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EQUATION OF MOTION OF THE NONLINEAR ROBOTIC SYSTEMS WITH CONSTRAINTS

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Abstract: *This paper presents a simple methodology for obtaining the entire set of continuous controllers that cause a nonlinear dynamical system to exactly track a given trajectory. The trajectory is provided as a set of algebraic differential equations that may or may not be explicitly dependent on time. The method provided is inspired by results from analytical dynamics and the close connection between nonlinear control and analytical dynamics is explored. The results provided in this paper here yield new and explicit methods for the control of highly nonlinear systems. The paper is based on previous work of the authors.*

Key words: *nonlinear dynamical system, robot analytical dynamics, robot trajectory tracking, controllers.*

1. Introduction

Most of the robotic applications are restricted to slow-motion operations without interactions with the environment. This is mainly due to limited performance of the available controllers in the market that are based on simplified system models. To increase the operation speed with more servo accuracy, advanced control strategies are needed.

The main specifically properties in the high speed motion control of the robots systems are the complexity of the dynamics and uncertainties, both parametric and dynamic. Parametric uncertainties arise from imprecise knowledge of kinematics parameters and inertia parameters, while dynamic uncertainties arise from joint and link flexibility, actuator dynamics, friction, sensor noise and unknown environment dynamics.

The operational-space formulation is particularly useful in the context of motion and force control systems. On the other hand, in the joint space control methods, is assumed that the reference trajectory is available in terms of the time history of joints positions and orientations of robot arm.

For design of the tracking controller, one assumes that the reference trajectory and path have been pre-computed.

Control of robot manipulators is naturally achieved in the joint space, since the control input are joint torques. But, the user specifies a motion in the task space, and thus it is important to extend the control problem to the task space. This can be achieved by different strategies. The more natural strategy consists of inverting the kinematics of the manipulator to compute the joint motion corresponding to the given end-effectors motion.

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Thus, the methods used to appointment primarily depend on linearization and/or PID-type control, and they envisage suppositions on the structure of the control effort.

2. Control of Nonlinear Dynamical Systems

Most physical robotic systems are inherently nonlinear. Thus, control of nonlinear systems is a subject of active research and increasing interest. However, most controller design techniques for nonlinear systems are not systematic and/or relate only to very specific cases.

Current systematic approach to design controllers for nonlinear systems is feedback linearization. The basic idea of this technique is to design a control law that cancels the nonlinearities of the plant and yields a closed-loop system with linear dynamics [5]. However, the technique is not robust to disturbances and uncertainties in the robot parameters, can yield to uncontrolled dynamics called zero dynamics and can only be applied to systems verifying certain vector field relations [3].

The development of controllers for nonlinear complex systems has been an area of intense research. Many controllers that have been developed for trajectory tracking of complex nonlinear and multi-body systems rely on some approximations and/or linearization [1]. Most control designs restrict controllers for nonlinear systems to be affine in the control inputs. Often, the system equations are linearized about the system's nominal trajectory and then the linearized equations are used along with various results from the well-developed theories of linear control. While this often works well in many situations, there are some situations in which better controllers may be needed. This is especially so when highly accurate trajectory tracking is required to be done in real time on systems that are highly nonlinear such the robotic systems.

In the robotics literature trajectory tracking using inverse dynamics and model reference control has been used for some time now, and the methods developed therein can be seen as particular subclasses of the formulation discussed in the present paper. Trajectory tracking in the adaptive control context (which is not the subject of this paper) has also been explored together with specific parameterizations to guarantee linearity in the unknown parameters of a system [4].

3. The Equation of Motion for Constrained Multi Body Robotic System

The equations of motion for constrained mechanical systems are based on the principle of Lagrange. The principle states that at each instant of time *t*, a constrained mechanical system evolves in such a manner that the total work done by all the forces of constraint under any set of virtual displacements is always zero. This principle, which in effect prescribes the nature of the forces of constraint which act upon a mechanical system, has been found to yield, in practice, adequate descriptions of the motion of large classes of mechanical systems, thereby making it an extremely useful and effective principle.

One considers the robot equation of motion for constrained robotic systems, given by the joint-space formulation, usually presented in the canonical forms:

$$\mathbf{M}(\mathbf{q},t)\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) + \mathbf{J}^{\mathbf{T}}\mathbf{F} = \boldsymbol{\tau}\,, \quad (1)$$

$\mathbf{M}$ is an *n* by *n* symmetric, positive-definite matrix and is called the generalized, or joint-space, inertia matrix, $\mathbf{C}$ is an *n* by *n* matrix such that $\mathbf{C}\dot{\mathbf{q}}$ is the vector of Coriolis and centrifugal terms - collectively known as velocity product terms - $\mathbf{g}$ is the vector of gravity terms and $\mathbf{F}$ is a vector of forces exerted by the end-effectors. More terms can be added to this equation, as required, to account for other dynamical effects (e.g., viscous friction).

The symbols $\mathbf{q}$, $\dot{\mathbf{q}}$, $\ddot{\mathbf{q}}$ and $\boldsymbol{\tau}$ represent *n*-dimensional vectors of joint position, velocity, acceleration and effort variables respectively, where n is the number of degrees of motion freedom (DoF) of the robot mechanism.

This equation shows the functional dependencies explicitly: $\mathbf{M}$ is a function of $\mathbf{q}$, $\mathbf{C}$ is a function of $\mathbf{q}$ and $\dot{\mathbf{q}}$, and so on. Once these dependencies are understood, they are usually omitted.

Consider an unconstrained nonlinear mechanical robot system described by the second order differential equation of motion under the form:

$$\mathbf{M}(\mathbf{q},t)\ddot{\mathbf{q}} = \mathbf{Q}(\mathbf{q},\dot{\mathbf{q}},t)\,, \tag{2}$$

$$\mathbf{q}(0) = \mathbf{q}_0\,,\ \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0\,, \tag{3}$$

where, $\mathbf{q_0}$ and $\dot{\mathbf{q}}_0$ are the position and velocity vectors at initial time of the robot with n DoF; the dots indicate differentiation with respect to time. Equations (1) and (2) can be obtained using Lagrangean model.

The *n*-nonlinear vector $\mathbf{Q}(\mathbf{q},\dot{\mathbf{q}},t)$, on the right hand side of Equation (2) is a 'known' vector in the sense that it is a known function of its arguments. By 'unconstrained' one means that the components of the initial velocity $\dot{\mathbf{q}}_0$ of the robot system can be independently assigned [2].

By 'unconstrained' one means here that the n coordinates, $\mathbf{q}$ are independent of one another, or are to be treated as being independent of each other.

One requires that this mechanical system will be controlled so that it tracks a trajectory that is described by the following constrained set of m equations:

$$\mathbf{\Phi}_i(\mathbf{q},t) = 0\,,\ \ i = 1..h\,, \tag{4}$$

and

$$\mathbf{\Psi}_i(\mathbf{q},\dot{\mathbf{q}},t) = 0\,,\ \ i = h+1,...m. \tag{5}$$

Suppose further that the unconstrained system is now subjected to the m constraints.

One assumes that the mechanical robot system's initial conditions are such as to satisfy these relations at the initial time. In order to control the system so that it exactly tracks the required trajectory i.e. satisfies Equations (3) and (4) one must apply an appropriate control *n*-vector $\mathbf{Q}(\mathbf{q},\dot{\mathbf{q}},t)$ so that the equation of motion of the controlled system becomes:

$$\begin{aligned}&\mathbf{M}(\mathbf{q},t)\ddot{\mathbf{q}} = \mathbf{Q}(\mathbf{q},\dot{\mathbf{q}},t) + \mathbf{Q}_c(\mathbf{q},\dot{\mathbf{q}},t)\,,\\ &\mathbf{q}(0) = \mathbf{q}_0\,,\ \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0\,,\end{aligned} \tag{6}$$

where now, the components of the *n*-vectors $\mathbf{q}_0$ and $\dot{\mathbf{q}}_0$, satisfy Equations (4) and (5) at the initial time, $t = 0$.

Throughout this paper, one shall, for brevity, drop the arguments of the various quantities, unless needed for clarity.

The controlled system is described by the Equation (6), where $\mathbf{Q}_c$ is the control vector.

One begins by expressing Equation (6) in terms of the accelerations of the system. For any positive-definite n by n matrix $\mathbf{P}$ ($\mathbf{q}$, t), one define the matrix:

$$\mathbf{G}(\mathbf{q},t)\mathbf{q} = [\mathbf{P}^{1/2}(\mathbf{q},t)\mathbf{M}(\mathbf{q},t)]^{-1}\,. \tag{7}$$

Pre-multiplying Equation (6) by $\mathbf{P}^{1/2}(\mathbf{q}, t)$, the ''pondered'' equation, which will indicate using the superscript p, is obtained as:

$$\ddot{\mathbf{q}}^p = \mathbf{a}^p + \ddot{\mathbf{q}}_c^p\,, \tag{8}$$

where:

$$\mathbf{a}^p = \mathbf{G}^{-1}\mathbf{a}, \text{ and } \ddot{\mathbf{q}}_c^p = \mathbf{G}^{-1}\ddot{\mathbf{q}}_c. \tag{9}$$

One designates the acceleration of the uncontrolled system by:

$$\mathbf{a}(\mathbf{q},\dot{\mathbf{q}},t) = \mathbf{M}^{-1}(\mathbf{q},t)\mathbf{Q}(\mathbf{q},\dot{\mathbf{q}},t). \tag{10}$$

In Equation (6), one identifies the expression:

$$\ddot{\mathbf{q}}_c(\mathbf{q},\dot{\mathbf{q}},t) = \mathbf{M}^{-1}(\mathbf{q},t)\mathbf{Q}_c(\mathbf{q},\dot{\mathbf{q}},t), \tag{11}$$

witch can be viewed as the deviation of the acceleration of the controlled system from that of the uncontrolled system.

From Equation (8), one obtains the expression:

$$\ddot{\mathbf{q}} = \mathbf{a} + \ddot{\mathbf{q}}_c. \tag{12}$$

One differentiates Equation (4) twice and Equation (5) once with respect to time t, giving the set of equations:

$$\mathbf{A}(\mathbf{q},\dot{\mathbf{q}},t)\ddot{\mathbf{q}} = \mathbf{b}(\mathbf{q},\dot{\mathbf{q}},t), \tag{13}$$

where $\mathbf{A}$ is an m by n matrix of rank k and $\mathbf{b}$ is an m-vector. With Equations (6), (8) and (12) Equation (13) can be further expressed as [5]:

$$\mathbf{B}(\mathbf{q},\dot{\mathbf{q}},t)\ddot{\mathbf{q}}_c = \mathbf{b}(\mathbf{q},\dot{\mathbf{q}},t), \tag{14}$$

where $\mathbf{B}$ is an m by n matrix who is calculated by the expression:

$$\begin{aligned}\mathbf{B}(\mathbf{q},\dot{\mathbf{q}},t)\ddot{\mathbf{q}}_c \\ = \mathbf{A}(\mathbf{q},\mathbf{q},t)[\mathbf{P}^{1/2}(\mathbf{q},t)\mathbf{M}(\mathbf{q},t)]^{-1}.\end{aligned} \tag{15}$$

One can now express the accelerations n-vector $\ddot{\mathbf{q}}$ in terms of its orthogonal projections on the range space of $\mathbf{B}^T$ and the null space of $\mathbf{B}$, so that:

$$\ddot{\mathbf{q}} = \mathbf{B}^+\mathbf{B}\ddot{\mathbf{q}} + (\mathbf{I} - \mathbf{B}^+\mathbf{B})\ddot{\mathbf{q}}. \tag{16}$$

In Equation (16), the matrix $\mathbf{B}^+$ denotes the Moore-Penrose generalized inverse of the matrix **B**. It should be noted that Equation (16) is a general identity that is always valid since it arises from the orthogonal partition of the identity matrix $\mathbf{I} = \mathbf{B}^+\mathbf{B} + (\mathbf{I} - \mathbf{B}^+\mathbf{B})$.

Using Eq. (14) in the first member on the right hand side of Eq. (16), and Eq. (12) to replace $\ddot{\mathbf{q}}$ in the second member, one gets:

$$\ddot{\mathbf{q}} = \mathbf{B}^+\mathbf{b} + (\mathbf{I} - \mathbf{B}^+\mathbf{b})(\mathbf{a} + \ddot{\mathbf{q}}_c), \tag{17}$$

which, due to Equation (11), yields:

$$\mathbf{B}^+\mathbf{B}\ddot{\mathbf{q}}_c = \mathbf{B}^+(\mathbf{b} - \mathbf{B}\mathbf{a}). \tag{18}$$

The general solution of the linear set of Equations (18) is given by [5]:

$$\begin{aligned}\ddot{\mathbf{q}}_c &= (\mathbf{B}^+\mathbf{B})^+\mathbf{B}^+(\mathbf{b} - \mathbf{B}\mathbf{a}) \\ &\quad + [\mathbf{I} - (\mathbf{B}^+\mathbf{B})^+(\mathbf{B}^+\mathbf{B})]\mathbf{z}.\end{aligned} \tag{19}$$

After any combination one obtains the second equality:

$$\ddot{\mathbf{q}}_c = \mathbf{B}^+(\mathbf{b} - \mathbf{B}\mathbf{a}) + (\mathbf{I} - \mathbf{B}^+\mathbf{B})\mathbf{z}, \tag{20}$$

where the n-vector $\mathbf{z}(\mathbf{q},\dot{\mathbf{q}},t)$ is any arbitrary n-vector. To obtain the second equality above, one used the property that $(\mathbf{B}^+\mathbf{B})^+ = (\mathbf{B}^+\mathbf{B})$ in the two members on the right hand side along, with the property so that $\mathbf{B}^+\mathbf{B}\mathbf{B}^+ = \mathbf{B}^+$.

The set of all possible controls $\mathbf{Q}_c(\mathbf{q},\dot{\mathbf{q}},t)$ (or controllers) that causes the controlled system to exactly track the required trajectory is explicitly given by:

$$\begin{aligned}\mathbf{Q}_c(\mathbf{q},\dot{\mathbf{q}},t) = \mathbf{P}^{-1/2}\ddot{\mathbf{q}}_c = \mathbf{P}^{1/2}\mathbf{B}^+(\mathbf{b} - \mathbf{B}\mathbf{a}) \\ + \mathbf{P}^{1/2}(\mathbf{I} - \mathbf{B}^+\mathbf{B})\mathbf{z}.\end{aligned} \tag{21}$$

The mechanical robotic system, described by the nonlinear Lagrange Equation (1), is explicitly controlled through the addition of a control, n-vector $\mathbf{Q}_c(\mathbf{q},\dot{\mathbf{q}},t)$, provided by Equation (21), in witch the n-vector $\mathbf{z}$ may be chosen still are exactly satisfied the imposed constrains.

4. Explicit Equations of Motion for General Constrained System

The constrained mechanical system described by Eqs. (1)…(4) evolves in time in such a manner that the total work done at any time, t, by the constraint force n-vector $\mathbf{Q}_c$ under virtual displacements at time t is given by (21).

The work done by the forces of constraints under virtual displacements at any instant of time t can be expressed as:

$$\mathbf{w}^T \mathbf{Q}_c(\mathbf{q},\dot{\mathbf{q}},t) = \mathbf{w}^T \mathbf{C}(\mathbf{q},\dot{\mathbf{q}},t), \quad (22)$$

where $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}},t)$ is an n-vector describing the nature of the non-ideal constraints, which could be obtained by experimentation and/or observation. The virtual displacement vector, $\mathbf{w}(t)$, is any non-zero n-vector that satisfies:

$$\mathbf{A}(\mathbf{q},\dot{\mathbf{q}},t)\mathbf{w} = 0. \quad (23)$$

Solving Equation (23), the n-vector $\mathbf{w}$ can be written as (we suppress the arguments for clarity):

$$\mathbf{w} = (\mathbf{I} - \mathbf{A}^+\mathbf{A})\,\boldsymbol{\gamma}, \quad (24)$$

where $\boldsymbol{\gamma}$ is any arbitrary n-vector, and $\mathbf{A}^+$ is the Moore-Penrose inverse of the matrix $\mathbf{A}$. Substituting Equation (24) in Equation (22), one obtains:

$$\begin{aligned}&\boldsymbol{\gamma}(\mathbf{I} - \mathbf{A}^+\mathbf{A})\mathbf{Q}_c(\mathbf{q},\dot{\mathbf{q}},t)\\ &\quad = \boldsymbol{\gamma}(\mathbf{I} - \mathbf{A}^+\mathbf{A})\mathbf{C}(\mathbf{q},\dot{\mathbf{q}},t).\end{aligned} \quad (25)$$

Since each component of the vector $\boldsymbol{\gamma}$ can be independently chosen, Equation (25) yields:

$$\begin{aligned}&(\mathbf{I} - \mathbf{A}^+\mathbf{A})\mathbf{Q}_c(\mathbf{q},\dot{\mathbf{q}},t)\\ &\quad = (\mathbf{I} - \mathbf{A}^+\mathbf{A})\mathbf{C}(\mathbf{q},\dot{\mathbf{q}},t).\end{aligned} \quad (26)$$

Pre-multiplying Equation (6) by $(\mathbf{I} - \mathbf{A}^+\mathbf{A})$ and using Equation (26), one gets:

$$\begin{aligned}&(\mathbf{I} - \mathbf{A}^+\mathbf{A})\mathbf{M}(\mathbf{q},t)\ddot{\mathbf{q}}\\ &\quad = (\mathbf{I} - \mathbf{A}^+\mathbf{A})[\mathbf{Q}(\mathbf{q},\dot{\mathbf{q}},t) + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}},t)].\end{aligned} \quad (27)$$

Equation (27) and Equation (13) can now be written together and will be obtained Equation (28). One can solve Equation (28) to get Equation (29), with $\boldsymbol{\eta}$ - an arbitrary n-vector:

$$\begin{bmatrix}(\mathbf{I} - \mathbf{A}^+\mathbf{A}) & \mathbf{M}(\mathbf{q},t)\\ & \mathbf{A}\end{bmatrix}\ddot{\mathbf{q}} = \begin{bmatrix}(\mathbf{I} - \mathbf{A}^+\mathbf{A}) & [\mathbf{Q}(\mathbf{q},\dot{\mathbf{q}},t) + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}},t)]\\ & \mathbf{b}\end{bmatrix}, \quad (28)$$

$$\ddot{\mathbf{q}} = \widehat{\mathbf{M}}^+(\mathbf{q},t)\begin{bmatrix}[\mathbf{Q}(\mathbf{q},\dot{\mathbf{q}},t) + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}},t)]\\ \mathbf{b}\end{bmatrix} + (\mathbf{I} - \widehat{\mathbf{M}}^+\widehat{\mathbf{M}})\boldsymbol{\eta}. \quad (29)$$

The matrix $\widehat{\mathbf{M}}$ defined above is a (m x n) x n rectangular matrix:

$$\widehat{\mathbf{M}} = \begin{bmatrix}(\mathbf{I} - \mathbf{A}^+\mathbf{A}) & \mathbf{M}(\mathbf{q},t)\\ & \mathbf{A}\end{bmatrix}. \quad (30)$$

Equation (29) is the general explicit equation of motion for constrained robotic systems with non-ideal constraints.

5. Conclusions

This paper presents the motion in terms of second-order differential equations. This methodology has been inspired by results in analytical dynamics. This paper takes a generally different approach that is based on recent results from analytical dynamics. Here the complete nonlinear problem is addressed with no assumptions on the type of controller that is to be used, except that it will be continuous.

Assuming that the system's initial conditions satisfy the description of the trajectory the explicit closed-form expression (17) provides the entire set of continuous tracking controllers that can exactly track a given trajectory description. The explicit closed-form expressions for the controllers can be computed in real time.

Closed-form expressions for all the continuous controllers required for trajectory tracking for nonlinear systems do not make approximations. Furthermore, no approximations or linearization are made here with respect to the trajectory that is being tracked, which may be described in terms of nonlinear algebraic equations or nonlinear differential equations.

Moreover, the approach arrives not just at one nonlinear controller for controlling a given nonlinear system, but also at the entire set of continuous controllers that would cause a given set of trajectory descriptions to be exactly satisfied.

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