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INFLUENCE OF FIBER ORIENTATION ON TORSIONAL VIBRATIONS OF THIN-WALLED LAMINATED COMPOSITE BEAMS

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Abstract: *In the paper was presented the influence of fiber orientation on torsional vibrations of thin-walled composite beams with open cross-section. Equations of motion are derived using the principle of virtual displacements. For a case of arbitrary cross-section and arbitrary stacking sequence differential equations are coupled. Closed form solution was derived for I cross section with arbitrary fiber orientation in laminate.*

Key words: *composite, thin-walled, vibrations.*

1. Introduction

Thin-walled composite structures are widely used in many fields of aerospace, automotive, nautical and other industries. Over a past few decades they became broadly adopted in civil engineering due to many advantages of this material, like lightweight feature in relation of resistance, corrosion resistance, low thermal expansion, good mechanical characteristics, etc.

The theory of thin walled beams was first investigated Vlasov [1]. Until now, research of stability and dynamic characteristics of these materials took a lot of attention and carried out intensively.

Wu and Sun [2] developed a simplified theory based on the Vlasov-our theory. It includes seven differential equations, which relate to the axial, horizontal and vertical displacement, torsion, derivatives of vertical and horizontal movement and rotation of the axis. Later these seven equations based on four coupled differential equations.

Lee [3] analyzed the free vibrations of thin-walled composite beams and cross-section with an arbitrary stacking sequence. To solve the problem, the finite element method was used and all the tones were included in the analysis.

Kim [4] represents the exact stiffness matrix that can be applied to the free vibration analysis of thin-walled and composite beams with symmetrical and arbitrary stacking sequence with respect to the center line. This work represents the first attempt to work with the exact frequencies of thin-walled composite beams with arbitrary stacking sequence.

Equations of motion are derived for a case of arbitrary cross-section and arbitrary stacking sequence, it is shown how suitable choice of cross-section affects on equations of motion. For a specific case, a

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closed-form solution for the natural frequences of free harmonic vibrations was derived. Results for the example are obtained numerically and using the commercial software Ansys, they verify the accuracy of the derived solutions.

2. Equations of Motion

A thin-walled beam of an arbitrary open cross-section is considered (Fig.1). Based on Vlasov`s theory [1], the displacements of an arbitrary point *S* of cross-section can be described by four components, three translations $u_p, v_{p,} w$ and rotation φ about the arbitrarily taken pole *P*.

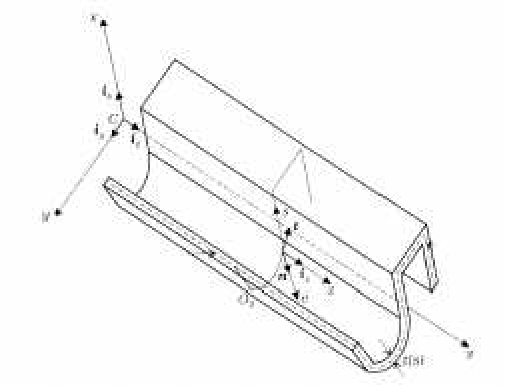

Fig. 1. *Arbitrary open cross section*

$$\begin{aligned} u_* &= u_p - \varphi(y - y_p) \\ v_* &= v_p + \varphi(x - x_p) \\ w_* &= w - u'_p x - v'_p y - \varphi' \omega_p \end{aligned} \tag{1}$$

- where ω_p represents warping function with respect to the pole P.

Based on the assumptions adopted in the theory of thin-walled beams, strain values that are different from zero are strain and shear.

$$\begin{aligned} \varepsilon_z &= \frac{\partial w_z}{\partial z} = w' - u''_P x - v''_P y - \varphi'' \omega_P \\ \gamma_{sz} &= \gamma_s = 2\varphi' e \end{aligned} \tag{2}$$

where *e* is the distance of the observed point from the middle surface measured along the normal n.

Reducing the normal stresses on the center of gravity and shear stresses on the pole *P*, for stress resultants the following expressions are obtained [2]:

$$\begin{aligned} N &= \iint_F \sigma_z \, dF \\ M_x &= \iint_F \sigma_z \, y dF \\ M_y &= -\iint_F \sigma_z \, x dF \\ V_x &= -\iint_F \tau_{zs} \sin \alpha dF \\ V_y &= \iint_F \tau_{zs} \cos \alpha dF \\ T_P &= \iint_F \tau_{zs} h_p dF \\ T_S &= 2\iint_F \tau_{zs} e dF \\ M_{\omega_P} &= \iint_F \sigma_z \omega_P dF \end{aligned} \tag{3}$$

In Eqs. (3), N represents the axial force, M_x and M_y the bending moments with respect to the x and y-axis, V_x and V_y the shear forces in the x and y directions, T_P the torsion moment, T_S the Saint Venant torque, $M_{\omega P}$ the bimoment.

Equations of motion are derived using the principle *of* virtual displacements. A small element subjected to external loads *p* per unit area is considered (Fig. 2.).

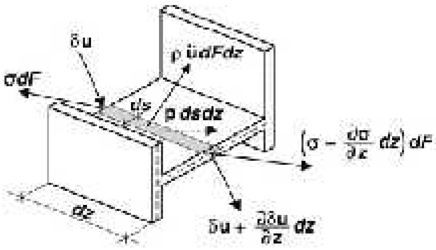

Fig. 2. *A small element*

Stress vector is defined by:

$$\sigma = \tau_{zs} t + \sigma_z i_z = \\ = -\tau_{zs} \sin\alpha i_x + \tau_{zs} \cos\alpha i_y + \sigma_z i_z \quad (4)$$

The vector of virtual displacements is adopted in the same form as a vector of real displacements:

$$\delta u = \delta u_* i_x + \delta v_* i_y + \delta w_* i_z = \\ = \left[\delta u_p - \delta\varphi(y - y_p)\right] i_x + \left[\delta v_p + \delta\varphi(x - x_p)\right] i_y \\ + (\delta w - \delta u'_p x - \delta v'_p y - \delta\varphi' \omega_p) i_z \quad (5)$$

Virtual displacements parameters are arbitrary functions of coordinates and do not depend upon external loads.

The virtual work expression is:

$$\delta W + \delta U = 0 \quad (6)$$

where δW is the virtual work of external loads and inertia forces through virtual displacements δu and δU the virtual work of actual stresses realized through virtual strains.

The virtual work of external loads and inertia forces per unit length of the element is:

$$\delta W = \iint_F (\sigma_{,z} \delta u + \sigma \delta u_{,z}) dF + \\ + \int_S \bar{p} \delta u ds - \rho \iint_F \ddot{u} \delta u dF \quad (7)$$

where ρ is the density, and $\ddot{u}$ is the acceleration vector given by:

$$\ddot{u} = \ddot{u}_* i_x + \ddot{v}_* i_y + \ddot{w}_* i_z = \\ = \left[\ddot{u}_p - \ddot{\varphi}(y - y_p)\right] i_x + \left[\ddot{v}_p + \ddot{\varphi}(x - x_p)\right] i_y \\ + (\ddot{w} - \ddot{u}'_p x - \ddot{v}'_p y - \ddot{\varphi}' \omega_p) i_z \quad (8)$$

Substituting (4), (5) and (8) into (7) we get following equation for δW: (5)

$$\delta W = \iint_F \Big\{ -\tau'_{zs} \sin\alpha \left[\delta u_p - \delta\varphi(y - y_p)\right] + \tau'_{zs} \cos\alpha \left[\delta v_p + \delta\varphi(x - x_p)\right] + \\ + \sigma'_z (\delta w - \delta v'_p y - \delta u'_p x - \delta\varphi' \omega_p) - \tau_{zs} \sin\alpha \left[\delta u'_p - \delta\varphi'(y - y_p)\right] + \\ + \tau_{zs} \cos\alpha \left[\delta v'_p + \delta\varphi'(x - x_p)\right] + \sigma_z (\delta w' - \delta v''_p y - \delta u''_p x - \delta\varphi'' \omega_p \Big\} dF + \\ + \int_S \left\{ \bar{p}_x \left[\delta u_p - \delta\varphi(y - y_p)\right] + \bar{p}_y \left[\delta v_p + \delta\varphi(x - x_p)\right] + \right. \\ \left. + \bar{p}_z (\delta w - \delta v'_p y - \delta u'_p x - \delta\varphi' \omega_p) \right\} dS - \rho \iint_F (\delta u_* \ddot{u}_* + \delta v_* \ddot{v}_* + \delta w_* \ddot{w}_*) dF \quad (9)$$

The virtual work of internal load due to the corresponding variation of deformation, per unit length of element, is:

$$\delta U = -\iint_F (\sigma_z \delta\varepsilon_z + \tau_s \delta\gamma_s) dF \quad (10)$$

Using expressions (2) for virtual strains we get:

$$\delta U = -\iint_F \left[\sigma_z (\delta w' - \delta u_p'' x - \delta v_p'' y - \delta \varphi'' \omega_p) + \tau_s 2\delta\varphi' e \right] dF \tag{11}$$

By suitable rearrangement of equations (9) and (11) in accordance with virtual displacement parameters, the principle of virtual work may be expresses as:

$$\begin{aligned}
&\delta w \left\{ \iint_F \sigma_z' dF - \rho \iint_F \ddot{w}_* dF + \int_s \bar{p}_z ds \right\} + \delta u_p \left\{ -\iint_F \tau_{zs}' \sin\alpha dF - \rho \iint_F \ddot{u}_* dF + \int_s \bar{p}_x ds \right\} + \\
&+\delta v_p \left\{ \iint_F \tau_{zs}' \cos\alpha dF - \rho \iint_F \ddot{v}_* dF + \int_s \bar{p}_y ds \right\} + \\
&+\delta\varphi \left\{ \iint_F \tau_{zs}' h_p dF + \rho \iint_F \left[(y - y_p)\ddot{u}_* - (x - x_p)\ddot{v}_* \right] dF + \int_s \left[\bar{p}_y (x - x_p) - \bar{p}_x (y - y_p) \right] ds \right\} \\
&-\delta u_p' \left\{ \iint_F (\sigma_z' x + \tau_{zs} \sin\alpha) dF - \rho \iint_F x \ddot{w}_* dF + \int_s \bar{p}_z x ds \right\} - \\
&-\delta v_p' \left\{ \iint_F (\sigma_z' y + \tau_{zs} \cos\alpha) dF - \rho \iint_F y \ddot{w}_* dF + \int_s \bar{p}_z y ds \right\} - \\
&-\delta\varphi' \left\{ \iint_F (\sigma_z' \omega_p + \tau_{zs} h_p + 2\tau_s e) dF - \rho \iint_F \omega_p \ddot{w}_* dF + \int_s \bar{p}_z \omega_p ds \right\} = 0
\end{aligned} \tag{12}$$

Since the virtual displacement parameters can have arbitrary values, equation (12) will be satisfied if the expressions in great brackets vanish.

Using the expressions for stress resultants (3) one obtains:

$$\begin{aligned}
&N' - \rho \iint_F \ddot{w}_* dF + p_z = 0 \\
&V_x' - \rho \iint_F \ddot{u}_* dF + p_x = 0 \\
&V_y' - \rho \iint_F \ddot{v}_* dF + p_y = 0 \\
&T_p' + \rho \iint_F \left[(y - y_p)\ddot{u}_* - (x - x_p)\ddot{v}_* \right] dF \\
&\quad + m_p = 0 \\
&M_y' + V_x + \rho \iint_F x \ddot{w}_* dF + m_y = 0 \\
&M_x' - V_y - \rho \iint_F y \ddot{w}_* dF + m_x = 0 \\
&M_{\omega P}' - T_p + T_s - \rho \iint_F \omega_p \ddot{w}_* dF + m_{\omega_p} = 0
\end{aligned} \tag{13}$$

The forces V_x, V_y, T_p can be eliminated from equation (13) in order to obtained four equations:

$$\begin{aligned}
&N' - \rho \iint_F \ddot{w}_* dF + p_z = 0 \\
&M_y'' + \rho \iint_F x \ddot{w}_*' dF + \rho \iint_F \ddot{u}_* dF - p_x + m_y' = 0 \\
&M_x'' - \rho \iint_F y \ddot{w}_*' dF - \rho \iint_F \ddot{v}_* dF + p_y + m_x' = 0 \\
&M_{\omega P}'' + T_s' - \rho \iint_F \omega_p \ddot{w}_*' dF + \rho \iint_F [(y - y_p)\ddot{u}_* \\
&\quad - (x - x_p)\ddot{v}_*] dF + m_p + m_{\omega_p}' = 0
\end{aligned} \tag{14}$$

The stress resultants can be expressed directly in terms of the displacements. The equations are written in matrix form:

$$\begin{bmatrix} N \\ M_y \\ -M_x \\ -M_{\omega P} \\ T_s \end{bmatrix} = \begin{bmatrix} F_{com} & -S_{x,com} & -S_{y,com} & -S_{\omega P,com} & S_{e,com} \\ -S_{x,com} & I_{xx,com} & I_{xy,com} & I_{x\omega P,com} & -I_{xe,com} \\ -S_{y,com} & I_{xy,com} & I_{yy,com} & I_{y\omega P,com} & -I_{ye,com} \\ -S_{\omega P,com} & I_{x\omega P,com} & I_{y\omega P,com} & I_{\omega P\omega P,com} & -I_{\omega Pe,com} \\ S_{e,com} & -I_{xe,com} & -I_{ye,com} & -I_{\omega Pe,com} & I_{ee,com} \end{bmatrix} \begin{bmatrix} w' \\ u_P'' \\ v_P'' \\ \varphi'' \\ \varphi' \end{bmatrix} \tag{15}$$

Selecting appropriate position and orientation of coordinate axis, and also appropriate position of the pole P and the point zero O_1 we can write:

$$\begin{aligned} S_{x,com} &= 0 \\ S_{y,com} &= 0 \\ I_{xy,com} &= 0 \end{aligned} \tag{16}$$

$$\begin{aligned} S_{\mu P,com} &= 0 \\ I_{x\mu P,com} &= 0 \\ I_{y\mu P,com} &= 0 \end{aligned} \tag{17}$$

Taking into account the conditions (16) and (17) we get simplified expressions for the cross section forces and substituting into (14) we get equations that represent equations of motion of thin-walled beam with arbitrary open cross-section:

$$\begin{aligned} &\begin{bmatrix} F_{com} & 0 & 0 & 0 \\ 0 & I_{xx,com} & 0 & 0 \\ 0 & 0 & I_{yy,com} & 0 \\ 0 & 0 & 0 & I_{\omega_D\omega_D,com} \end{bmatrix} \begin{bmatrix} w''' \\ u_P'''' \\ v_P'''' \\ \varphi'''' \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & S_{e,com} \\ 0 & 0 & 0 & -I_{xe,com} \\ 0 & 0 & 0 & -I_{ye,com} \\ -S_{e,com} & I_{xe,com} & I_{ye,com} & 0 \end{bmatrix} \begin{bmatrix} w'' \\ u_P''' \\ v_P''' \\ \varphi''' \end{bmatrix} - \\ &- \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{ee,com} \end{bmatrix} \begin{bmatrix} w' \\ u_P'' \\ v_P'' \\ \varphi'' \end{bmatrix} - \rho \begin{bmatrix} F & -S_x & -S_y & -S_{\omega_P} \\ -S_x & I_{xx} & I_{xy} & I_{x\omega_P} \\ -S_y & I_{xy} & I_{yy} & I_{y\omega_P} \\ -S_{\omega_P} & I_{x\omega_P} & I_{y\omega_P} & I_{\omega_P\omega_P} \end{bmatrix} \begin{bmatrix} \ddot{w}' \\ \ddot{u}_P'' \\ \ddot{v}_P'' \\ \ddot{\varphi}'' \end{bmatrix} + \\ &+ \rho \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & F & 0 & y_P F - S_y \\ 0 & 0 & F & -x_P F + S_x \\ 0 & y_P F - S_y & -x_P F + S_x & I_P \end{bmatrix} \begin{bmatrix} \ddot{w} \\ \ddot{u}_P \\ \ddot{v}_P \\ \ddot{\varphi} \end{bmatrix} = \begin{bmatrix} -p_z' \\ p_x - m_y' \\ p_y + m_x' \\ m_P + m_{\omega_P}' \end{bmatrix} \end{aligned} \tag{18}$$

Analyzing this system of equations we can conclude that the differential equations of motion cannot be separated, meaning that the equations describing the axial, transverse and torsional vibrations must be solved simultaneously.

3. I Cross Section

If we observe the beams symmetrical around two axes with antisymmetric orientations of the laminas in relation to geometrical axis, elements of the matrix equal to zero are:

$$I_{xe,com} = I_{ye,com} = I_{\omega_D e,com} = 0 \tag{19}$$

As the main central axis of the composite cross-section overlap with the axis of homogenius cross-section we can write:

$$S_x = S_y = S_{\omega_D} = I_{xy} = I_{x\omega_D} = I_{y\omega_D} = 0 \tag{20}$$

For double symmetry center of gravity overlaps with shear center, meaning

$$x_D = y_D = 0 \tag{21}$$

4. Solution for I Cross Section

The solution of equation (18) with assumptions (19) and (20) may be expressed in the form:

$$\begin{bmatrix} w(z,t) \\ \varphi(z,t) \end{bmatrix} = \begin{bmatrix} W(z) \\ \Phi(z) \end{bmatrix} \sin pt \tag{22}$$

Substituting Eq (22) into (18) yields:

$$\begin{aligned} &\begin{bmatrix} F_{com} & 0 \\ 0 & I_{\omega_D\omega_D,com} \end{bmatrix} \begin{bmatrix} W'''' \\ \Phi'''' \end{bmatrix} + \begin{bmatrix} 0 & S_{e,com} \\ -S_{e,com} & 0 \end{bmatrix} \begin{bmatrix} W'' \\ \Phi''' \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & I_{ee,com} \end{bmatrix} \begin{bmatrix} W' \\ \Phi'' \end{bmatrix} + \\ &+\rho p^2 \begin{bmatrix} -F & 0 \\ 0 & I_{\omega_D\omega_D} \end{bmatrix} \begin{bmatrix} W' \\ \Phi'' \end{bmatrix} - \rho p^2 \begin{bmatrix} 0 & 0 \\ 0 & I_D \end{bmatrix} \begin{bmatrix} W \\ \Phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned} \tag{23}$$

In the case of a beam with simply supported ends the end conditions are:

$$\begin{bmatrix} W \\ \Phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} W'' \\ \Phi'' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{24}$$

Equation (24) will be satisfied by taking

$$\begin{bmatrix} W(z) \\ \Phi(z) \end{bmatrix} = \begin{bmatrix} C_W \\ C_\Phi \end{bmatrix} \sin \lambda_n z \tag{25}$$

Where C_W, C_U, C_V, C_Φ are the constants and $\lambda_n = \dfrac{n\pi}{L}$

Substituting Eq (25) into (23) results in:

$$\begin{aligned} &\lambda_n^4 \begin{bmatrix} 0 & 0 \\ 0 & I_{\omega_D\omega_D,com} \end{bmatrix} \begin{bmatrix} C_W \\ C_\Phi \end{bmatrix} + \lambda_n^2 \begin{bmatrix} -F_{com} & -S_{e,com} \\ S_{e,com} & I_{ee,com} \end{bmatrix} \begin{bmatrix} C_W \\ C_\Phi \end{bmatrix} + \\ &-\rho p^2 \lambda_n^2 \begin{bmatrix} 0 & 0 \\ 0 & I_{\omega_D\omega_D} \end{bmatrix} \begin{bmatrix} C_W \\ C_\Phi \end{bmatrix} - \rho p^2 \begin{bmatrix} -F & 0 \\ 0 & I_D \end{bmatrix} \begin{bmatrix} C_W \\ C_\Phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned} \tag{26}$$

Setting the determinant of the above system equal to zero:

$$\begin{vmatrix} -\lambda_n^2 F_{com} + \rho p^2 F & -\lambda_n^2 S_{e,com} \\ \lambda_n^2 S_{e,com} & \lambda_n^4 I_{\omega_D\omega_D,com} + \lambda_n^2 I_{ee,com} - \rho p^2 \lambda_n^2 I_{\omega_D\omega_D} - \rho p^2 I_D \end{vmatrix} = 0 \tag{27}$$

yields the following algebraic frequency equation

$$a \cdot p^4 + b \cdot p^2 + c = 0 \tag{28}$$

with the coefficients:

$$\begin{aligned} a &= -\rho^2 \lambda_n^2 F I_{\omega_D \omega_D} - \rho^2 I_D F \\ b &= \rho \lambda_n^4 F_{com} I_{\omega_D \omega_D} + \rho \lambda_n^4 F I_{\omega_D \omega_D, com} + \rho \lambda_n^2 I_D F_{com} + \rho \lambda_n^2 F I_{ee, com} \\ c &= -\lambda_n^6 F_{com} I_{\omega_D \omega_D, com} - \lambda_n^4 F_{com} I_{ee, com} + \lambda_n^4 S_{e, com}^2 \end{aligned} \tag{29}$$

5. Numerical Example

Torsional frequences for the simply supported I-beam with arbitrary lamination in relation to the midline are evaluated. We consider I-beam that has a flange width 50cm and the height 60cm. Total thicknesses of the web and flanges are 3cm. It is assumed to have eight layers in laminate, material properties are:

$E_1 = 144\ GPa,\ E_2 = 9.65\ GPa,$

$G_{12} = 4.14\ GPa,\ \upsilon = 0.3$

Frequency (Hz) *of the beam in observed example* Table 1.

	$[0/30/60/90]_S$			$[0/90/0/90]_S$			$[30/-30/30/-30]_S$		
	Theory	Ansys	Error %	Theory	Ansys	Error %	Theory	Ansys	Error %
I	13.475	14.253	5.78	14.968	14.711	1.72	15.079	16.240	7.70
II	47.885	48.105	0.46	55.905	52.616	5.88	47.990	49.032	2.16

6. Conclusion

Principal of virtual displacements is used for solving system of equations of thin-walled laminated beams.

Solution is derived for natural frequences of free harmonic vibrations. Accuracy of derived solution is confirmed with numerical example.

Torsional vibrations of simply supported beam obtained theoretically coincide with the solution obtained with the software package Ansys [5] and that confirms validation of derived solution. Fiber orientation has influence on the results.

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