following matrices:

$$T_{0,i} = T_{0,i-1} \cdot T_{i-1,i},\; i = 2,\ldots,n\,. \quad (11)$$

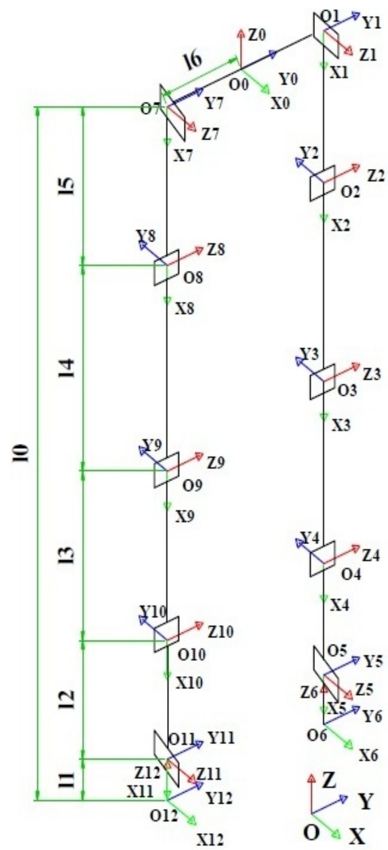

Fig. 3. *Kinematic model*

4. Kinematic Modelling

Direct kinematics problem is to define all relationships that end-effector position (foot of biped robot) based on joint coordinates practically [3], [5], it ensures internal coordinates conversion (joint) Coordinate external (operational).

Biped robot kinematics equations are (Figure 3):

$$T_{0,1} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ \sin(q_1) & \cos(q_1) & 0 & l_6 \\ \cos(q_1) & -\sin(q_1) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (12)$$

$$T_{1,2}=\begin{bmatrix}\cos(q_2) & -\sin(q_2) & 0 & l_5\\ 0 & 0 & -1 & 0\\ \sin(q_2) & \cos(q_2) & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \quad (13)$$

$$T_{2,3}=\begin{bmatrix}\cos(q_3) & -\sin(q_3) & 0 & l_4\\ \sin(q_3) & \cos(q_3) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \quad (14)$$

$$T_{3,4}=\begin{bmatrix}\cos(q_4) & -\sin(q_4) & 0 & l_3\\ \sin(q_4) & \cos(q_4) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \quad (15)$$

$$T_{4,5}=\begin{bmatrix}\cos(q_5) & -\sin(q_5) & 0 & l_2\\ 0 & 0 & -1 & 0\\ -\sin(q_5) & -\cos(q_5) & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \quad (16)$$

$$T_{5,6}=\begin{bmatrix}0 & 0 & 1 & l_1\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}. \quad (17)$$

Employ the equation (7) this Biped robot model (Figure 3) and the following backwards progressing relations provide the partial results:

$$T_{4,6}=T_{4,5}\cdot T_{5,6}=\begin{bmatrix}0 & -\sin(q_5) & \cos(q_5) & \cos(q_5)\cdot l_1+l_2\\ -1 & 0 & 0 & 0\\ 0 & -\cos(q_5) & -\sin(q_5) & -\sin(q_5)\cdot l_1\\ 0 & 0 & 0 & 1\end{bmatrix}, \quad (18)$$

$$T_{3,6}=T_{3,4}\cdot T_{4,6}=\begin{bmatrix}s(q_4) & -c(q_4)\cdot s(q_5) & c(q_4)\cdot s(q_5) & c(q_4)\cdot c(q_5)\cdot l_1+c(q_4)\cdot l_2+l_3\\ -c(q_4) & -s(q_4)\cdot s(q_5) & s(q_4)\cdot c(q_5) & c(q_4)\cdot c(q_5)\cdot l_1+c(q_4)\cdot l_2\\ 0 & -c(q_5) & -s(q_5) & -s(q_5)\cdot l_1\\ 0 & 0 & 0 & 1\end{bmatrix}, \quad (19)$$

$$T_{2,6}=T_{2,3}\cdot T_{3,6}=\begin{bmatrix}s(q_3+q_4) & -s(q_5)\cdot c(q_3+q_4) & c(q_5)\cdot c(q_3+q_4) & c(q_5)\cdot c(q_3+q_4)\cdot l_1+c(q_3+q_4)\cdot l_2+c(q_3)\cdot l_3+l_4\\ c(q_3+q_4) & -s(q_5)\cdot s(q_3+q_4) & c(q_5)\cdot s(q_3+q_4) & c(q_5)\cdot s(q_3+q_4)\cdot l_1+s(q_3+q_4)\cdot l_2+s(q_3)\cdot l_3\\ 0 & c(q_5) & -s(q_5) & s(q_5)\cdot l_1\\ 0 & 0 & 0 & 1\end{bmatrix}, \quad (20)$$

$$T_{1,6} = T_{1,2} \cdot T_{2,6} = \begin{bmatrix} s(q_2+q_3+q_4) & -s(q_5)\cdot c(q_2+q_3+q_4) & c(q_5)\cdot c(q_2+q_3+q_4) & c(q_5)\cdot c(q_2+q_3+q_4)\cdot l_1 + c(q_2+q_3+q_4)\cdot l_2 + c(q_2+q_3)\cdot l_3 + c(q_2)\cdot l_4 + l_5 \\ 0 & c(q_5) & s(q_5) & s(q_5)\cdot l_1 \\ c(q_2+q_3+q_4) & -s(q_5)\cdot s(q_2+q_3+q_4) & c(q_5)\cdot s(q_2+q_3+q_4) & c(q_5)\cdot s(q_2+q_3+q_4)\cdot l_1 + s(q_2+q_3+q_4)\cdot l_2 + s(q_2+q_3)\cdot l_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (21)$$

$$T_{0,6} = T_{0,1} \cdot T_{1,6} = \begin{bmatrix} -c(r) & s(q_5)\cdot s(r) & c(q_5)\cdot s(r) & -c(q_5)\cdot c(r)\cdot l_1 + s(r)\cdot l_2 + s(t)\cdot l_3 - s(q_2)\cdot l_4 \\ s(q_1)\cdot s(r) & c(q_5)\cdot c(q_1) - s(q_1)\cdot s(q_5)\cdot s(r) & s(q_5)\cdot c(q_1) + s(q_1)\cdot c(q_5)\cdot c(r) & s(q_1)\cdot l_5 + l_6 + (s(q_5)\cdot c(q_1) + c(q_5)\cdot s(q_1)\cdot c(r))\cdot l_1 + s(q_1)\cdot c(r)\cdot l_2 + s(q_1)\cdot c(t)\cdot l_3 + c(q_2)\cdot s(q_1)\cdot l_4 \\ c(q_1)\cdot s(r) & c(q_5)\cdot s(q_1) - c(q_1)\cdot s(q_5)\cdot c(r) & -s(q_5)\cdot s(q_1) + c(q_1)\cdot c(q_5)\cdot c(r) & c(q_1)\cdot l_5 + l_6 + (s(q_5)\cdot c(q_1) + c(q_5)\cdot s(q_1)\cdot c(r))\cdot l_1 + c(q_1)\cdot c(r)\cdot l_2 + c(q_1)\cdot c(t)\cdot l_3 + c(q_2)\cdot c(q_1)\cdot l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (22)$$

$c(r)$, $s(r)$, $c(t)$, $s(t)$, $c(q_i)$ and $s(q_i)$ is symbol, where $c(q_i) = \cos(q_i)$, $s(q_i) = \sin(q_i)$, $c(r) = \cos(q_2 + q_3 + q_4)$, $s(r) = \sin(q_2 + q_3 + q_4)$, $c(t) = \cos(q_2 + q_3)$, $s(t) = \sin(q_2 + q_3)$.

Convert coordinate joint operational details is done by solving the direct kinematics problem and coordinate joint operational coordinate conversion is done by solving the inverse kinematics problem.

Inverse kinematics problem allows the calculation [4], [2] coordinates of the joints, which provide end-effector in the desired position and orientation, given the absolute coordinates (operational). When the problem is the inverse kinematics solution, it is the inverse geometrical model. If we cannot find an analytical solution for inverse kinematics problem (which happens quite frequently) we resort to numerical methods, but whose weakness is the sheer volume of calculations. The most common method is Newton-Raphson method. Among these features is remarkable for the way it offers and Khalil Pieper and Paul's method. Pieper and Khalil's method allows solving inverse kinematics problem regardless of the values of the robot geometrical.

5. Inverse Kinematics Problem Solving Opportunities

In other word we can say that we are searching for the equations solution, where $\vec{q}$ the searched joint is coordinates vector, and x_q is the known external coordinate's vector:

$$\vec{q} = f^{-1}(x_q)\,. \quad (23)$$

The solution for the inverse kinematics problem, generally speaking is unclear. In the case of the so-called cinematically redundant robots [6], [1], [3], where the mobility number is greater than the dimension of the external coordinates, an infinite number of vector $\vec{q}$ can be found, which satisfies the $f(\vec{q}\,) = x_q$ equation.

In the case of the cinematically non-redundant robots only a finite number of solutions exist. Easy to imagine, for example, we can approach the same point with the same hand-coordination with raised and lowered elbow position as well. Furthermore the possibilities for solving the inverse problem are discussed only for non-redundant manipulators.

During the solution of the problem, non-linear equations have to be solved. For the determination of the roots, two modes of approach are found in the literature: a general numerical solution method is applied (eg. Newton's method) and other techniques the given biped robot in closed form searches for symbolic solutions.

6. The Inverse Matrix Method

The inverse matrix method [1], [7], can be applied for most biped robots, if homogeneous transformation's description leads to trigonometric equations. During the solution we try to choose such equations, of which the joint variables can be expressed in the form of a two variable arctangent function.

If for example the 5 mobility degree biped robot limb's $T_{0,6}$ transformation matrix, is known from the direct task symbolic solution, then the next 5, can be calculated with the known matrix multiplication methods:

$$T_{0,6} = T_{0,1} \cdot T_{1,2} \cdot T_{2,3} \cdot T_{3,4} \cdot T_{4,5} \cdot T_{5,6} . \qquad (24)$$

In the case of the inverse task, the left side matrix is known in the following form:

$$T_{0,6}^{*} = \begin{bmatrix} a_x & b_x & c_x & f_x \\ a_y & b_y & c_y & f_y \\ a_z & b_z & c_z & f_z \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (25)$$

where $T_{0,6}^{*}$ are boards control homogenous transformation matrix:

$$T_{0,6}^{*} = T_{0,6} . \qquad (26)$$

If we multiply the equation respectively with the inverse of matrices from the left, then 4 matrix equations are obtained. To determine the inverses, the properties of ortho-normal matrices can be used:

$$T_{0,1}^{-1} \cdot T_{0,6}^{*} = T_{0,6} , \qquad (27)$$

$$T_{1,2}^{-1} \cdot T_{0,1}^{-1} \cdot T_{0,6}^{*} = T_{0,6} , \qquad (28)$$

$$T_{2,3}^{-1} \cdot T_{1,2}^{-1} \cdot T_{0,1}^{-1} \cdot T_{0,6}^{*} = T_{0,6} , \qquad (29)$$

$$T_{3,4}^{-1} \cdot T_{2,3}^{-1} \cdot T_{1,2}^{-1} \cdot T_{0,1}^{-1} \cdot T_{0,6}^{*} = T_{0,6} , \qquad (30)$$

$$T_{4,5}^{-1} \cdot T_{3,4}^{-1} \cdot T_{2,3}^{-1} \cdot T_{1,2}^{-1} \cdot T_{0,1}^{-1} \cdot T_{0,6}^{*} = T_{0,6} . \qquad (31)$$

To find the corresponding joint variables q_1, q_2, q_3, q_4, and q_5 we must solve the following simultaneous set of nonlinear trigonometric equations (26).

The equations this example is, of course, much too difficult to solve directly in closed form. This is the case for most biped robot leg's. Therefore, we need to develop efficient and systematic techniques that exploit the particular kinematic structure [7], [4] of the biped robot. Whereas the forward or backward kinematics problem always has a unique solution that can be obtained simply by evaluating the forward and backward equations, the inverse kinematics problem may or may not have a solution. Even if a symbolic solution exists, it may or may not be unique.

Furthermore, because these forward or backward kinematic equations are in general complicated nonlinear functions of the joint variables, the symbolic solutions may be difficult to obtain even when they exist.

Employ the equation (26) and the following a symbolic solution symbolic solution is obtained from q_1, q_5 and q_2, q_3, q_4 this amount:

$$q_1 = \operatorname{arctg}\left(\frac{a_y}{a_z}\right), \tag{32}$$

$$q_5 = \operatorname{arctg}\left(\frac{b_x}{c_x}\right), \tag{33}$$

$$q_2 + q_3 + q_4 = \operatorname{arctg}\left(\frac{b_x}{c_x}\right). \tag{34}$$

On the left side of the (27…31) equations the known $T^*_{0,6}$ matrix and the we get symbolic solution's in a q_2, q_3 and q_4 joint variable is used. This matrix is known from the backward progressing symbolic solution. If so, we look for the first element on the left-hand side containing the joint coordinates, for which the suitable element on the right-hand side is constant or 0. Thus (27…31) equation the joint variable q_2, q_3 and q_4 can be determined.

In equation (27…31) we once again look for a matrix element on the left-hand side, from which we express the q_2, q_3 and q_4 joint coordinate in simple form, in turn the right-hand side is constant. Continuing this technique step-by-step, from each matrix element newer and newer joint variables can be expressed. We obtained the joint coordinates from the solution of the equations:

$$q_2 + q_3 = \operatorname{arctg}\left(\frac{m}{\sqrt{1-m^2}}\right), \tag{35}$$

where m is:

$$\begin{aligned} m = \frac{1}{l_3} \cdot (&-f_x - \sin(q_2+q_3+q_4)\cdot l_2 \\ &- \cos(q_5)\cdot \sin(q_2+q_3+q_4)\cdot l_1), \end{aligned} \tag{36}$$

$$q_2 = \operatorname{arctg}\left(\frac{\sqrt{1-k^2}}{k}\right), \tag{37}$$

where k is:

$$\begin{aligned} k = \frac{1}{l_4} \cdot (&-\sin(q_1)\cdot f_y + \cos(q_1)\cdot f_z \\ &- \sin(q_1)\cdot l_6 - \cos(q_2+q_3)\cdot l_3 \\ &+ \cos(q_5)\cdot \cos(q_2+q_3+q_4)\cdot l_1 \\ &- \cos(q_2+q_3+q_4)\cdot l_2 - l_5), \end{aligned} \tag{38}$$

$$q_3 = \operatorname{arctg}\left(\frac{m}{\sqrt{1-m^2}}\right) - q_2, \tag{39}$$

$$q_4 = \operatorname{arctg}\left(\frac{b_x}{c_x}\right) - q_2 - q_3. \tag{40}$$

7. Conclusions

The method described in this paper is well suited to obtain symbolical inverse functions for the practical relevant redundant kinematics with parallel and/or perpendicular joint axes. The symbolic solution is derived in a closed form and contain the parameters of the additional geometrical constraints. They allows a simple selection of the solution sets, which describe the different possible biped robot poses. The optimization of these poses to the given biped robots task may be fulfilled through any criterion, including nonanalytic.

If a biped robot's inverse kinematics problem is symbolic solved well, then this helps a lot on the stability, because the well positioned ligament's overall centre of weight has to fall in the given sole's polygon, so that the robot wouldn't tumble over.

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